ダイナミカルなランダム行列と棲み分けの問題 (part 3/3)

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生命ダイナミクスの数理とその応用
理論からのさらなる深化

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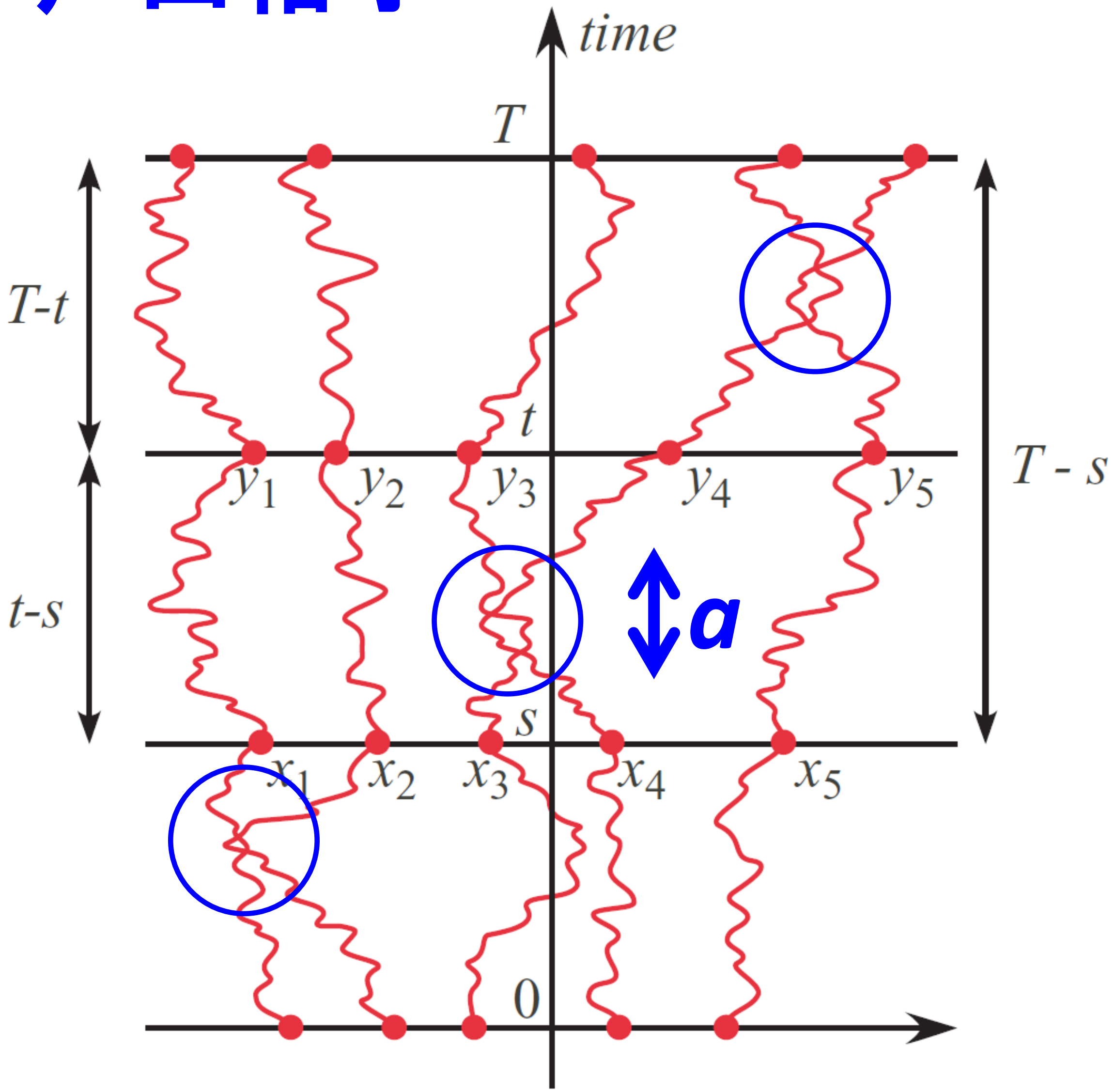
5. 少し friendly な vicious walkers と量子戸田格子



$$\mathbb{W}_N = \left\{ \boldsymbol{x} = (x_1, x_2, \dots, x_N) : x_1 < x_2 < \cdots < x_N \right\}$$

vicious BM

$$P_N(t|\boldsymbol{x}) = \text{survival probab. of all } N\text{-particles}$$

$$\frac{\partial}{\partial t} P_N(t|\boldsymbol{x}) = \frac{1}{2} \Delta P_N(t|\boldsymbol{x}) - \infty \times \mathbf{1}(\boldsymbol{x} \notin \mathbb{W}_N) P_N(t|\boldsymbol{x})$$

$a \to 0$ **(tropicalization)**
combinatorial limit

drift term $\boldsymbol{\nu} \in \mathbb{W}_N$
+
'softening' with a parameter $a > 0$
geometric lifting

$$\frac{\partial}{\partial t} P_N^{a,\nu}(t|\boldsymbol{x}) = \frac{1}{2} \Delta P_N^{a,\nu}(t|\boldsymbol{x}) + \boldsymbol{\nu} \cdot \nabla P_N^{a,\nu}(t|\boldsymbol{x}) - \frac{1}{a^2} \sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)/a} P_N^{a,\nu}(t|\boldsymbol{x})$$

(diffusion term) (drift term) (killing term)

$$\frac{\partial}{\partial t} P_N^{a,\nu}(t|\boldsymbol{x}) = \frac{1}{2}\Delta P_N^{a,\nu}(t|\boldsymbol{x}) + \boldsymbol{\nu} \cdot \nabla P_N^{a,\nu}(t|\boldsymbol{x}) - \frac{1}{a^2}\sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)/a} P_N^{a,\nu}(t|\boldsymbol{x})$$

(diffusion term) (drift term) (killing term)

$$\text{coefficient of} - \text{(killing term)} = \frac{1}{a^2}\sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)/a}$$

$$= \text{Toda-lattice potential}$$

Schrödinger 方程式

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right\}\psi(t,x)$$

拡散方程式 (backward Kolmogorov equation)

$$\frac{\partial}{\partial x}Q(t,y|x) = \left\{\frac{1}{2}\frac{\partial^2}{\partial x^2} - V(x)\right\}Q(t,y|x)$$

ポテンシャル $\Longleftrightarrow$ killing term

Hamiltonian of $\mathrm{GL}(N,\mathbb{R})$-quantum Toda lattice

$$\mathcal{H}_N^a = -\frac{1}{2}\Delta + \frac{1}{a^2}\sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)/a}$$

eigenfunction problem

$$\mathcal{H}_N^a \psi_\nu^{(N)}(\boldsymbol{x}/a) = \lambda(\boldsymbol{\nu})\psi_\nu^{(N)}(\boldsymbol{x}/a)$$

with eigenvalue $\lambda(\nu) = -\frac{1}{2}\sum_{j=1}^{N}\nu_j^2$

under the condition

$$\lim_{a\to 0} e^{-\boldsymbol{\nu}\cdot\boldsymbol{x}/a}\psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}/a) = \prod_{1\le j<k\le N}\Gamma(\nu_k-\nu_j), \quad \boldsymbol{x},\boldsymbol{\nu}\in\mathbb{W}_N$$

has a unique solution

$\psi_\nu^{(N)}(\boldsymbol{x}/a)$ class-one Whittaker function

The class-one Whittaker function has Givental's integral representation:

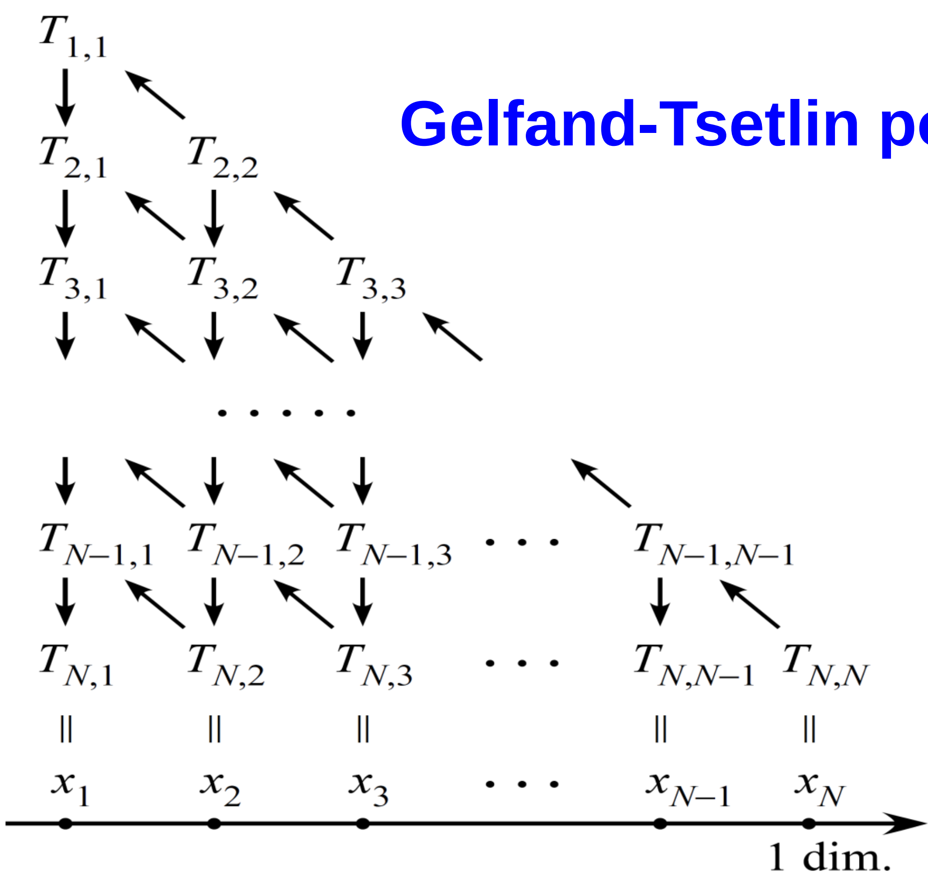
$$\psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}) = \int_{\mathbb{T}_N(\boldsymbol{x})} \exp\left(\mathcal{F}_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{T})\right) d\boldsymbol{T}.$$

Here the integral is performed over the space $\mathbb{T}_N(\boldsymbol{x})$ of all real lower triangular arrays with size N, $\boldsymbol{T} = (T_{j,k}, 1 \le k \le j \le N)$, with $T_{N,k} = x_k, 1 \le k \le N$, and

$$\mathcal{F}_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{T}) = \sum_{j=1}^{N} \nu_j \left(\sum_{k=1}^{j} T_{j,k} - \sum_{k=1}^{j-1} T_{j-1,k} \right) - \sum_{j=1}^{N-1} \sum_{k=1}^{j} \left\{ e^{-(T_{j,k}-T_{j+1,k})} + e^{-(T_{j+1,k+1}-T_{j,k})} \right\}.$$

A. Givental: Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture, In: *Topics in Singular Theory*, AMS Trans. Ser. 2, vol. 180, pp.103-115, AMS, Rhode Island (1997)

$$\psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}) = \int_{\Gamma_N(\boldsymbol{x})} \exp\left[\sum_{k=1}^{N} \nu_k \left(\sum_{j=1}^{k} T_{k,j} - \sum_{j=1}^{k-1} T_{k-1,j}\right) - \sum_{k=1}^{N-1}\sum_{j=1}^{k}\left\{e^{-(T_{k,j}-T_{k+1,j})} + e^{-(T_{k+1,j+1}-T_{k,j})}\right\}\right] d\boldsymbol{T}.$$



Gelfand-Tsetlin polytope

$$\lim_{t\to\infty} P_N^{a,\nu}(t|\boldsymbol{x}) = c_1(N,\boldsymbol{\nu},a)e^{-\boldsymbol{\nu}\cdot\boldsymbol{x}/a}\psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}/a)$$

$$\text{if } \boldsymbol{x},\boldsymbol{\nu} \in \mathbb{W}_N, \boldsymbol{\nu} \neq 0$$

「外向き」のドリフトの効果

$$P_N^{a,\nu}(t|\boldsymbol{x}) \sim c_2(N,a)t^{-N(N-1)/4}\psi_0^{(N)}(\boldsymbol{x}/a)$$

$$\text{as } t \to \infty, \text{ if } \boldsymbol{\nu} = 0$$

O'Connell process ≡ 'softened version' of vicious walker model
(with the killing term of Toda-lattice potential)
conditoned that all N particle survive forever

N. O'Connell: Directed polymers and the quantum Toda lattice, *Ann. Probab.* **40**, 437-458 (2012).

M. Katori: O'Connell's process as a vicious Brownian motion, *Phys. Rev.* E **84**, 061144 (2011).

M. Katori: Survival probability of mutually killing Brownian motion and the O'Connell process, *J. Stat. Phys.* **147**, 206-223 (2012).

6. Ginibre 行列式点過程と 2次元平面上の棲み分け問題

- $N \times N$ のエルミート値のガウス型ランダム行列 (Gaussian Unitary Ensemble: GUE) 固有値の確率密度関数 ($\boldsymbol{x}_j \in \mathbb{R}, 1 \le j \le N$)

$$p(\boldsymbol{x};\sigma) = c_N^{\mathrm{GUE}} e^{-\sum_{j=1}^N x_j^2/2\sigma^2} \prod_{1 \le j < k \le N} (x_k - x_j)^2.$$

- $N \times N$ の実（あるいは複素）行列値のガウス型ランダム行列 (Ginibre アンサンブル) 固有値の確率密度関数 ($\boldsymbol{z}_j \in \mathbb{C}, 1 \le j \le N$)

$$p(\boldsymbol{z};\sigma) = c_N^{\mathrm{Ginibre}} e^{-\sum_{j=1}^N |z_j|^2/2\sigma^2} \prod_{1 \le j < k \le N} |z_k - z_j|^2.$$

ただし，$|z|^2 = z\overline{z} = (\Re z)^2 + (\Im z)^2$ （複素数としての２乗）．

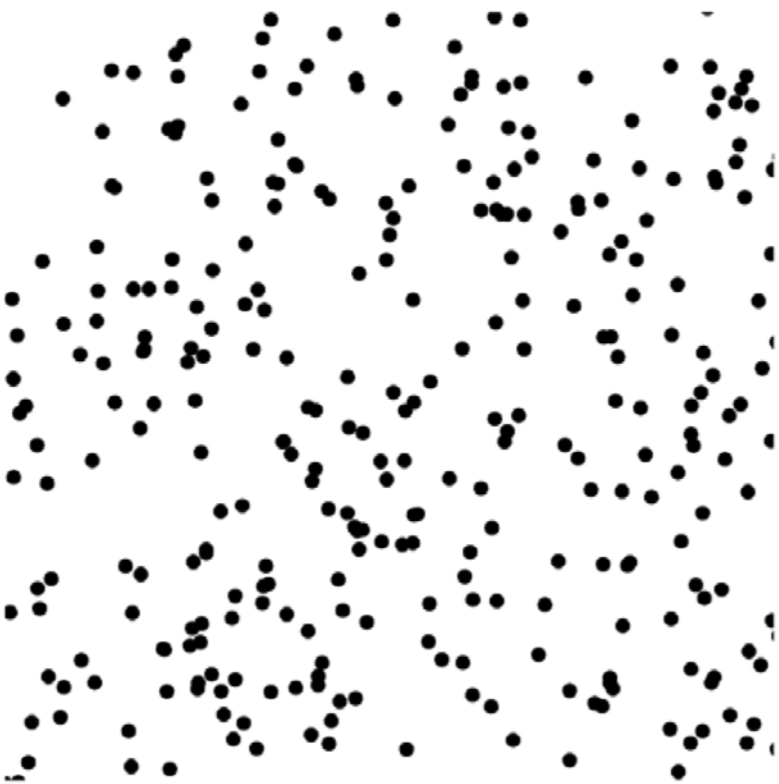


図１：Poisson 点過程．平面上の一様ランダムな点の配置であるが，実現した配置には粒子分布の空間的な粗密が見られる．

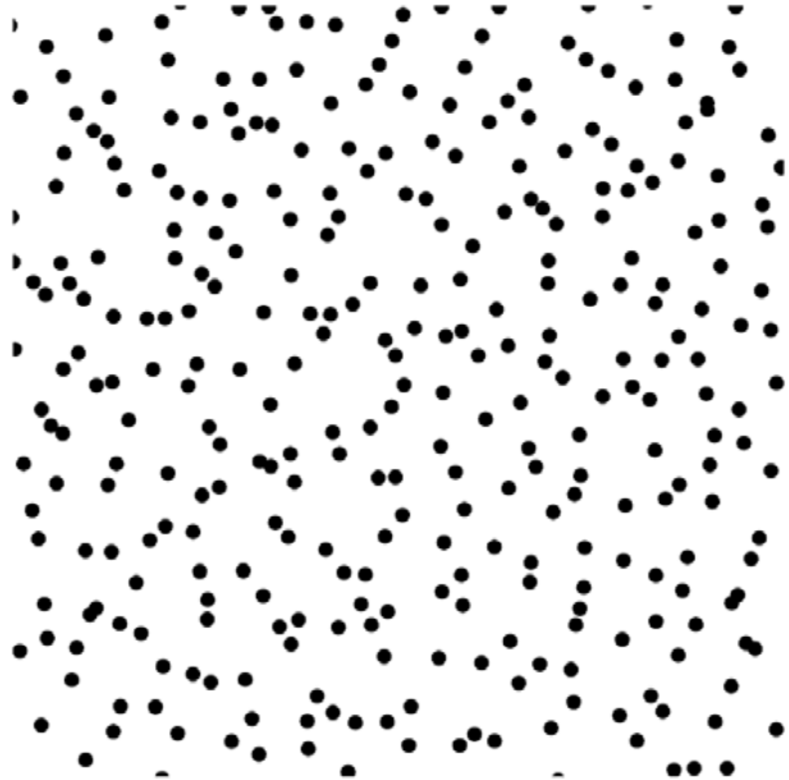


図２：複素ランダム行列の複素平面上の固有値分布として実現されるGinibre 点過程．粒子間に斥力相互作用が働き，棲み分けが実現している．（粒子数密度は図１と同じ．）

２次元上の静的な棲み分けの問題

原子物理学で量子多体系のエネルギー準位統計の理論として導入されたランダム行列理論．
その負の相関（排他的な相互作用）は，多種（diverse）な系をシミュレートするのに有効である．

- 行列式点過程（フェルミオン点過程）の実世界の問題への応用：
  machine learning technologies
  Kulesza, A., Taskar, B.: Determinantal point processes for machine learning.
  Foundations Trends Mach. Learn. **5**, 123–286 (2012)

特に，平面上の排他的点過程である Ginibre 点過程は，すでにいろいろな問題に応用されている．

- **Ginibre-Voronoi tessellation on the plane (平面の分割問題)**

  G Le Caert, G. Le, Ho, J. S.: The Voronoi tessellation generated from eigenvalues of complex random matrices.
  J. Phys. A: Math. Gen. **23**, 3279-3295 (1990)

  Goldman, A.: The Palm measure and the Voronoi tessellation for the Ginibre process.
  Ann. Appl. Probab. **20**, 90–128 (2010)

- **applications to cellular network modeling**
  Miyoshi, N., Shirai, T.: A cellular network model with Ginibre configured base stations.
  Adv. Appl. Probab. **46**, 832–845 (2014)

生態系の棲み分けの問題への応用も考えられるのではないだろうか．

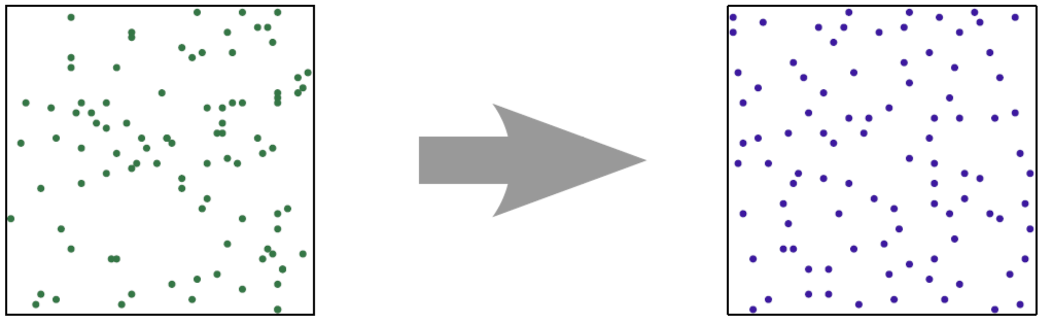


Figure 2: On the left, points are sampled randomly; on the right, repulsion between points leads to the selection of a diverse set of locations.

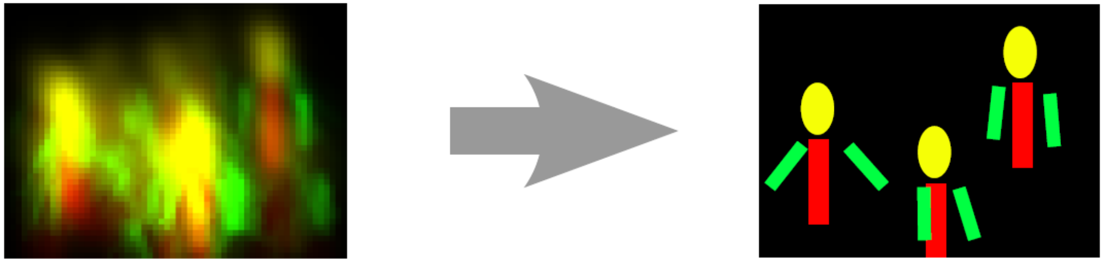


Figure 3: On the left, the output of a human pose detector is noisy and uncertain; on the right, applying diversity as a filter leads to a clean, separated set of predictions.

Kulesza, A., Taskar, B.: Determinantal point processes for machine learning. Foundations Trends Mach. Learn. **5**, 123–286 (2012)

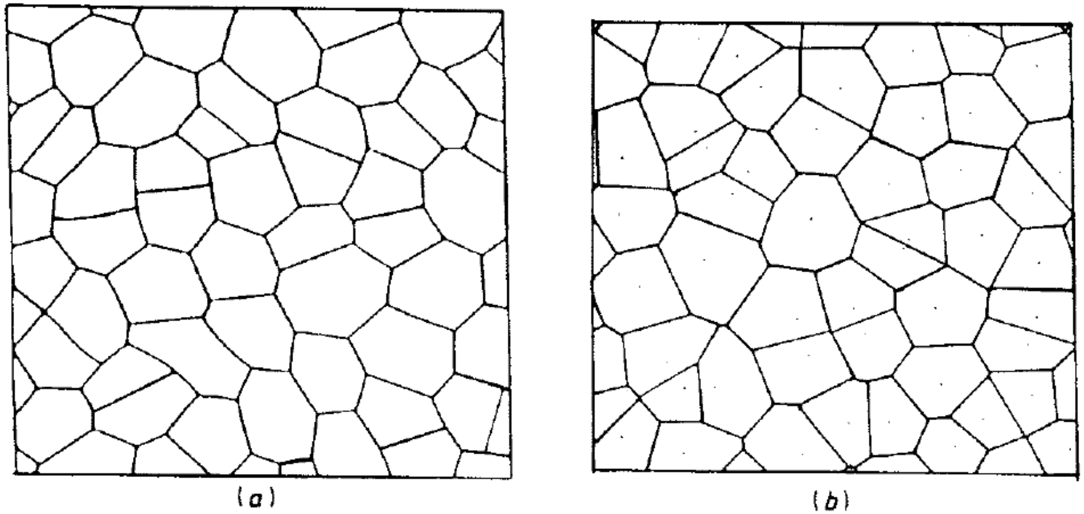
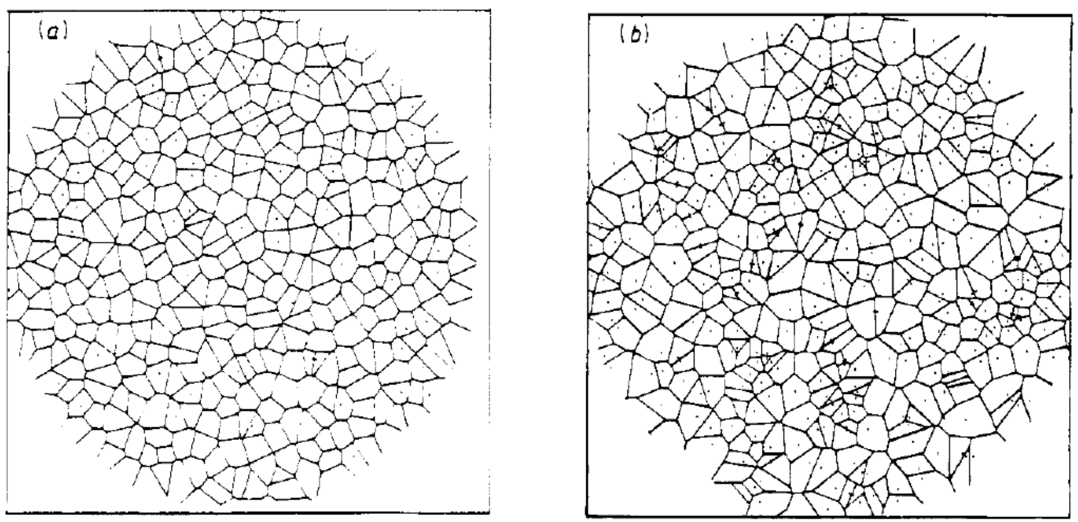


Figure 1. (*a*) Cells from the epidermal epithelium of the cucumber (after Lewis 1928). (*b*) Voronoi tessellation generated from a complex Gaussian random matrix.

Ginibre →



← Poisson

Figure 3. (*a*) Random matrix Voronoi froth, $N = 400$, $\sigma = 1$, (*b*) Random Voronoi froth, $N = 400$ points distributed according to a Poisson process in a circle of radius 20.

G Le Caert, G. Le, Ho, J. S.: The Voronoi tessellation generated from eigenvalues of complex random matrices.
J. Phys. A: Math. Gen. **23**, 3279-3295 (1990)

- ただし，2 次元上のブラウン運動は再帰的ではないので，1 次元系において成り立った
  「動的ランダム行列模型」＝「非衝突条件件を課したブラウン粒子系」
  という構図は成り立たない.

- 静的な「行列式点過程」である Ginibre アンサンブルを，動的な（非平衡な）「行列過程」に拡張することには，まだ成功していない.

A Lecture Note entitled

`Bessel Process, Schramm-Loewner Evolution, and the Dyson Model'

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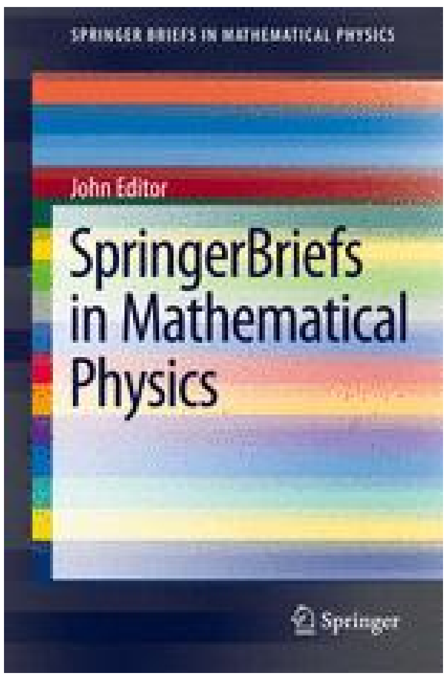

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