PP2-3-17 **Conformal Martingale**

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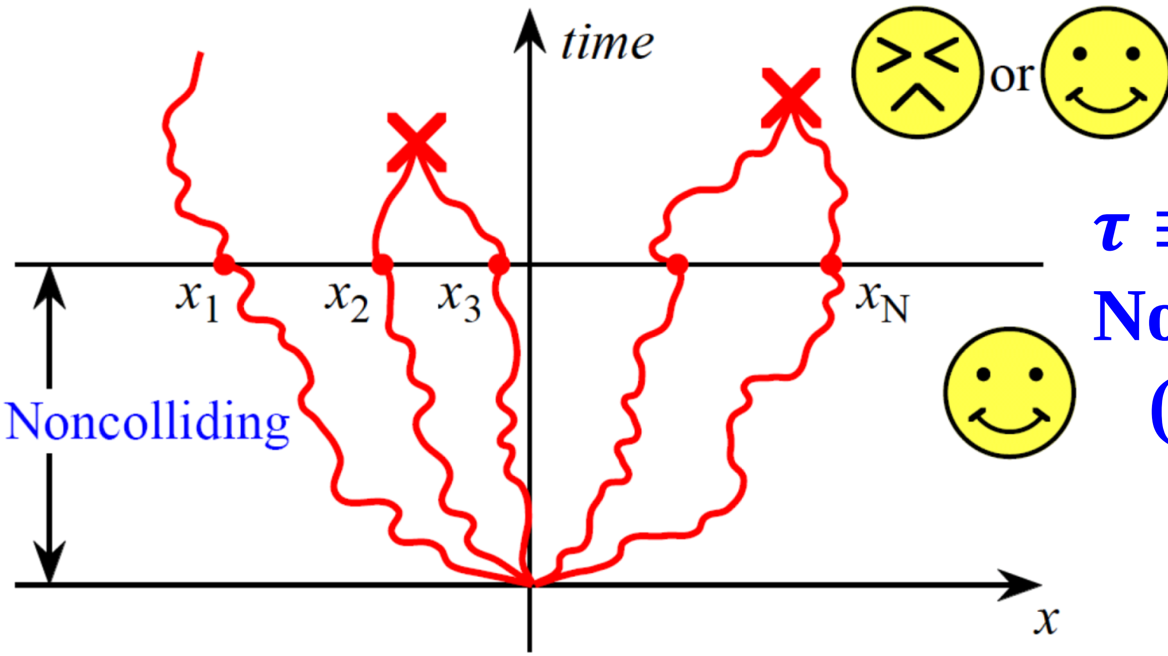

$\tau \equiv$ first collision time

Non-colliding condition
(for peace-keeping !)

$\tau \to \infty$

[KT10] M.K. and H. Tanemura: *Commun. Math. Phys.* 293 (2010) 469
[KT13] M.K. and H. Tanemura: *Electron. Commun. Probab.* 18 (2013) 1
[K13a] M.K. arXiv:math.PR/1305.4412
[K13b] M.K. arXiv:math.PR/1307.1856

Representations of Log-Gases

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1. Dyson's Brownian Motion Model

= Eigenvalue Process of Hermitian Matrix-valued Brownian motion (BM) ← **Random Matrix Theory**

= BMs Conditioned Never to Collide with Each Other **(Noncolliding BMs)**

= Interacting BMs with Repulsive Log-Potentials **(Log-Gase)**

Stochastic Differencial Equations

$$\boldsymbol{X}(t) = (X_1(t), X_2(t), \dots, X_N(t))$$

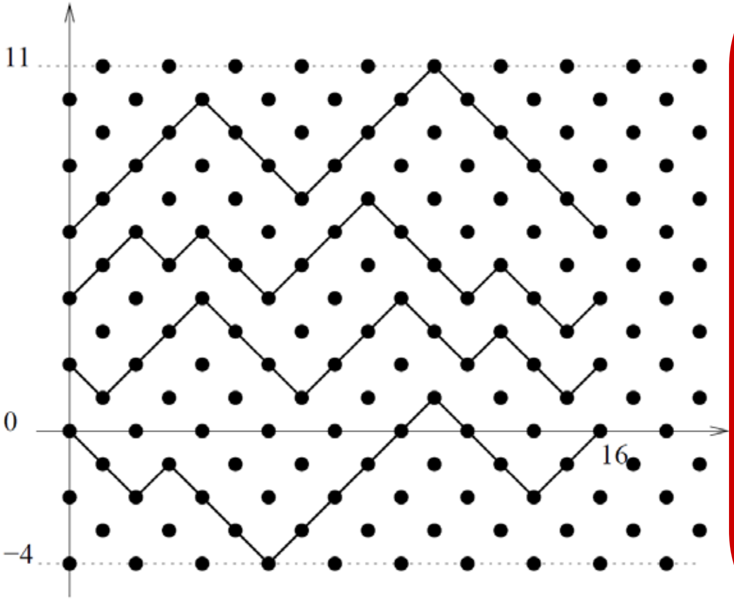
$$dX_j(t) = dB_j(t) + \sum_{1 \le k \le N: k \ne j} \frac{dt}{X_j(t) - X_k(t)},$$

$$1 \le j \le N, \quad t \ge 0$$

2. Fisher's Vicious Walker Model

= Random Walks (RWs) Conditioned Never to Collide with Each Other **(Noncolliding RWs)**

= Discrete Version of Dyson's BM model



$X_j(0) = v_j \in 2\mathbf{Z}$

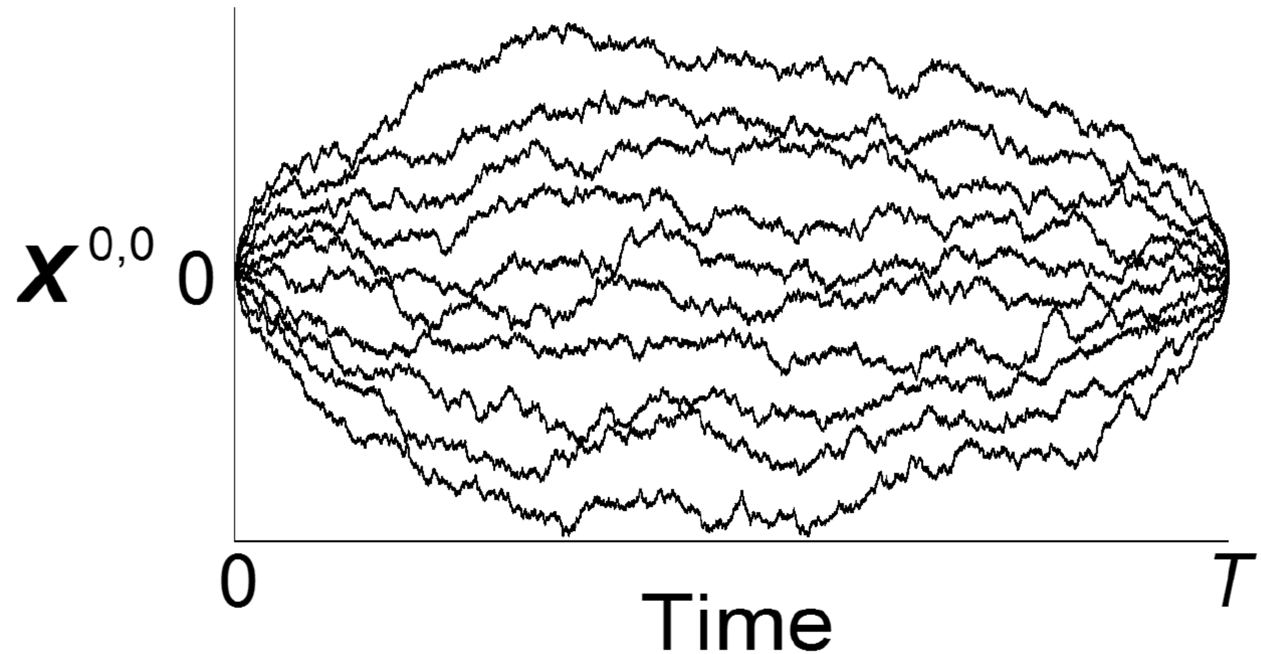
$X_j(t) = v_j + \zeta_j(1) + \cdots \zeta_j(t)$

$\zeta_j(t)$: i.i.d. $t \in \mathbf{N}, \quad 1 \le j \le N$

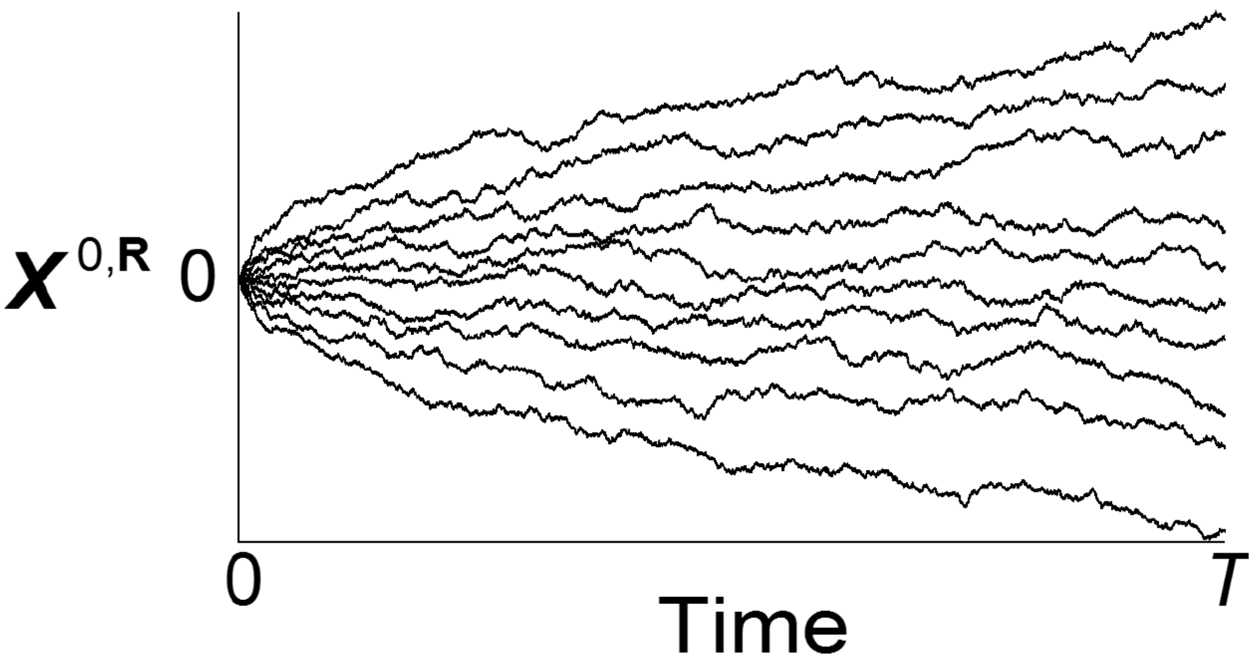
$\mathrm{P}[\zeta_j(1) = 1] = \mathrm{P}[\zeta_j(1) = -1] = \frac{1}{2},$

Non-Colliding Condition

$X_1(t) < X_2(t) < \cdots < X_N(t), \forall t \in \mathbf{N}_0$

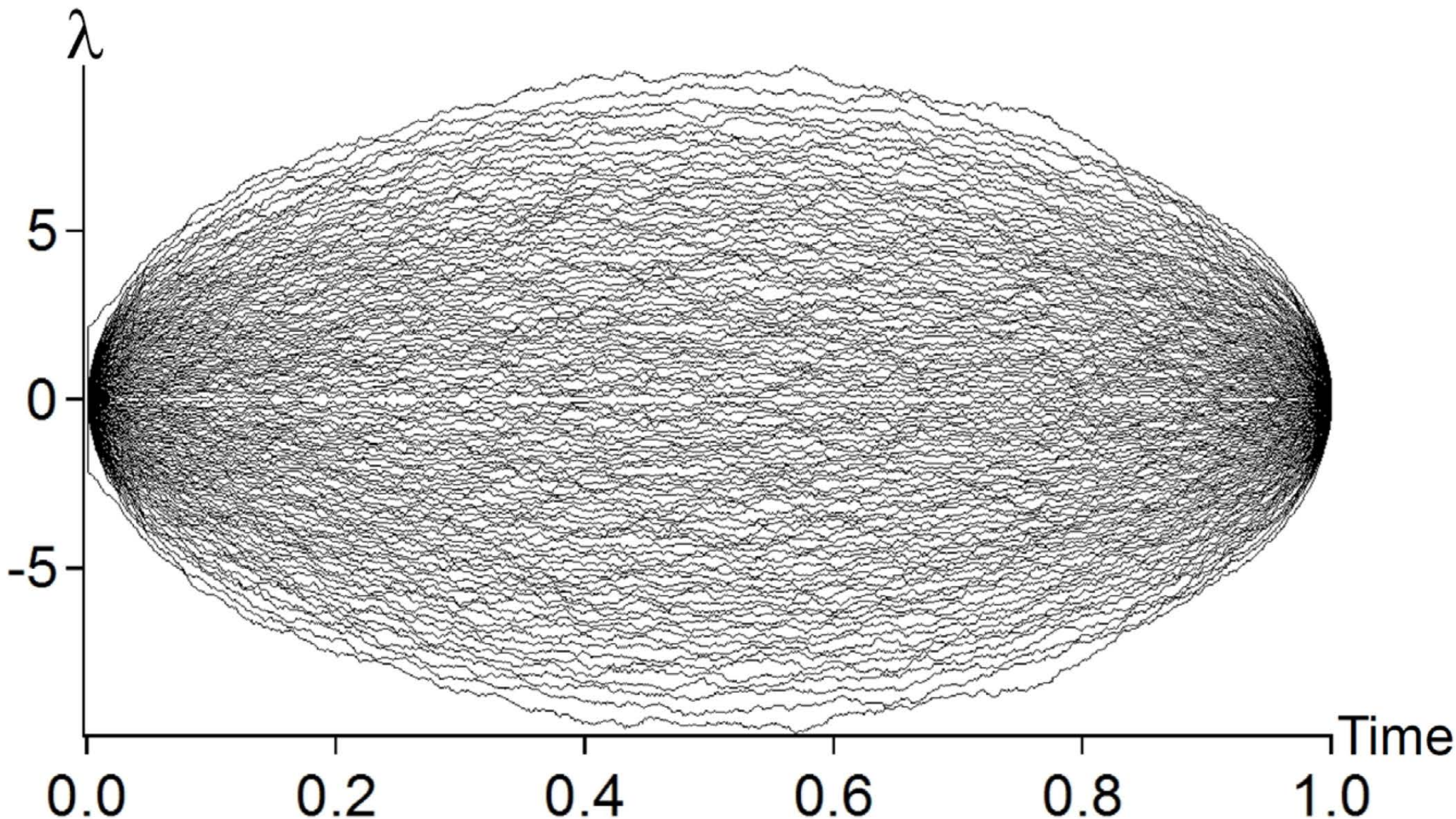


Watermelon Configuration



Star Configuration

100 paths of noncolliding Brownian bridges with reflection symmetry



3. Conformal Martingales (CM)

- A **complex process** $Z(t) = V(t) + iW(t)$
  for Dyson's model: $V(t), W(t)$: indep. **BMs**
  for Fisher's model: $V(t)$: **RW**, $W(t)$: **a Lévy process**
- N independent copies $\boldsymbol{Z}(t) = (Z_1(t), Z_2(t), \dots, Z_N(t))$
  with distinct initial points $Z_j(0) = v_j \in \mathbf{R}$, $\boldsymbol{v} = (v_j)$
- Polynomials (→ **entire functions** in $N \to \infty$)

$$\boldsymbol{\Phi}_{\boldsymbol{v}}{}^{v_k}\left(Z_j(t)\right) = \prod_{1 \le \ell \le N, \ell \ne k} \frac{Z_j(t) - v_\ell}{v_k - v_\ell}, 1 \le j, k \le N$$

- **Conformal martingales**, $0 \le \forall t < \infty$

$$\left\langle \boldsymbol{\Phi}_{\boldsymbol{v}}{}^{v_k}\left(Z_j(t)\right) \right\rangle = \left\langle \boldsymbol{\Phi}_{\boldsymbol{v}}{}^{v_k}\left(Z_j(0)\right) \right\rangle = \boldsymbol{\Phi}_{\boldsymbol{v}}{}^{v_k}\left(v_j\right) = \delta_{jk}$$

4. CM Representations

- Assume $\boldsymbol{X}(0) = \boldsymbol{v} \in \mathbb{W}_N$ (Weyl chamber $v_1 < v_2 < \cdots < v_N$)
- $F\left[\left(\boldsymbol{X}(t)\right)_{0 \le t \le T}\right]$: any observable up to time $T < \infty$

Theorem

$$\left\langle F\left[\left(\boldsymbol{X}(t)\right)_{0 \le t \le T}\right]\right\rangle = \left\langle F\left[\left(\mathrm{Re}\,\boldsymbol{Z}(t)\right)_{0 \le t \le T}\right] \det_{1 \le j,k \le N}\left[\boldsymbol{\Phi}_{\boldsymbol{v}}{}^{v_k}\left(Z_j(T)\right)\right]\right\rangle$$

(LHS) = expectation for the present **interacting particle systems** (Dyson's and Fisher's Log-Gases)

(RHS) = expectation for **independent complex processes** with **weight** $\det_{1 \le j,k \le N}\left[\boldsymbol{\Phi}_{\boldsymbol{v}}{}^{v_k}\left(Z_j(T)\right)\right]$ **at the final time** $\boldsymbol{T}$

5. Determinantal Processes

From our Conformal Martingale Representations, we can prove the following;

1. The present log-gases are **determinantal processes** for any finite initial configuration. That is, all spatio-temporal correlation functions are given by determinants.
2. These determinants are specified by a single function called the **correlation kernel**, which is **directly determined by the conformal martingales** $\boldsymbol{\Phi_v}^{v_k}(\boldsymbol{Z_j}(\boldsymbol{T})), 1 \leq j, k \leq N$.
3. The results are extended to (i) the case that the initial config. ***v*** has **multiple points**, to (ii) **infinite particle systems**, and to (iii) other systems including **noncolliding Bessel processes** and **O'Connell's processes** (geometric liftings of Log-Gases).
4. A variety of **Eynard-Mehta type dynamical correlation kernels** are readily derived by choosing ***v*** appropriately.

An example of application of CMR: Relaxation Phenomena to Equilibrium Dynamics

If we start the processes from the **infinite equidistant points**, *a* **Z**, we can trace the **non-equilibrium dynamics with infinite numbers of particles** showing **relaxation phenomena** to equilibrium dynamics. The equilibrium dynamics are determinantal processes with **extended (continuous and discrete) sine kernels**.