Elliptic Determinantal Processes

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1. Introduction

We have studied **Noncolliding Diffusion Processes** in state spaces S.
(In this talk we will consider only systems with finite numbers of particles $N < \infty$.)

Examples

(1) Noncolliding Brownian motion (Dyson's Brownian motion model with $\beta = 2$**);**

$$\begin{aligned} S &= \mathbb{R}, \\ X_j(t) &= u_j + B_j(t) + \sum_{\substack{1 \le k \le N, \\ k \ne j}} \int_0^t \frac{ds}{X_j(s) - X_k(s)}, \quad 1 \le j \le N, \quad t \ge 0. \end{aligned}$$

(2) Noncolliding squared Bessel process (BESQ$^{(\nu)}$**) with** $\nu > -1$**;**

$$\begin{aligned} S &= \mathbb{R}_+ \cup \{0\}, \quad \mathbb{R}_+ \equiv \{x \in \mathbb{R} : x > 0\}, \\ X_j(t) &= u_j + \int_0^t 2\sqrt{X_j(s)} dB_j(s) + 2(\nu + 1)t + \sum_{\substack{1 \le k \le N, \\ k \ne j}} \int_0^t \frac{4X_j(s) ds}{X_j(s) - X_k(s)}, \quad 1 \le j \le N, \quad t \ge 0, \end{aligned}$$

where if $-1 < \nu < 0$ the reflection boundary condition is assumed at the origin.

(3) Noncolliding BM on a circle with a radius $r > 0$**;**
(trigonometric extension of Dyson's Brownian motion model with $\beta = 2$**);**
We solve the SDEs

$$\check{X}_j(t) = u_j + B_j(t) + \frac{1}{2r} \sum_{\substack{1 \le k \le N, \\ k \ne j}} \int_0^t \cot \left(\frac{\check{X}_j(s) - \check{X}_k(s)}{2r} \right) ds, \quad 1 \le j \le N, \quad t \ge 0,$$

on $\mathbb{R}$ and then define the process on $S = [0, 2\pi r)$ by

$$X_j(t) = \check{X}_j(t) \mod 2\pi r, \quad 1 \le j \le N, \quad t \ge 0.$$

Weyl chamber

$$\mathbb{W}_N(S) = \{\boldsymbol{x} = (x_1, \dots, x_N) \in S^N : x_1 < x_2 < \cdots < x_N\}.$$

Noncolliding Property

$$\boldsymbol{X}(0) = \boldsymbol{u} = (u_1, \dots, u_N) \in \mathbb{W}_N(S) \quad \Longrightarrow \quad \boldsymbol{X}(t) \in \mathbb{W}_N(S) \quad {}^{\forall} t \geq 0 \quad \text{with probability 1.}$$

$$\mathfrak{M}(S) = \left\{ \xi = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot) : \xi(K) = \sharp\{x_j : x_j \in K\} < \infty, \ {}^{\forall} \text{ compact subset } K \subset S \right\},$$

$$\mathfrak{M}_0(S) = \{\xi \in \mathfrak{M}(S) : \xi(\{x\}) \leq 1, \ {}^{\forall} x \in S\}.$$

labeled configuration
$\boldsymbol{X}(t), t \geq 0$

⟺

unlabeled configuration
$$\Xi(t, \cdot) = \sum_{j=1}^{N} \delta_{X_j(t)}(\cdot), t \geq 0$$

$$\xi = \sum_{j=1}^{N} \delta_{u_j} \in \mathfrak{M}_0(S) \quad \Longrightarrow \quad \Xi(t) \in \mathfrak{M}_0(S) \quad {}^{\forall} t \geq 0 \quad \text{with probability 1.}$$

Two Aspects of Noncolliding Diffusion Processes

- Although they are originally introduced as eigenvalue processes of matrix-valued diffusions, they are realized as **harmonic Doob transforms** of absorbing particle systems in the Weyl chambers.

generalization

$$\mathrm{BES}(3) = \text{harmonic Doob transform of BM in } \mathbb{R}_+ \text{ with an absorbing wall at } 0$$

- They are **exactly solvable (stochastic integrable models)** in the sense that any spatio-temporal correlation function can be explicitly expressed by a determinant specified by a single continuous function called the correlation kernel. Such sustems are called **determinantal processes**.

Recently we clarified the connection between these two aspects by introducing a notion of **determinantal martingale**.

harmonic function $h(\boldsymbol{x})$ ⇒ martingale function $\mathcal{M}_\xi^u(t,x)$ ⇒ correlation kernel $\mathbb{K}_\xi(s,x;t,y)$

K. : Determinantal martingales and noncolliding diffusion processes. SPA **124**, 3724-3768 (2014)

2. Trigonometric & Elliptic Extensions of BES(3)

BES(3) $(X(t), t \geq 0, \mathbb{P}_u)$

$$S = \mathbb{R}_+,$$

$$X(t) = u + B(t) + \int_0^t \frac{ds}{X(s)}, \quad t \geq 0,$$

$$u = X(0) \in \mathbb{R}_+ \quad \Longrightarrow \quad X(t) \in \mathbb{R}_+, \quad {}^{\forall} t \geq 0, \quad \text{w.p.1.}$$

♣ $(W(t), t \geq 0, \mathrm{P}_u)$: one-dim standard BM started at $u > 0$.

$$T'_W = \inf\{t > 0 : W(t) = 0\}$$

$$\mathbb{P}_u(X(t) \in dx) = \mathrm{P}_u(T'_W > t, W(t) \in dx) \frac{x}{u}, \quad t \in [0, \infty).$$

Trigonometric extension of BES(3) $(X(t), t \geq 0, \mathbb{P}_u)$

$$S = (0, 2\pi r),$$

$$\check{X}(t) = u + B(t) + \frac{1}{2r} \int_0^t \cot\left(\frac{\check{X}(s)}{2r}\right) ds, \quad t \geq 0,$$

$$X(t) = \check{X}(t) \mod 2\pi r, \quad t \in [0, \infty)$$

$r \to \infty$

♣ $(W(t), t \geq 0, \mathrm{P}_u)$: one-dim standard BM started at $u > 0$.

$$\mathbb{P}_u(X(t) \in dx) = \mathrm{P}_u(T_W > t, W(t) \in dx) \frac{\sin(x/2r)}{\sin(u/2r)}, \quad t \in [0, \infty), \quad u, x \in (0, 2\pi r),$$

$$T_W = \inf\{t > 0 : W(t) \in \{0, 2\pi r\}\}.$$

Elliptic extension of BES(3) Assume $0 < t_* < \infty$. $(X(t), t \in [0, t_*), \mathbb{P}_u)$

$S = (0, 2\pi r)$,

$$\check{X}(t) = u + B(t) + \int_0^t A_1^{2\pi r}(t_* - s, \check{X}(s))ds, \quad t \in [0, t_*).$$

Here

$$A_N^\alpha(t_* - t, x) = \frac{1}{\alpha}\left[\frac{d}{dv}\log \vartheta_1(v;\tau)\right]_{v=x/\alpha, \tau=2\pi i N(t_*-t)/\alpha^2} = \frac{1}{\alpha}\frac{\vartheta_1'(x/\alpha; 2\pi i N(t_* - t)/\alpha^2)}{\vartheta_1(x/\alpha; 2\pi i N(t_* - t)/\alpha^2)},$$

where $N \in \mathbb{N}$ and $\vartheta_1'(v;\tau) = d\vartheta_1(v;\tau)/dv$. (I will explain Jacobi's theta function ϑ_1 shortly.)

♣ $(W(t), t \geq 0, \mathrm{P}_u)$: one-dim standard BM started at $u > 0$.

$$\mathbb{P}_u(X(t) \in dx) = \mathrm{P}_u(T_W > t, W(t) \in dx)\frac{\vartheta_1(x/2\pi r; i(t_* - t)/2\pi r^2)}{\vartheta_1(u/2\pi r; it_*/2\pi r^2)}, \quad u, x \in (0, 2\pi r),$$

$T_W = \inf\{t > 0 : W(t) \in \{0, 2\pi r\}\}$.

♣
$$A_N^{2\pi r}(t_* - t, x) \sim \frac{1}{x} \quad \text{as } x \downarrow 0,$$
$$A_N^{2\pi r}(t_* - t, x) \sim -\frac{1}{2\pi r - x} \quad \text{as } x \uparrow 2\pi r.$$

Elliptic BES(3) $\xrightarrow{t_* \to \infty}$ Trigonometric BES(3) $\xrightarrow{r \to \infty}$ BES(3)

Elliptic functions and their related functions

- $i = \sqrt{-1}, v, \tau \in \mathbb{C}$

$$z = z(v) = e^{\pi i v}, \quad q = q(\tau) = e^{\pi i \tau}.$$

- The **Jacobi theta function** ϑ_1 is defined as

$$\begin{aligned}\vartheta_1(v;\tau) &= i \sum_{n\in\mathbb{Z}} (-1)^n q^{(n-(1/2))^2} z^{2n-1} \\ &= -iq^{1/4} z \prod_{k=1}^{\infty} (1-q^{2k}) \prod_{j=1}^{\infty} (1-q^{2j}z^2)(1-q^{2j-2}/z^2).\end{aligned}$$

- For $\Im\tau > 0$, $\vartheta_1(v;\tau)$ is holomorphic for $|v| < \infty$ and satisfies the partial differential equation

$$\frac{\partial \vartheta_1(v;\tau)}{\partial \tau} = \frac{1}{4\pi i} \frac{\partial^2 \vartheta_1(v;\tau)}{\partial v^2}.$$

- For $N \in \mathbb{N} \equiv \{1, 2, \dots\}, \alpha > 0$, and $0 < t_* < \infty$,

$$A_N^\alpha(t_* - t, x) \equiv \left[\frac{1}{\alpha}\frac{d}{dv}\log\vartheta_1(v;\tau)\right]_{v=x/\alpha,\tau=2\pi i N(t_*-t)/\alpha^2}$$
$$= \frac{\pi}{\alpha}\cot\left(\frac{\pi x}{\alpha}\right) + \frac{4\pi}{\alpha}\sum_{n=1}^{\infty}\frac{e^{-4\pi^2 nN(t_*-t)/\alpha^2}}{1 - e^{-4\pi^2 nN(t_*-t)/\alpha^2}}\sin\left(\frac{2\pi n x}{\alpha}\right),$$
$$(t, x) \in [0, t_*) \times \mathbb{R}.$$

- $t \in [0, t_*) \implies \Im\tau > 0 \implies \vartheta_1$ is holomorphic for $|v| < \infty$.
- As a function of $x \in \mathbb{R}$, it is odd; $A_N^\alpha(t_* - t, -x) = -A_N^\alpha(t_* - t, x)$, and periodic with period α; $A_N^\alpha(t_* - t, x + m\alpha) = A_N^\alpha(t_* - t, x), \quad m \in \mathbb{Z}$.
- It has only simple poles at $x = m\alpha, m \in \mathbb{Z}$, and simple zeroes at $x = (m + 1/2)\alpha, m \in \mathbb{Z}$.

Another Expression

$$A_N^\alpha(t_* - t, x) = \left[\zeta(x|2\omega_1, 2\omega_3) - \frac{\eta_1 x}{\omega_1}\right]_{\omega_1=\alpha/2, \omega_3=\pi i N(t_*-t)/\alpha}.$$

- Let ω_1 and ω_3 be fundamental periods and set $\omega_2 = -(\omega_1 + \omega_3)$,

$$\tau = \frac{\omega_3}{\omega_1}, \quad \Im\tau > 0, \quad \Omega_{m,n} = 2m\omega_1 + 2n\omega_3, \quad m, n \in \mathbb{Z}.$$

- The **Weierstrass $\wp$ function** and **zeta function** ζ are defined as the following meromorphic functions

$$\begin{aligned} \wp(z) &= \wp(z|2\omega_1, 2\omega_3) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left[\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{{\Omega_{m,n}}^2}\right], \\ \zeta(z) &= \zeta(z|2\omega_1, 2\omega_3) = \frac{1}{z} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left[\frac{1}{z-\Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{{\Omega_{m,n}}^2}\right]. \end{aligned}$$

- The function $\wp(z)$ is an **elliptic function** (a meromorphic and doubly periodic function) with fundamental periods ω_1, ω_2 and ω_3;
  $$\wp(z + 2\omega_\nu) = \wp(z), \quad \nu = 1, 2, 3,$$
- By definition, $\wp(z) = -\zeta'(z)$.
- Let $\eta_\nu = \zeta(\omega_\nu), \quad \nu = 1, 2, 3.$
- The function ζ is quasi-periodic in the sense $\zeta(z + 2\omega_\nu) = \zeta(z) + 2\eta_\nu, \quad \nu = 1, 2, 3.$

- Let

$$\mathcal{K}_q(z^2) \equiv i\frac{\alpha}{\pi} A_N^{\alpha}(t_* - t, x),$$

where

$$q = e^{-2\pi^2 N(t_*-t)/\alpha^2} \quad (0 < q < 1), \quad z^2 = e^{2\pi i x/\alpha}.$$

- Then

$$\mathcal{K}_q(z) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1+q^{2n}z}{1-q^{2n}z}.$$

It is **Villat's kernel for an annulus**, $\mathbb{A}_q = \{z \in \mathbb{C} : q < |z| < 1\}$.

- The **radial Komatu-Loewner evolution** in $\mathbb{A}_q$ is given by (Zhan 2004, Bauer-Friedrich 2008)

$$\frac{d}{dt} \log g_t(z) = i\mathcal{K}_q(g_t(z)/\xi_t),$$

where ξ_t is a driving function on the unit circle (the outer circle of $\mathbb{A}_q$).

- We note that

$$\mathcal{F}_q(z) = i\mathcal{K}_q(z): \quad \text{conformal map} \quad \mathbb{A}_q \to D(\boldsymbol{s}) \equiv \mathbb{H} \setminus [-a(q)+i, a(q)+i],$$

$$a(q) = \frac{1}{2} q_0^2 (q_2^4 - q_3^4) \quad \text{with} \quad q_0 = \prod_{n=1}^{\infty} (1-q^{2n}), \quad q_2 = \prod_{n=1}^{\infty} (1+q^{2n-1}), \quad q_3 = \prod_{n=1}^{\infty} (1-q^{2n-1}).$$

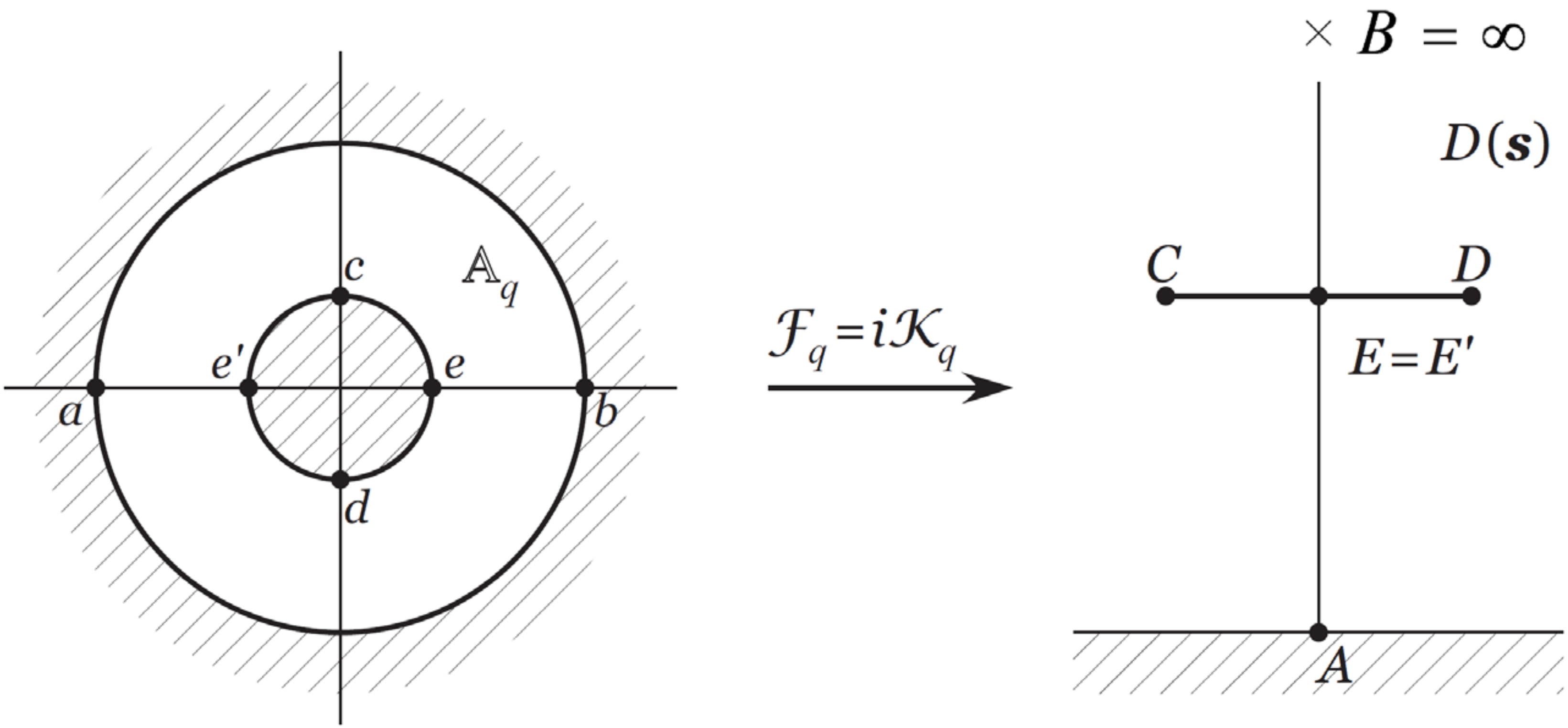

Figure 1: Conformal map $\mathcal{F}_q = i\mathcal{K}_q$ from $\mathbb{A}_q$ to $D(\boldsymbol{s}) = \mathbb{H} \setminus [-a(q)+i, a(q)+i]$. Points $a, b, c, \ldots$ are mapped to points $A, B, C, \ldots$, where $a = -1, b = 1, c = iq, d = -iq, e = q, e' = -q$, and $A = 0, B = \infty, C = -a(q)+i, D = a(q)+i, E = E' = i$.

3. Elliptic Dyson Model and Generalized *h*-Transform

Elliptic extension of Dyson model

$$\check{X}_j^A(t) = u_j + B_j(t) + \sum_{\substack{1 \le k \le N, \\ k \neq j}} \int_0^t A_N^{2\pi r}(t_* - s, \check{X}_j^A(s) - \check{X}_k^A(s))ds + \int_0^t A_N^{2\pi r}(t_* - s, \overline{X}_\delta^A(s))ds,$$

$1 \le j \le N, t \in [0, t_*)$, where $\overline{X}_\delta^A(t) = \delta + \sum_{j=1}^N \check{X}_j^A(t)$.

- Initial conditions

  $$\check{X}^A(0) = \boldsymbol{u} \in \mathcal{A}_{2\pi r}^{A_{N-1}} \equiv \{\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N < x_1 + 2\pi r\},$$

  which is called a **scaled alcove of the affine Weyl group of type** A_{N-1} (with scale $2\pi r$).

- Choose an index $\delta \in \pi r \mathbb{Z}$, so that $\overline{u}_\delta \equiv \delta + \sum_{j=1}^N u_j \in (0, 2\pi r)$.

- We set $X_j^A(t) = \check{X}_j^A(t) \bmod 2\pi r$, $t \in [0, t_*)$.

- The unlabeled process is denoted by $(\Xi^A(t), t \in [0, t_*), \mathbb{P}_\xi^A)$, where $\Xi^A(t, \cdot) = \sum_{j=1}^N \delta_{X_j^A(t)}(\cdot)$, and the initial configuration is $\xi = \sum_{j=1}^N \delta_{u_j}$. The expectation is denoted by $\mathbb{E}_\xi^A$.

- Let $\mathcal{F}_{\Xi^A}(t) = \sigma(\Xi^A(s), 0 \le s \le t), t \in [0, t_*)$.

- Let $\check{\boldsymbol{W}}(t) = (\check{W}_1(t), \dots, \check{W}_N), t \geq 0$ be N-dimensional Brownian motion on $(\mathrm{S}^1(r))^N$ started at $\boldsymbol{u} \in \mathcal{A}_{2\pi r}^{A_{N-1}}$. Here $\mathrm{S}^1(r) = \{x \in \mathbb{R} : x + 2\pi r = x\}$ (a circle with radius $r > 0$). The expectation with respect to this process is denoted by $\check{\mathrm{E}}_{\boldsymbol{u}}$.

- Consider a stopping time $T_{\check{\mathbf{W}}} = \inf\{t > 0 : \check{\boldsymbol{W}}(t) \notin \mathcal{A}_{2\pi r}^{A_{N-1}}\}$. Put $\overline{W}_\delta = \delta + \sum_{j=1}^N \check{W}_j(t)$. Consider also $T_{\overline{W}_\delta} = \inf\{t > 0 : \overline{W}_\delta \in \{0, 2\pi r\}\}$.

- Let

$$
\begin{aligned}
h_N^A(t_* - t, \boldsymbol{x}) &= h_N^A(t_* - t, \boldsymbol{x}; r, t_*) \\
&= e^{-N(N-1)(N-2)t_*/48r^2} \eta(e^{-N(t_*-t)/r^2})^{-(N-1)(N-2)/2} \\
&\quad \times \vartheta_1\left(\frac{\overline{x}_\delta}{2\pi r}; \frac{iN(t_*-t)}{2\pi r^2}\right) \prod_{1 \leq j < k \leq N} \vartheta_1\left(\frac{x_k - x_j}{2\pi r}; \frac{iN(t_*-t)}{2\pi r^2}\right),
\end{aligned}
$$

$t \in [0, t_*), \boldsymbol{x} \in \mathcal{A}_{[0,2\pi r)^N} \equiv \mathcal{A}_{2\pi r}^{A_{N-1}} \cap \{\boldsymbol{x} \in \mathbb{R}^N : x_1 \geq 0\}$,

where $\eta(x) = x^{1/24} \prod_{n=1}^{\infty} (1 - x^n)$ (Dedekind's η-function).

Proposition 1

Suppose $\xi = \sum_{j=1}^N \delta_{u_j} \in \mathfrak{M}_0([0, 2\pi r))$. Let $T \in [0, t_*)$. For any $\mathcal{F}_{\Xi^A}(T)$-measurable function F,

$$
\mathbb{E}_\xi^A[F(\Xi^A(\cdot))] = \check{\mathrm{E}}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^N \delta_{\check{W}_j(\cdot)}\right) \mathbf{1}(T_{\check{\mathbf{W}}} \wedge T_{\overline{W}_\delta} > T) \frac{h_N^A(t_* - T, \check{\boldsymbol{W}}(T))}{h_N^A(t_*, \boldsymbol{u})}\right].
$$

4. Determinantal Structure

Proposition 1

Suppose $\xi = \sum_{j=1}^{N} \delta_{u_j} \in \mathfrak{M}_0([0, 2\pi r))$. Let $T \in [0, t_*)$. For any $\mathcal{F}_{\Xi^A}(T)$-measurable function F,

$$\mathbb{E}_{\xi}^{A}[F(\Xi^A(\cdot))] = \check{\mathrm{E}}_{\boldsymbol{u}} \left[F\left(\sum_{j=1}^{N} \delta_{\check{W}_j(\cdot)} \right) \mathbf{1}(T_{\check{\mathbf{W}}} \wedge T_{\overline{W}_\delta} > T) \frac{h_N^A(t_* - T, \check{\boldsymbol{W}}(T))}{h_N^A(t_*, \boldsymbol{u})} \right].$$

- We consider the Brownian motion $\boldsymbol{V}^r(\cdot)$ started at $\boldsymbol{u} \in \mathcal{A}_{[0,2\pi r)^N}$ with an index $\delta \in \pi r\mathbb{Z}$ chosen as $\overline{u}_\delta \in (0, 2\pi r)$.
- It is killed when it arrives at the boundary of $\mathcal{A}_{2\pi r}^{A_{N-1}}$ and when $\overline{V}_\delta^r(\cdot) \in \{0, 2\pi r\}$.
- Let $q_N^A(t, \boldsymbol{y}|\boldsymbol{x})$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}_{[0,2\pi r)^N}$, $t \in [0, t_*)$ be the transition probability density of $\boldsymbol{V}^r(\cdot)$, which satisfies

$$\lim_{t \downarrow 0} q_N^A(t, \boldsymbol{y}|\boldsymbol{x}) = \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^{N} \delta_{y_{\sigma(j)}}(\{x_j\}),$$

Problem Find $q_N^A(t - s, \boldsymbol{y}|\boldsymbol{x})$.

This problem was solved only for the following special initial condition;

$$\eta = \sum_{j=1}^{N} \delta_{v_j} \quad \text{with} \quad v_j = \frac{2\pi r}{N}(j-1), \quad 1 \le j \le N, \quad \delta = -\pi r(N-2).$$

It is the configuration with equidistant spacing on $\mathrm{S}^1(r)$.

- We write the transition probability density of BM on $\mathbb{R}$ as

$$p_{\mathrm{BM}}(t, y|x) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}, \quad x, y \in \mathbb{R}, \quad t \in [0, \infty).$$

- By wrapping it on $\mathrm{S}^1(r)$, we define

$$p^r_{A_{N-1}}(t, y|x) = \begin{cases} \displaystyle\sum_{\ell \in \mathbb{Z}} p_{\mathrm{BM}}(t, y + 2\pi r\ell|x), & \text{if } N \text{ is even}, \\ \displaystyle\sum_{\ell \in \mathbb{Z}} (-1)^\ell p_{\mathrm{BM}}(t, y + 2\pi r\ell|x), & \text{if } N \text{ is odd}, \end{cases} \qquad x, y \in [0, 2\pi r), \quad t \ge 0.$$

Proposition 2

For $N \in \{2, 3, \dots\}$, $\boldsymbol{v}$ is the N-particle configuration with equidistant spacing on $\mathrm{S}^1(r)$. Then for $\boldsymbol{y} \in \mathcal{A}_{[0,2\pi r)^N}$, $t > 0$,

$$q^A_N(t, \boldsymbol{y}|\boldsymbol{v}) = \det_{1 \le j,k \le N} \left[p^r_{A_{N-1}}(t, y_j|v_k) \right].$$

Sketch of Proof

- By Jacobi's imaginary transform, we have the expression

$$p^r_{A_{N-1}}(t,y|x) = \begin{cases} \dfrac{1}{2\pi r}\vartheta_3\left(\dfrac{y-x}{2\pi r}; \dfrac{it}{2\pi r^2}\right), & \text{if } N \text{ is even,} \\ \dfrac{1}{2\pi r}\vartheta_2\left(\dfrac{y-x}{2\pi r}; \dfrac{it}{2\pi r^2}\right), & \text{if } N \text{ is odd,} \end{cases}$$

  where ϑ_2 and ϑ_3 are some modified versions of ϑ_1.

- Then we can use the determinantal equalities found in the textbook of Forrester (Proposition 5.6.3 in Forrester, *Log-gases and Random Matrices* (2010)).

- We find

$$\begin{aligned} q^A_N(t,\boldsymbol{y}|\boldsymbol{v}) &= \det_{1\le j,k\le N}\left[p^r_{A_{N-1}}(t,y_j|v_k)\right] = \left(\frac{\sqrt{N}}{2\pi r}\right)^N \eta(e^{-Nt/r^2})^{-(N-1)(N-2)/2} e^{Nt/8r^2} \\ &\quad \times \vartheta_1\left(\frac{\overline{y}_{-\pi r(N-2)}}{2\pi r}; \frac{iNt}{\pi r^2}\right) \prod_{1\le j<k\le N} \vartheta_1\left(\frac{y_k-y_j}{2\pi r}; \frac{iNt}{2\pi r^2}\right), \end{aligned}$$

- Since

$$\frac{\partial \vartheta_\mu(v;\tau)}{\partial \tau} = \frac{1}{4\pi i}\frac{\partial^2 \vartheta_\mu(v;\tau)}{\partial v^2}, \quad \mu = 0,1,2,3,$$

  it is obvious that q^A_N solves the diffusion equation.

- This expression guarantees the positivity and finiteness of $\det_{1\le j,k\le N}[p^r_{A_{N-1}}(t,y_j|v_k)]$ for $\boldsymbol{y} \in \mathcal{A}_{[0,2\pi r)^N}$ and $\overline{y}_{-\pi r(N-2)} \in (0, 2\pi r)$.

- It also shows that it vanishes when $y_j = y_k$ for any $j \neq k$ and when $\overline{y}_{-\pi r(N-2)} \in \{0, 2\pi r\}$.

- By the argument given by Liechty and Wang (arXiv:math.PR/1312.7390), we can prove that the moderated initial configuration $\lim_{t\downarrow 0}\det_{1\le j,k\le N}[p^r_{A_{N-1}}(t,y_j|v_k)] = \sum_{\sigma\in\mathcal{S}_N}\prod_{j=1}^N \delta_{v_{\sigma(j)}}(\{y_j\})$. is satisfied.

- Then the proof is completed. ∎

- Given $N \in \mathbb{N}$, let $W^r(t), t \geq 0$ be a Markov process in $[0, 2\pi r)$ such that its transition density is given by

$$p^r_{A_{N-1}}(t, y|x) = \begin{cases} \sum_{\ell \in \mathbb{Z}} p_{\rm BM}(t, y + 2\pi r\ell | x), & \text{if } N \text{ is even}, \\ \sum_{\ell \in \mathbb{Z}} (-1)^\ell p_{\rm BM}(t, y + 2\pi r\ell | x), & \text{if } N \text{ is odd}, \end{cases}$$

$x, y \in [0, 2\pi r)$, $t \geq 0$.

- Then we introduce an N independent copies of $W^r(t), t \geq 0$, denoted by $W^r_j(t), t \geq 0, 1 \leq j \leq N$ and let $\boldsymbol{W}^r(t) = (W^r_1(t), \dots, W^r_N(t))$, $t \geq 0$.

- The probability space of the process is denoted by $(\Omega_{W^r}, \mathcal{F}_{W^r}, {\rm P}^r_{\boldsymbol{v}})$, and the expectation is written as ${\rm E}^r_{\boldsymbol{v}}$, where the initial configuration is given by $\boldsymbol{v}$ with equidistant spacing on ${\rm S}^1(r)$.

- A filtration $\{\mathcal{F}_{W^r}(t) : t \geq 0\}$ is generated by $\boldsymbol{W}^r(t), t \geq 0$, which satisfies the usual conditions.

Theorem 3

Suppose that $N \in \mathbb{N}$, $\eta = \sum_{j=1}^{N} \delta_{v_j}$ (the equidistant initial configuration). Let $T \in [0, t_*)$. For any $\mathcal{F}_{\Xi^A}(T)$-measurable observable F,

$$\mathbb{E}_{\eta}^{A}\left[F\left(\Xi^A(\cdot)\right)\right] = \mathrm{E}_{\boldsymbol{v}}^{r}\left[F\left(\sum_{j=1}^{N} \delta_{W_j^r(\cdot)}\right) \mathcal{D}_{\eta}^{A}(T, \boldsymbol{W}^r(T))\right],$$

where

$$\mathcal{D}_{\eta}^{A}(t, \boldsymbol{x}) = \det_{1 \le j,k \le N}[\mathcal{M}_{\eta,u_k}^{A}(t, x_j)], \quad t \in [0, t_*)$$

with

$$\begin{aligned}\mathcal{M}_{\eta,u_k}^{A}(t, x) &= \int_{\mathbb{R}} d\widetilde{w} \frac{e^{-\widetilde{w}^2/2t}}{\sqrt{2\pi t}} \Phi_{\eta,u_k}^{A}(x + i\widetilde{w}) \\ &= \widetilde{\mathrm{E}}[\Phi_{\eta,u_k}^{A}(x + i\widetilde{W}(t))],\end{aligned}$$

where $\widetilde{W}$ denotes a BM on $\mathbb{R}$ started at 0, which is independent of $\boldsymbol{W}^r$, and $\widetilde{\mathrm{E}}$ does the expectation for $\widetilde{W}$. Here

$$\Phi_{\eta,u_k}^{A}(z) = \frac{\vartheta_1((\overline{u}_\delta + z - u_k)/2\pi r; iNt_*/2\pi r^2)}{\vartheta_1(\overline{u}_\delta/2\pi r; iNt_*/2\pi r^2)} \prod_{\substack{1 \le \ell \le N, \\ \ell \ne k}} \frac{\vartheta_1((z - u_\ell)/2\pi r; iNt_*/2\pi r^2)}{\vartheta_1((u_k - u_\ell)/2\pi r; iNt_*/2\pi r^2)}, \quad z \in \mathbb{C}.$$

Lemma 4

(i) $\mathcal{M}^A_{\eta,v_k}(t, W^r(t)), 1 \le k \le N, t \in [0, t_*)$ are continuous-time martingales;

$$\mathrm{E}^r[\mathcal{M}^A_{\eta,v_k}(t, W^r(t))|\mathcal{F}_{W^r}(s)] = \mathcal{M}^A_{\eta,v_k}(s, W^r(s)) \quad \text{a.s.}$$

for any two bounded stopping times with $0 \le s \le t < t_*$.

(ii) For any $t \in [0, t_*), \mathcal{M}^A_{\eta,v_k}(t, x), 1 \le k \le N$, are linearly independent functions of $x \in [0, 2\pi r)$,

(iii) $\mathcal{M}^A_{\eta,v_k}(0, v_j) = \delta_{jk}, \quad 1 \le j, k \le N.$

Theorem 3

Suppose that $N \in \mathbb{N}$, $\eta = \sum_{j=1}^N \delta_{v_j}$ (the equidistant initial configuration). Let $T \in [0, t_*)$. For any $\mathcal{F}_{\Xi^A}(T)$-measurable observable F,

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where

$$\mathcal{D}_\eta^A(t, \boldsymbol{x}) = \det_{1 \le j,k \le N} [\mathcal{M}_{\eta,u_k}^A(t, x_j)], \quad t \in [0, t_*)$$

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(iii) $\mathcal{M}^A_{\eta,v_k}(0, v_j) = \delta_{jk}, \quad 1 \le j, k \le N.$

Theorem 1.3 in [K:SPA(2014)]

Collorary 5

For $\eta = \sum_{j=1}^N \delta_{v_j}$ (the equidistant spacing initial configuration), the process $(\Xi^A(t), t \in [0, t_*), \mathbb{P}^A_\eta)$ is determinantal with the correlation kernel

$$\mathbb{K}^A_\eta(s, x; t, y) = \int_0^{2\pi r} \eta(du)\, p^r_{A_{N-1}}(s, x|u) \mathcal{M}^A_{\eta,u}(t, y) - \mathbf{1}(s > t) p^r_{A_{N-1}}(s - t, x|y),$$

$(s, x), (t, y) \in [0, t_*) \times [0, 2\pi r)$.

5. Future Problems

(1) **Solve the Problem for general initial configuration.**

(2) **On additional terms in the elliptic Dyson model and its reduced process.**

$$\check{X}_j^A(t) = u_j + B_j(t) + \sum_{\substack{1 \le k \le N, \\ k \ne j}} \int_0^t A_N^{2\pi r}(t_* - s, \check{X}_j^A(s) - \check{X}_k^A(s))ds + \int_0^t A_N^{2\pi r}(t_* - s, \overline{X}_\delta^A(t))ds,$$

$$1 \le j \le N, t \in [0, t_*), \text{ where } \quad \overline{X}_\delta^A(t) = \delta + \sum_{j=1}^N \check{X}_j^A(t).$$

$t_* \to \infty$

$$\check{X}_j^A(t) = u_j + B_j(t) + \frac{1}{2r} \sum_{\substack{1 \le k \le N, \\ k \ne j}} \int_0^t \cot\left(\frac{\check{X}_j^A(s) - \check{X}_k^A(s)}{2r}\right) ds + \frac{1}{2r} \int_0^t \cot\left(\frac{\overline{X}_\delta^A(s)}{2r}\right) ds,$$

$$1 \le j \le N, t \ge 0.$$

$r \to \infty$

$$X_j^A(t) = u_j + B_j(t) + \sum_{\substack{1 \le k \le N, \\ k \ne j}} \int_0^t \frac{1}{X_j^A(s) - X_k^A(s)} ds, \quad 1 \le j \le N, \quad t \ge 0.$$

(3) **Infinite-particle limits.**

(4) **On the other root systems and determinantal processes.**

[RS06] Rosengren, H., Schlosser, M.: Elliptic determinant evaluations and the Macdonald identities for affine root systems. Compositio Math. **142**, 937-961 (2006)

By using θ; 'multiplicative notation' for theta functions.

$$W_{A_{n-1}}(x) = \prod_{1\leqslant i<j\leqslant n} x_j \theta(x_i/x_j),$$

$$W_{B_n}(x) = \prod_{i=1}^{n} \theta(x_i) \prod_{1\leqslant i<j\leqslant n} x_i^{-1}\theta(x_i x_j^{\pm}),$$

$$W_{B_n^\vee}(x) = \prod_{i=1}^{n} x_i^{-1}\theta(x_i^2;p^2) \prod_{1\leqslant i<j\leqslant n} x_i^{-1}\theta(x_i x_j^{\pm}),$$

$$W_{C_n}(x) = \prod_{i=1}^{n} x_i^{-1}\theta(x_i^2) \prod_{1\leqslant i<j\leqslant n} x_i^{-1}\theta(x_i x_j^{\pm}),$$

$$W_{C_n^\vee}(x) = \prod_{i=1}^{n} \theta(x_i;p^{\frac{1}{2}}) \prod_{1\leqslant i<j\leqslant n} x_i^{-1}\theta(x_i x_j^{\pm}),$$

$$W_{BC_n}(x) = \prod_{i=1}^{n} \theta(x_i)\theta(px_i^2;p^2) \prod_{1\leqslant i<j\leqslant n} x_i^{-1}\theta(x_i x_j^{\pm}),$$

$$W_{D_n}(x) = \prod_{1\leqslant i<j\leqslant n} x_i^{-1}\theta(x_i x_j^{\pm}).$$

(5) **Connection between the elliptic determinantal processes and probabilistic discrete models with elliptic weights.**

[Sch07] Schlosser, M.: Elliptic enumeration of nonintersecting lattice paths. J. Combin. Theory Ser. A **114**, 505-521 (2007)

[BGR10] Borodin, A., Gorin, V., Rains, E. M.: q-distributions on boxed plane partitions. Sel. Math. (N. S.) **16**, 731-789 (2010)

[Bet11] Betea, D.: Elliptically distributed lozenge tilings of a hexagon. arXiv:math-ph/1110.4176

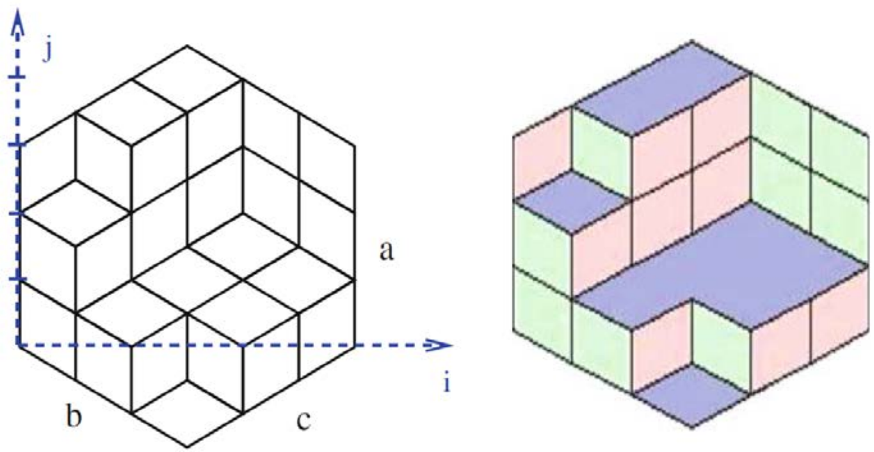

Fig. 1 Tiling of a $3 \times 3 \times 3$ hexagon

$$w(\Diamond) = \frac{(u_1 u_2)^{1/2} q^{j-1/2} \theta_p(q^{2j-1} u_1 u_2)}{\theta_p(q^{j-3i/2-1} u_1, q^{j-3i/2} u_1, q^{j+3i/2-1} u_2, q^{j+3i/2} u_2)},$$

copied from [BGR10]