Relaxation driven by the exchange interaction in Dunkl processes (*using graphs*)

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0. Introduction

- Dyson model: well-known one-dimensional stochastic Coulomb gas. First formulation as eigenvalue process of matrices with independent Brownian motions (up to symmetry) as entries.
- An alternate formulation can be obtained as the symmetrized part of the Dunkl process of type A.
- This formulation depends on the differential-difference Dunkl operators; when the Dunkl process is not symmetrized, an exchange term remains.
- The non-symmetrized Dunkl process can be interpreted as a Dyson model with exchange interaction.
- The present work is the first attempt to give a physical interpretation to the exchange interaction as a means of information flow.
- *We construct random graphs using the exchange interaction and consider the time required for them to become connected.*

1.1 The Dyson model

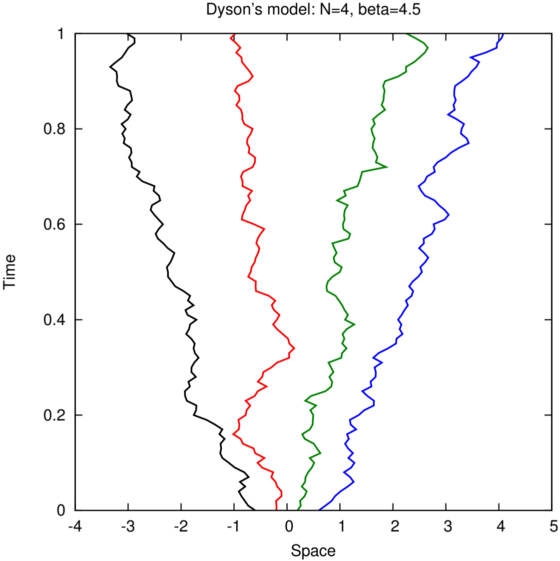
The Dyson model [Dyson 1962] $\boldsymbol{X}^S(t)$ of N particles and parameter β is the multivariate stochastic process given by the SDE

$$\mathrm{d}X^S(t) = \mathrm{d}\hat{B}_i(t) + \frac{\beta}{2}\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\mathrm{d}t}{X_i^S(t) - X_j^S(t)}. \tag{1}$$

β can be understood as the square root of the charge or as the inverse temperature. Equivalently, the Dyson model can be defined by its Fokker-Planck equation; if its transition density is denoted by $p(t, \boldsymbol{y}|\boldsymbol{x})$, the Fokker-Planck equation is

$$\frac{\partial}{\partial t} p(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{1}{2}\Delta_x p(t, \boldsymbol{y}|\boldsymbol{x}) + \frac{\beta}{2}\sum_{1\leq i\neq j\leq N} \frac{1}{x_i - x_j}\frac{\partial}{\partial x_i} p(t, \boldsymbol{y}|\boldsymbol{x}). \tag{2}$$

The Dyson model was first formulated [Dyson 1962] as the eigenvalue process of a Gaussian random matrix with independent Brownian motions as entries, up to symmetry. In this case $\beta = 1, 2$ or 4.



A sample path of the 4-particle Dyson model with $\beta = 4.5$

1.2 Dunkl operators and the exchange interaction

The Dunkl operator [Dunkl 1989] of type A and parameter $\beta > 0$ is defined by

$$T_i f(\boldsymbol{x}) = \frac{\partial}{\partial x_i} f(\boldsymbol{x}) + \frac{\beta}{2} \sum_{j=1:j\neq i}^{N} \frac{f(\boldsymbol{x}) - f(\sigma_{ij}\boldsymbol{x})}{x_i - x_j}. \quad (3)$$

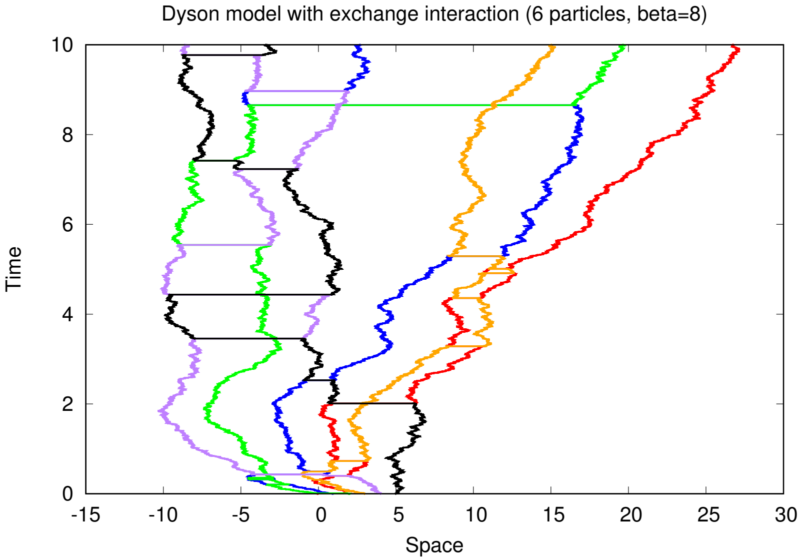
Here, σ_{ij} permutes the variables x_i and x_j. Consider the Markov process obtained from the generalized heat equation [Rösler-Voit 1998]

$$\frac{\partial}{\partial t} P(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{1}{2} \sum_{i=1}^{N} T_i^2 P(t, \boldsymbol{y}|\boldsymbol{x}). \quad (4)$$

This is the Dunkl process of type A. Explicitly, the Fokker-Planck equation reads

$$\frac{\partial}{\partial t} P(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{1}{2} \Delta_x P(t, \boldsymbol{y}|\boldsymbol{x}) + \frac{\beta}{2} \sum_{1\leq i\neq j\leq N} \left[\frac{1}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{1 - {}_x\sigma_{ij}}{(x_i - x_j)^2} \right] P(t, \boldsymbol{y}|\boldsymbol{x}). \quad (5)$$

With $p(t, \boldsymbol{y}|\boldsymbol{x}) = \sum_{\rho\in S_N} P(t, \boldsymbol{y}|\rho\boldsymbol{x})$, we recover the Dyson model ($\beta > 0$). We see that the Dunkl process of type A is the Dyson model with exchange interaction.



A sample path of the 6-particle Dyson model with interaction for $\beta = 8$

2.1 The exchange interaction - what we know

- Through a gauge transformation and a change to (or from) imaginary time, the Dyson model with exchange is taken to a Calogero-Moser system of spins with inverse-squared distance couplings [Hikami-Wadati 1993].

$$\mathcal{H}_{\mathrm{CMR}} = -\frac{1}{2}\Delta + \frac{\beta}{2}\sum_{1\leq i<j\leq N}\frac{\beta/2-\sigma_{ij}}{(x_i-x_j)^2} + \frac{1}{2}\omega^2\sum_{i=1}^{N}x_i^2. \qquad (6)$$

  The spins are su(ν), so they are generalized spins.

- Via the scaling $\boldsymbol{y} = \sqrt{\beta t}\boldsymbol{Y}$, the transition density converges to the steady state distribution

$$(\beta t)^{N/2}P(t,\sqrt{\beta t}\boldsymbol{Y}|\boldsymbol{x}) \stackrel{t\to\infty}{\longrightarrow} \frac{\mathrm{e}^{-\beta F(\boldsymbol{Y})}}{z_\beta}, \qquad F(\boldsymbol{Y}) = \frac{Y^2}{2} - \frac{\beta}{2}\sum_{i<j}\log|Y_i - Y_j|.$$

  The convergence to the steady-state distribution is of order $O(t^{-1/2})$, and it is due to the exchange interaction [SA-Miyashita 2015].

2.2 Exchange number expectation

Theorem

Denote the jump counting process by $\mathcal{N}(t)$ *and set* $\boldsymbol{x}_0$ *such that* $x_{0,1} \neq x_{0,j}$ $^{\forall}1 \leq i,j \leq N$, $\beta > 1$. *Then, for* $t \gg x_0^2$,

$$\mathbb{E}_{\boldsymbol{x}}[\mathcal{N}(t)] = O[\log(t)] \text{ and } \frac{d}{dt}\mathbb{E}_{\boldsymbol{x}}[\mathcal{N}(t)] = O(1/t). \quad (7)$$

Proof: the jump process is a non-homogeneous Poisson process with rate function

$$\lambda[\boldsymbol{X}(t)] := \frac{\beta}{2} \sum_{1\leq i\neq j\leq N} \frac{1}{[X_i(t) - X_j(t)]^2},$$

provided $\beta \geq 1$ [Demni 2008]. $\mathbb{E}_{\boldsymbol{x}}[\mathcal{N}(t)]$ can be written in terms of $f(\boldsymbol{x}) := \log \prod_{1\leq i<j\leq N} |x_i - x_j|$ [Lépingle 2012], using the Itô formula:

$$\frac{N(N-1)}{4}\log t + L(\boldsymbol{x}_0/\sqrt{t}) - f(\boldsymbol{x}_0) = \frac{\beta - 1}{\beta}\mathbb{E}_{\boldsymbol{x}_0}[\mathcal{N}(t)], \text{ where}$$

$$L(\boldsymbol{x}) = \int_{y_1<\cdots<y_N} \log \prod_{1\leq i<j\leq N} |y_i - y_j| p(1,\boldsymbol{y}|\boldsymbol{x})\, \mathrm{d}^N y.$$

2.3 The skew-product representation

Suppose that the Dyson model with exchange is driven by $\boldsymbol{B}(t)$ and that the continuous Dyson model $\boldsymbol{X}^S(t)$ is driven by $\hat{\boldsymbol{B}}(t)$, with

$$\hat{\boldsymbol{B}}(t) - \hat{\boldsymbol{B}}(T_n) = \rho(t)^{-1}[\boldsymbol{B}(t) - \boldsymbol{B}(T_n)] \tag{8}$$

for all exchange times $T_n < t < T_{n+1}$ and all $n \geq 0$.

Theorem

Denote by $C_A := \{\boldsymbol{y} \in \mathbb{R}^N : y_j - y_i > 0, ^\forall i < j\}$ the Weyl chamber of $W = S_N$. A Dyson model with exchange interaction $\boldsymbol{X}(t)$ with $\beta > 1$ can be written as

$$\boldsymbol{X}(t) \overset{law}{=} \rho(t)\boldsymbol{X}^S(t). \tag{9}$$

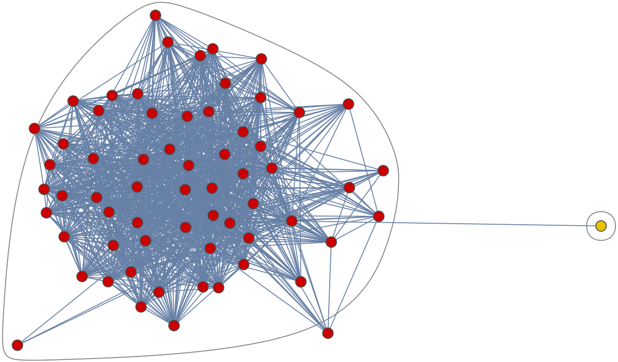
$\rho(t)$: continuous-time random walk on S_N; $\boldsymbol{X}^S(t)$: Dyson model on C_A. The equality holds pathwise when the driving Brownian motions are related by (8).

Proof: whenever an exchange occurs, perform the inverse exchange to bring the process back to C_A. Due to the finiteness of $\mathbb{E}_{\boldsymbol{x}}[\mathcal{N}(t)]$, we can perform a mathematical induction, and the proof is complete. All the important information about the exchange interaction lies in the random walk $\rho(t) = \rho(\boldsymbol{X}^S(t), t)$.

3.1 The exchange interaction and information propagation

- We perform numerical simulations at large β to examine the exchange interaction.
- Probing $\rho(\boldsymbol{X}^S(t), t) \in S_N$ directly is time consuming.
- We propose three random-graph construction models with equally-spaced ($\sim \sqrt{\beta}$) and centered initial conditions on the Dyson model:
  1. "Networking" model (we add an edge between any two particles who make an exchange).
  2. Contagion (we add an edge when a non-infected particle and an infected particle make an exchange).
- Here, we assume that the particles (agents) are more prone to develop links with particles that are closer, so every particle behaves differently depending on its position within the (Dyson) configuration.

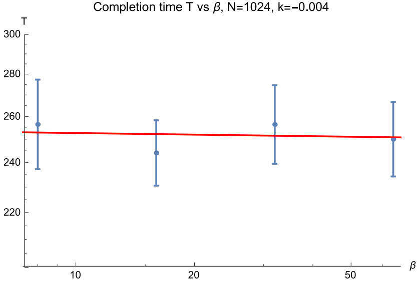
3.2 Networking - sample



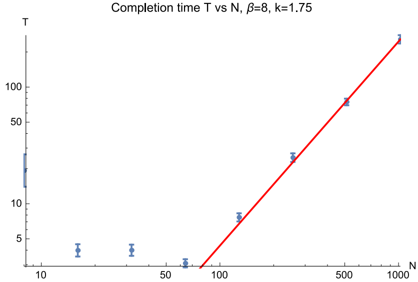
Sample graph for $N = 64$, $\beta = 8$

- With high probability, there is at least one node of strength 1.
- Completion time T: time needed to get a connected graph.

3.2 Networking - completion times



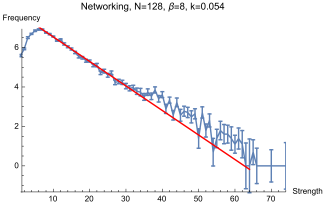
$N = 1024$



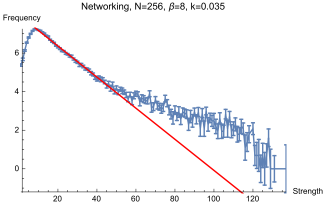
$\beta = 8$

- The dependence of T on β is weak ($\sqrt{\beta}$ scaling).
- At large N, it seems that there is a power-law relationship between T and N.

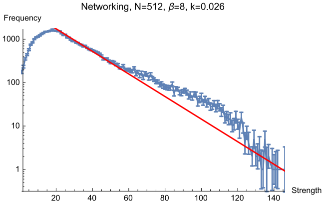
3.2 Networking - strength frequencies



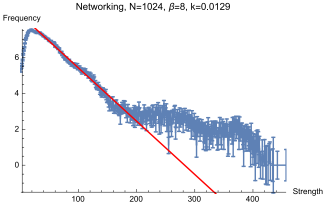
$N = 128, k = 0.054$



$N = 256, k = 0.035$



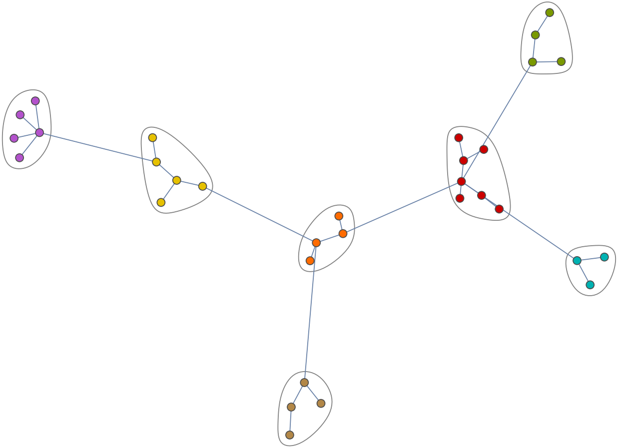
$N = 512, k = 0.026$



$N = 1024, k = 0.013$

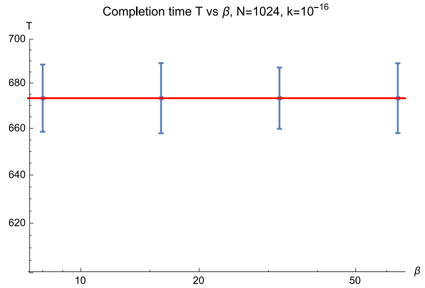
- Fixed β, varying N. Fitting function is of the form $f = A \times 10^{-ks}$.
- Strength distribution seems non-trivial; k seems to fall with N.

3.3 Contagion - sample

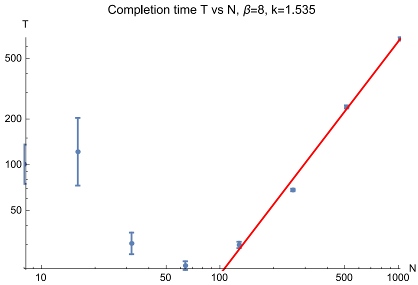


Sample graph for $N = 32$, $\beta = 8$

3.3 Contagion - completion times



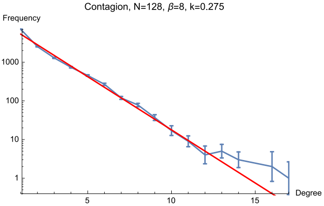
$N = 1024$



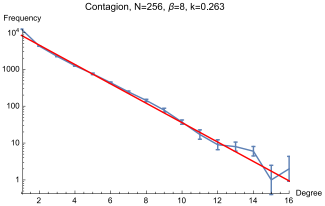
$\beta = 8$

- As before, the dependence of T on β is weak.
- At large N, it seems that there is a power-law relationship between T and N.

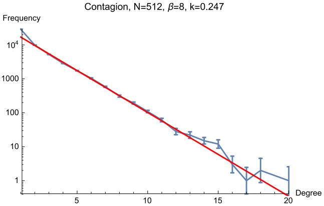
3.3 Contagion - degree frequencies



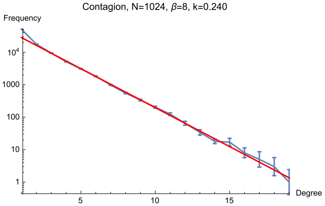
$N = 128, k = 0.275$



$N = 256, k = 0.263$



$N = 512, k = 0.247$



$N = 1024, k = 0.240$

- Fixed β, varying N. Degree frequencies show an exponential behavior.
- Fitting function: $f = A \times 10^{-kd}$; k decreases only slightly with N.

4. Summary and outlook

Results:

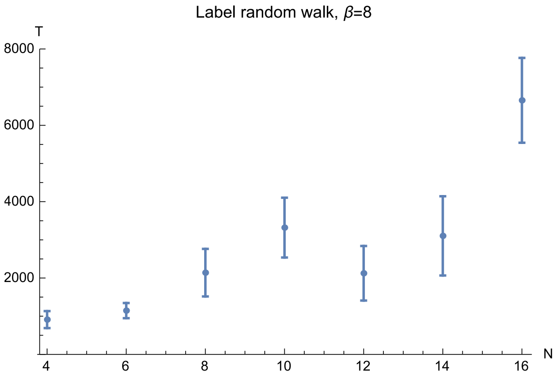
- At large N, it seems that $T = A \times 10^{kN}$.
- With an initial condition scaled by $\sqrt{\beta}$, T has a very weak dependence on β (if any) when β is large.
- The strength distribution shows a complex structure in the networking model.
- The degree distribution in the contagion model shows an exponential behavior ($f = A \times 10^{-kd}$).

Current work:

- Analytics.
- Exponential behavior in the contagion model: is it due to the rules of the model, or intrinsic to the exchange interaction?
- Improvement of the simulation code to deal with smaller values of β (what happens when β is close to 1?)
- How do these results translate to the behavior of $\rho(\boldsymbol{X}^S(t), t)$?
- Formulation of other types of models.

Thank you!

3. Appendix - Label random walk



Label walk times for $\beta = 8$

- T: time until the graph is connected (completion time).
- Dependence on β is extremely weak (see previous slides).
- Dependence on N is non-monotonic. T seems to grow too rapidly to probe numerically.