Exchange dynamics in finite-particle Dunkl processes of type A

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1. Setting

We consider the Dunkl process of type A. Transition density: $p(t, \boldsymbol{y}|\boldsymbol{x})$. Backward Fokker-Planck eq:

$$\begin{aligned}
\frac{\partial}{\partial t} p(t, \boldsymbol{y}|\boldsymbol{x}) &= \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} p(t, \boldsymbol{y}|\boldsymbol{x}) \\
&\quad + \frac{\beta}{2} \sum_{1 \le i \neq j \le N} \left\{ \frac{1}{x_i - x_j} \frac{\partial}{\partial x_i} p(t, \boldsymbol{y}|\boldsymbol{x}) - \frac{1}{2} \frac{p(t, \boldsymbol{y}|\boldsymbol{x}) - p(t, \boldsymbol{y}|\sigma_{ij}\boldsymbol{x})}{(x_i - x_j)^2} \right\}.
\end{aligned}$$

- β-Dyson model with particle exchange: Brownian particles in one dimension repel mutually and exchange positions spontaneously.
- Widely known in random matrix theory.
- Gauge transform of the Calogero-Moser system of type A with exchange interaction.

2. Background (1)

$$\frac{\partial}{\partial t} p(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} p(t, \boldsymbol{y}|\boldsymbol{x}) + \frac{\beta}{2} \sum_{1 \le i \ne j \le N} \left\{ \frac{1}{x_i - x_j} \frac{\partial}{\partial x_i} p(t, \boldsymbol{y}|\boldsymbol{x}) - \frac{1}{2} \frac{p(t, \boldsymbol{y}|\boldsymbol{x}) - p(t, \boldsymbol{y}|\sigma_{ij}\boldsymbol{x})}{(x_i - x_j)^2} \right\}.$$

- Note that the particle exchange does not change particle trajectories.
- Trajectories given by the Dyson model: for $N \to \infty$ and $\beta = 2$, relaxation to steady-state goes like $1/t$ [Katori-Tanemura 2011, 2014].
- For N finite, the relaxation of any scaled Dunkl process occurs in two steps [Andraus-Miyashita, 2015]:
  - Drift mechanism (responsible for trajectories): $1/t$.
  - Exchange mechanism (responsible for particle exchange): $1/\sqrt{t}$.
  - Correspondence of asymptotics obtained by an indirect observation!
  - Not suitable in the $N \to \infty$ limit!

2. Background (2)

Denote the process by $\boldsymbol{X}_t$. Its stochastic differential equation is given by [Chybiryakov-Gallardo-Yor, 2008]

$$\begin{aligned} \boldsymbol{X}_t &= \boldsymbol{x} + \boldsymbol{B}_t + \frac{\beta}{2} \sum_{1 \le i < j \le N} (\boldsymbol{e}_i - \boldsymbol{e}_j) \Big\{ \int_0^t \frac{\mathrm{d}\tau}{X_{i,\tau^-} - X_{j,\tau^-}} \\ & \quad - \frac{1}{2} \sum_{\tau \le t} (X_{i,\tau^-} - X_{j,\tau^-}) \mathbf{1}_{[\boldsymbol{X}_\tau = \sigma_{ij} \boldsymbol{X}_{\tau^-} \neq \boldsymbol{X}_{\tau^-}]} \Big\}. \end{aligned}$$

- This SDE is dependent on the timing of the jumps, which are random variables.
- A claim is made in [Chybiryakov-Gallardo-Yor, 2008] that in a finite time interval, the number of exchanges is finite with probability 1 for $\beta > 1$. More on this later.

3. Relationship with the Dyson model

Based on the previous claims, we have proved the following:

- Denote the β-Dyson model by $\boldsymbol{X}_t^{\mathrm{D}}$. Then, the equivalence

$$\boldsymbol{X}_t = \rho_t \boldsymbol{X}_t^{\mathrm{D}}$$

  holds in law. Here, ρ_t is a continuous-time random walk on S_N. The probability of an exchange in the interval $[t, t + \mathrm{d}t]$ is (see next slide)

$$\frac{\beta}{2} \sum_{1 \leq i < j \leq N} \frac{\mathrm{d}t}{(X_{i,t} - X_{j,t})^2}.$$

- Proof: induction on the exchange times. The Brownian motions driving the processes are not equal, as they must be transformed by ρ_t. Note that the finite exchange assumption is critical for this result.

4. Exchange probability and finite exchange assumption (1)

Probability of a particle exchange in $[t, t + \mathrm{d}t]$: for $\epsilon > 0$,

$$
\begin{aligned}
P[|\boldsymbol{X}_{t+\mathrm{d}t} - \boldsymbol{X}_t| > \epsilon | \boldsymbol{X}_t = \boldsymbol{x}] &= \int_{\mathbb{R}^N \setminus B_\epsilon(\boldsymbol{x})} p(\mathrm{d}t, \boldsymbol{y}|\boldsymbol{x})\,\mathrm{d}\boldsymbol{y} \\
&= \int_{\mathbb{R}^N \setminus B_\epsilon(\boldsymbol{x})} \frac{\partial}{\partial t} p(0, \boldsymbol{y}|\boldsymbol{x})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}t = \frac{\beta}{2} \sum_{1 \le i < j \le N} \frac{\mathrm{d}t}{(x_i - x_j)^2}.
\end{aligned}
$$

This follows from the BFPE and $p(0, \boldsymbol{y}|\boldsymbol{x}) = \delta(\boldsymbol{y} - \boldsymbol{x})$. Denote the number of jumps in $[0, t]$ by

$$
\mathcal{N}_t = \frac{\beta}{2} \sum_{1 \le i < j \le N} \int_0^t \frac{\mathrm{d}\tau}{(X_{i,\tau} - X_{j,\tau})^2} > 0.
$$

This is a positive and strictly increasing process.

4. Exchange probability and finite exchange assumption (2)

- Chybiryakov's argument: note that, because Dunkl processes are homogeneous [Rösler-Voit, 1998],

$$\mathbf{E}_{\boldsymbol{x}=\mathbf{0}}[\mathcal{N}_t] = \mathbf{E}_{\boldsymbol{x}=\mathbf{0}}[\mathbf{E}_{\boldsymbol{X}_s}[\mathcal{N}_{t-s}]] + \mathbf{E}_{\boldsymbol{x}=\mathbf{0}}[\mathcal{N}_s]$$

for $s < t$. Then, if $\mathbf{E}_{\boldsymbol{x}=\mathbf{0}}[\mathcal{N}_t] < \infty$, it follows that

$$\mathbf{E}_{\boldsymbol{x}=\mathbf{0}}[\mathbf{E}_{\boldsymbol{X}_s}[\mathcal{N}_{t-s}]] = \int_{\mathbb{R}^N} \mathbf{E}_{\boldsymbol{y}}[\mathcal{N}_{t-s}] p(s, \boldsymbol{y}|\mathbf{0})\, \mathrm{d}\boldsymbol{y} < \infty.$$

- Because $p(t, \boldsymbol{y}|\boldsymbol{x})$ is positive, the boundedness of $\mathbf{E}_{\boldsymbol{x}}[\mathcal{N}_t]$ should follow.
- However, $\mathbf{E}_{\boldsymbol{x}=\mathbf{0}}[\mathcal{N}_t]$ is unbounded! Set $w_\beta(\boldsymbol{x}) = \prod_{1\le i<j\le N} |x_i - x_j|^\beta$. Then,

$$\begin{aligned} \mathbf{E}_{\boldsymbol{x}=\mathbf{0}}[\mathcal{N}_t] &= \frac{1}{c_\beta}\frac{\beta}{2}\sum_{1\le i<j\le N}\int_{\mathbb{R}^N}\int_0^t \frac{w_\beta(\boldsymbol{y}/\sqrt{\tau})\mathrm{e}^{-y^2/2\tau}}{\tau^{N/2}(y_i-y_j)^2}\,\mathrm{d}\tau\,\mathrm{d}\boldsymbol{y} \\ &= \frac{1}{c_\beta}\frac{\beta}{2}\sum_{1\le i<j\le N}\int_{\mathbb{R}^N}\frac{w_\beta(\boldsymbol{z})\mathrm{e}^{-z^2/2}}{(z_i-z_j)^2}\,\mathrm{d}\boldsymbol{z}\int_{0^+}^t\frac{\mathrm{d}\tau}{\tau} \to \infty. \end{aligned}$$

5. Finite exchange and a conjecture (1)

We must consider $\mathbf{E}_{\boldsymbol{x}\neq\mathbf{0}}[\mathcal{N}_t]$. Introduce the Dunkl kernel $E_\beta(\boldsymbol{x},\boldsymbol{y})$, defined by $E_\beta(\mathbf{0},\boldsymbol{y})=1$ and the relationship

$$\frac{\partial}{\partial x_i}E_\beta(\boldsymbol{x},\boldsymbol{y})+\frac{\beta}{2}\sum_{j:j\neq i}\frac{E_\beta(\boldsymbol{x},\boldsymbol{y})-E_\beta(\sigma_{ij}\boldsymbol{x},\boldsymbol{y})}{x_i-x_j}=y_iE_\beta(\boldsymbol{x},\boldsymbol{y}),\ 1\leq i\leq N.$$

Then,

$$\begin{aligned}\mathbf{E}_{\boldsymbol{x}}[\mathcal{N}_t] &= \frac{1}{c_\beta}\frac{\beta}{2}\sum_{1\leq i<j\leq N}\int_{\mathbb{R}^N}\int_0^t\frac{w_\beta(\boldsymbol{y}/\sqrt{\tau})}{\tau^{N/2}(y_i-y_j)^2}\mathrm{e}^{-(y^2+x^2)/2\tau}E_\beta\Big(\frac{\boldsymbol{x}}{\sqrt{\tau}},\frac{\boldsymbol{y}}{\sqrt{\tau}}\Big)\,\mathrm{d}\tau\,\mathrm{d}\boldsymbol{y}\\ &= \frac{\beta}{c_\beta^2}\sum_{1\leq i<j\leq N}\int_{\mathbb{R}^N}\frac{w_\beta(\boldsymbol{z})}{(z_i-z_j)^2}\int_{\mathbb{R}^N}\frac{w_\beta(\boldsymbol{u})}{u^2}(1-\mathrm{e}^{-tu^2/2})\\ &\qquad\times E_\beta(\mathrm{i}\boldsymbol{u},\boldsymbol{z})E_\beta(-\mathrm{i}\boldsymbol{u},\boldsymbol{x})\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{z}.\end{aligned}$$

Two cases ($\beta>1$):

- Convergence: number of exchanges is finite a.s. as $t\to\infty$.
- Divergence: we must consider multiple simultaneous exchanges (unlikely).

5. Finite exchange and a conjecture (2)

- It is strongly believed that the integral converges. It is known that $|E_\beta(\mathrm{i}\boldsymbol{x}, \boldsymbol{y})| \leq 1$.
- However, we need a bound for $|E_\beta(\mathrm{i}\boldsymbol{x}, \boldsymbol{y})|$ at large $|\boldsymbol{x}||\boldsymbol{y}|$.
- Conjecture: there exist a function $\phi(\beta) > 0$ and a constant $C > 0$ s.t.

$$|E_\beta(\mathrm{i}\boldsymbol{x}, \boldsymbol{y})| \leq (|\boldsymbol{x}||\boldsymbol{y}|)^{-\phi(\beta)} \text{ for } |\boldsymbol{x}||\boldsymbol{y}| > C.$$

- The conjecture is true for $N = 2$ and large β.

$$\lim_{\beta\to\infty} E_\beta(\mathrm{i}\sqrt{\beta}\boldsymbol{x}, \boldsymbol{y}) = \exp\Big(-\frac{x^2y^2}{N(N-1)}\Big), \text{ if } \sum_i x_i = 0.$$

- Convergence will depend on the form of $\phi(\beta)$.

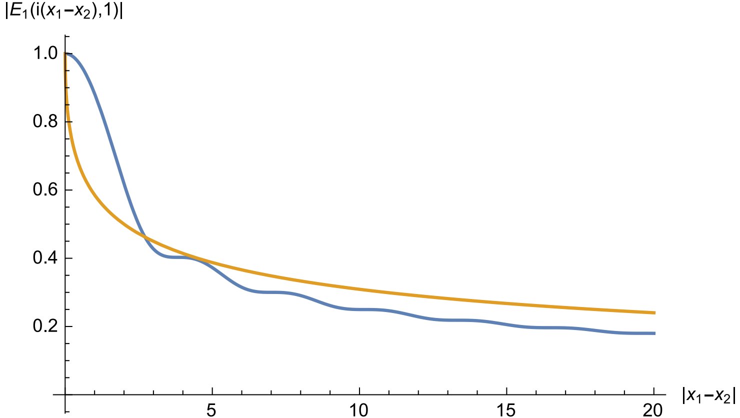

Figure: Norm of $E_{\beta=1}(\mathrm{i}(x_1 - x_2), 1)$ (blue) compared with $(1 + |(x_1 - x_2)/2|^{1/2})^{-1}$ (yellow) for $N = 2$.

6. Summary and Outlook

- We studied the particle exchange dynamics in Dunkl processes of type A.
- We wrote the exchange process as a continuous time random walk on S_N.
- We found that, if the finite exchange assumption is correct, the exchange number tends to a finite value as $t \to \infty$ w.p. 1.

Current work:

- Proof of conjecture on $|E_\beta(\mathrm{i}\boldsymbol{x}, \boldsymbol{y})|$.
- Further focus on the process ρ_t.
- Limit $N \to \infty$.

To be continued...

Thanks for your attention!