RIMS Research Project（RIMS共同研究（公開型））：

Gaussian Free Fields and Related Topics（ガウス自由場および関連する話題）

Dates: 18 September 2018 (Tuesday) 9:45 – 21 September 2018 (Friday) 16:50
Venue: Room 420, Research Institute for Mathematical Sciences (RIMS), Kyoto University
Organizers: Naotaka Kajino (Kobe University, Chair)
Takashi Kumagai (Kyoto University)
Daisuke Shiraishi (Kyoto University)

Program

	Sept. 18 (Tue.)	Sept. 19 (Wed.)	Sept. 20 (Thu.)	Sept. 21 (Fri.)
9:50–10:40	Wu	Berestycki	Sheffield	Louidor
10:40–11:00	(Tea Break)	(Tea Break)	(Tea Break)	(Tea Break)
11:00–11:40	Sakagawa	Katori	Nakashima	Abe
11:50–12:40	Vargas	Vargas	Miller	Miller
12:40–14:00	(Lunch Break)	(Lunch Break)	(Lunch Break)	(Lunch Break)
14:00–14:50	Wu	Berestycki	Sheffield	Louidor
14:50–15:10	(Tea Break)	(Tea Break)	(Tea Break)	(Tea Break)
15:10–15:50	Murayama	Nakajima	de Gier	M. Fukushima
16:00–16:40	R. Fukushima	Kawabi	Sasamoto	Oshima
16:50–17:30			Kusuoka	
18:00–		Workshop Banquet		

18 September 2018 (Tuesday)

- 9:45–9:50 Opening Remarks
- 9:50–10:40 Hao Wu (Tsinghua University)
  Gaussian free field: level lines and connection probabilities (I)

(Tea break for 20 minutes)

- 11:00–11:40 Hironobu Sakagawa (Keio University)
  Localization of a Gaussian membrane model with weak pinning potentials
- 11:50–12:40 Vincent Vargas (École normale supérieure)
  An introduction to Liouville conformal field theory (I)

(Lunch break for 80 minutes)

- 14:00–14:50 Hao Wu (Tsinghua University)
  Gaussian free field: level lines and connection probabilities (II)

(Tea break for 20 minutes)

- 15:10–15:50 Takuya Murayama (Kyoto University)
  Characterization of the explosion time for the Komatu–Loewner evolution
- 16:00–16:40 Ryoki Fukushima (Kyoto University)
  Tail estimates for the random walk in random scenery

19 September 2018 (Wednesday)

- 9:50–10:40 Nathanaël Berestycki (University of Vienna)
  Dimers and imaginary geometry (I)

(Tea break for 20 minutes)

- 11:00–11:40 Makoto Katori (Chuo University)
  Elliptic DPP, one-component plasma, and GFF
- 11:50–12:40 Vincent Vargas (École normale supérieure)
  An introduction to Liouville conformal field theory (II)

(Lunch break for 80 minutes)

- 14:00–14:50 Nathanaël Berestycki (University of Vienna)
  Dimers and imaginary geometry (II)

(Tea break for 20 minutes)

- 15:10–15:50 Shuta Nakajima (Kyoto University)
  Ergodic property of the number of infinite geodesics in first-passage percolation
- 16:00–16:40 Hiroshi Kawabi (Keio University)
  Riemannian Wasserstein geometry on the space of Gaussian measures over the Wiener space

18:00– Workshop Banquet

20 September 2018 (Thursday)

- 9:50–10:40 Scott Sheffield (Massachusetts Institute of Technology)
  Universal randomness in 2D (I)

(Tea break for 20 minutes)

- 11:00–11:40 Makoto Nakashima (Nagoya University)
  Free energy of directed polymers in random environment

- 11:50–12:40 Jason P. Miller (University of Cambridge)
  Random walks on random planar maps (I)

(Lunch break for 80 minutes)

- 14:00–14:50 Scott Sheffield (Massachusetts Institute of Technology)
  Universal randomness in 2D (II)

(Tea break for 20 minutes)

- 15:10–15:50 Jan de Gier (University of Melbourne)
  Six vertex limit shape outside the free fermion point

- 16:00–16:40 Tomohiro Sasamoto (Tokyo Institute of Technology)
  Fluctuations of the stochastic higher spin six vertex model

- 16:50–17:30 Seiichiro Kusuoka (Okayama University)
  The invariant measure and flow associated to the Phi4-quantum field model on the three-dimensional torus

21 September 2018 (Friday)

- 9:50–10:40 Oren Louidor (Technion - Israel Institute of Technology)
  Extreme and large values of the discrete Gaussian free field (I)

(Tea break for 20 minutes)

- 11:00–11:40 Yoshihiro Abe (Gakushuin University)
  Thick points for simple random walk on two-dimensional lattice

- 11:50–12:40 Jason P. Miller (University of Cambridge)
  Random walks on random planar maps (II)

(Lunch break for 80 minutes)

- 14:00–14:50 Oren Louidor (Technion - Israel Institute of Technology)
  Extreme and large values of the discrete Gaussian free field (II)

(Tea break for 20 minutes)

- 15:10–15:50 Masatoshi Fukushima (Osaka University)
  Gaussian fields, equilibrium potentials and Liouville random measures for Dirichlet forms I

- 16:00–16:40 Yoichi Oshima (Kumamoto University)
  Gaussian fields, equilibrium potentials and Liouville random measures for Dirichlet forms II

(End of Workshop)

Abstracts

Keynote Talks

Gaussian free field: level lines and connection probabilities

Hao Wu (Tsinghua University)

The 2D Gaussian free field (GFF) is a natural 2-dimensional time analogue of Brownian motion. Like Brownian motion, it is conformally invariant and satisfies a certain domain Markov property. It plays an important role in statistical physics: for instance, it is the scaling limit of the height function of the dimer model, and has connections with numerous other models. In the physics literature, the GFF is also known as the free bosonic field, a very fundamental and well-understood object. It is a starting point for many constructions in quantum field theory.

In a series of works by O. Schramm, S. Sheffield and J. Miller, the authors studied the level lines and flow lines of the GFF. The level lines are SLE(4) and the flow lines are SLE(κ) for general κ. In this series of two talks, we focus on the level lines. In the first talk, we study the geometric properties of the level lines, for instance the monotonicity, the intersection, and the reversibility of level lines. In the second talk, we study the connection probabilities of the level lines with alternating boundary conditions. We relate these connection probabilities to the pure partition functions of multiple SLEs, and give explicit formulas for them.

References

[SS13] O. Schramm, S. Sheffield. A Contour Line of the Continuum Gaussian Free Field. Probab. Theory Related Fields 157(1-2): 47-80, 2013

[MS16] J. Miller, S. Sheffield. Imaginary Geometry I: Interacting SLEs. Probab. Theory Related Fields 164(3-4): 553-705, 2016

[WW17] M. Wang, H. Wu. Level Lines of Gaussian Free Field I: Zero-Boundary GFF. Stochastic Process. Appl. 127(4): 1045-1124, 2017

[PW17] E. Peltola, H. Wu. Global and Local Multiple SLEs for $\kappa \leq 4$ and Connection Probabilities for Level Lines of GFF

An introduction to Liouville conformal field theory

Vincent Vargas (École normale supérieure)

Liouville conformal field theory (LCFT hereafter), introduced by Polyakov in his 1981 seminal work "Quantum geometry of bosonic strings", can be seen as a random version of the theory of Riemann surfaces. LCFT appears in Polyakov's work as a 2d version of the Feynman path integral with an exponential interaction term. Since then, LCFT has emerged in a wide variety of contexts in the physics literature and in particular recently in relation with 4d supersymmetric gauge theories (via the AGT conjecture).

The purpose of the course (based on joint works with F. David, A. Kupiainen and R. Rhodes) is to present a rigorous probabilistic construction of Polyakov's path integral formulation of LCFT. The construction is based on the Gaussian Free Field.

Second, if time permits, I will explain another approach to LCFT, the so-called conformal bootstrap approach. This approach which is very popular in physics is based on the celebrated DOZZ formula (after Dorn–Otto and Zamolodchikov–Zamolodchikov) and a recursive procedure. It is an ongoing program to show the equivalence between the probabilistic and the conformal bootstrap approach to LCFT.

Reference (course at IHES): `https://arxiv.org/abs/1712.00829`

Dimers and imaginary geometry

Nathanaël Berestycki (University of Vienna)

The dimer model on a finite bipartite graph is a uniformly chosen perfect matching, i.e., a set of edges which cover every vertex exactly once. It is a classical model of mathematical physics, going back to work of Kasteleyn and Temeperley/Fisher in the 1960s.

A central object for the dimer model is a notion of height function introduced by Thurston, which turns the dimer model into a random discrete surface. I will discuss a series of recent results with Benoit Laslier (Paris) and Gourab Ray (Victoria) where we establish the convergence of the height function to a scaling limit in a variety of situations. This includes simply connected domains of the plane with linear boundary conditions for the height, in which case the limit is the Gaussian free field, and Temperleyan graphs drawn on Riemann surfaces. In all these cases the scaling limit is universal (i.e., independent of the details of the graph) and conformally invariant.

A key new idea in our approach is to exploit "imaginary geometry" couplings between the Gaussian free field and SLE curves.

Universal randomness in 2D

Scott Sheffield (Massachusetts Institute of Technology)

We introduce several universal and canonical random objects that are planar in the sense that they can be embedded in or parameterized by a two dimensional surface. These objects include trees, distributions, curves, loop ensembles, surfaces, and growth trajectories. We discuss the intricate and surprising relationships between these universal objects. We explain how to use generalized functions to construct curves and vice versa; how to conformally weld a pair of surfaces to produce a surface decorated by a simple curve; how to conformally mate a pair of trees to obtain a surface decorated by a non-simple curve; and how to shuffle certain mating and welding operations to produce random growth trajectories on random surfaces. We present both discrete and continuum analogs of these constructions. Some of these constructions are inspired and motivated by physics, especially string theory, conformal field theory, gauge theory, and statistical mechanics. The mathematics can nonetheless be understood independently of the physical motivation.

Random walks on random planar maps

Jason P. Miller (University of Cambridge)

I will describe some recent developments on the study of random walks on random planar maps (spectral dimension, resistance growth, Green's function bounds, and typical displacement). The mathematical tools to derive these results are based on the tree mating constructions which relate different types of SLE curves with the Gaussian free field and Liouville quantum gravity. I will also describe some related work on the conformal embedding problem for certain types of random planar maps, which in this framework reduces to an invariance principle for a type of random walk in random environment. This is based on joint works with Bertrand Duplantier, Ewain Gwynne, and Scott Sheffield.

Extreme and large values of the discrete Gaussian free field

Oren Louidor (Technion - Israel Institute of Technology)

I will survey recent (mainly last 5 years) progress in the study of extreme and large values of the discrete Gaussian free field in two dimensions. Some new results concerning the geometry of extremal level sets will be presented towards the end.

Invited Talks

Localization of a Gaussian membrane model with weak pinning potentials

Hironobu Sakagawa (Keio University)

We consider a class of effective model on $\mathbb{Z}^d$ called Gaussian membrane model with weak pinning potentials. It is known that when $d = 1$ this model exhibits localization/delocalization transition depending on the strength of pinning. In this talk we show that when $d \geq 2$, once we impose pinning potentials the field is always localized in the sense that the corresponding free energy is always positive.

Characterization of the explosion time for the Komatu–Loewner evolution

Takuya Murayama (Kyoto University)

The chordal Komatu-Loewner equation extends the chordal Loewner equation in the upper half-plane, which is well known in the SLE theory, to standard slit domains. As the Loewner equation generates a family of bounded sets by generating a family of corresponding conformal maps, the Komatu-Loewner equation also generates such a family, which we call the Komatu-Loewner evolution. In this talk, I will characterize the explosion time for the Komatu-Loewner equation for the slits (of the image domain) in a natural fashion with relation to the hitting time of the evolution to the slits, boundary of the domain. My result is a refinement of a part of the study by Bauer and Friedrich (2008).

Tail estimates for the random walk in random scenery

Ryoki Fukushima (Kyoto University)
(Joint work with Jean-Dominique Deuschel (TU Berlin))

Let $(\{S_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{Z}^d})$ be the continuous time simple random walk on $\mathbb{Z}^d$ and $(\{z(x)\}_{x\in\mathbb{Z}^d}, \mathbb{P})$ non-negative IID random variable with a power law tail

$$\mathbb{P}(z(x) > r) = r^{-\alpha+o(1)}, \quad r \to \infty$$

for some $\alpha > 0$. Random walk in random scenery is a process defined as

$$A_t = \int_0^t z(S_u)\mathrm{d}u.$$

This process first appeared in the independent works by Borodin and Kesten-Spitzer in 1979, who aimed at constructing a new class of self-similar processes by taking scaling limits. Beside this initial motivation, it naturally appears in models of diffusion in radom media, for example in the parabolic Anderson model and the Bouchaud trap model.

In this talk, I discuss the tail estimates for A_t with the *quenched* scenery. To illustrate the results, here I focus on the case $d = 1$ and $\alpha < 1$. In this case, A_t is shown to scale like $t^{(\alpha+1)/2\alpha}$ by Borodin and Kesten-Spitzer. For the upper deviation probability, we have more or less complete result.

Theorem 1. *For any $\rho > (\alpha+1)/2\alpha$, $\mathbb{P}$-almost surely,*

$$P_0\left(A_t \geq t^{\rho}\right) = \exp\left\{-t^{p(\alpha,\rho)+o(1)}\right\}$$

as $t \to \infty$, where

$$p(\alpha,\rho) = \begin{cases} \frac{2\alpha\rho}{\alpha+1} - 1, & \rho \in \left(\frac{\alpha+1}{2\alpha}, \frac{\alpha+1}{\alpha}\right], \\ \alpha(\rho-1), & \rho > \frac{\alpha+1}{\alpha}. \end{cases}$$

As for the lower deviation probability, we know it decays stretched exponentially but have not identified the exponent.

Theorem 2. *For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $\mathbb{P}$-almost surely,*

$$P_0\left(A_t \leq t^{\frac{\alpha+1}{2\alpha}-\epsilon}\right) = \exp\left\{-t^{\delta(\epsilon)}\right\}$$

for all sufficiently large $t > 0$.

In the talk, I also present applications of these results to the study of random walk in a random conductance which has a layered structure.

Elliptic DPP, one-component plasma, and GFF

Makoto Katori (Chuo University)

We introduce new families of determinantal point processes (DPPs) on a complex plane $\mathbb{C}$, which are classified into seven types following the irreducible reduced affine root systems, $R_N = A_N$, B_N, $B_N^{\vee}$, C_N, $C_N^{\vee}$, BC_N, D_N, $N \in \mathbb{N}$. Their multivariate probability densities are totally elliptic functions with periods (L, iW), $0 < L, W < \infty$, $i = \sqrt{-1}$. The construction is based on the orthogonality relations with respect to the double integrals over the fundamental domain, $[0, L) \times [0, iW)$, which are proved in this paper for the R_N-theta functions introduced by Rosengren and Schlosser. In the scaling limit $N \to \infty, L \to \infty$ with constant density $\rho = N/(LW)$ and constant W, we obtain four types of DPPs with an infinite number of points on $\mathbb{C}$, which have periodicity with period iW. In the further limit $W \to \infty$ with constant ρ, they are degenerated into three infinite-dimensional DPPs. One of them is uniform on $\mathbb{C}$ and equivalent with the Ginibre point process studied in random matrix theory, while other two systems are isotropic viewed from the origin, but non-uniform on $\mathbb{C}$. We show that the elliptic DPP of type A_N is identified with the particle section, obtained by subtracting the background effect, of the two-dimensional exactly solvable model for one-component plasma studied by Forrester. Other two exactly solvable models of one-component plasma are constructed associated with the elliptic DPPs of types C_N and D_N. Relationship to the Gaussian free field (GFF) on a torus is discussed for these three exactly solvable plasma models. (For more details, see arXiv:math-ph/1807.08287.)

Ergodic property of the number of infinite geodesics in first-passage percolation

Shuta Nakajima (Kyoto University)

First-passage percolation is a random growth model which has a metric structure. Infinite geodesics is an infinite sequence whose all sub-sequences are shortest paths. One of the classical problem is the number of infinite geodesics originating from the origin. When the dimension is two and an edge distribution is continuous, it is proved to be almost surely constant [D. Ahlberg, C. Hoffman. Random coalescing geodesics in first-passage percolation]. In this talk, we will discuss the above result for other dimensions and general distributions.

Riemannian Wasserstein geometry on the space of Gaussian measures over the Wiener space

Hiroshi Kawabi (Keio University)

The space of Gaussian measures on an abstract Wiener space being equivalent to the Wiener measure becomes a Hilbert manifold, and the manifold admits a non-positive Riemannian metric derived from the information geometry. We consider another geometric structure on the manifold, so-called the Wasserstein geometry, which is a metric geometry on the space of probability measures. We first show the convexity of the manifold with respect to the Wasserstein geometry, which enables us to restrict the Wasserstein geometry to the manifold naturally. We then construct a Riemannian metric on the manifold, which induces the Wasserstein distance function. The Riemannian manifold has a non-negative sectional curvature, which provides the difference from the information geometry. Finally, we mention a brief idea of a construction of Brownian motion on this infinite dimensional Riemannian manifold. This talk is based on joint work with Asuka Takatsu (Tokyo Metropolitan University).

Free energy of directed polymers in random environment

Makoto Nakashima (Nagoya University)

We consider the free energy $F(\beta)$ of the directed polymers in random environment in $1+1$-dimension. It is known that $F(\beta)$ is of order $-\beta^4$ as $\beta \to 0$. In this talk, we will see that under a certain concentration condition of the environment,

$$\lim_{\beta\to 0}\frac{F(\beta)}{\beta^4} = \lim_{T\to\infty}\frac{1}{T}P_{\mathcal{Z}}\left[\log \mathcal{Z}_{\sqrt{2}}(T)\right] = -\frac{1}{6},$$

where $\{\mathcal{Z}_\beta(t,x) : t \geq 0, x \in \mathbb{R}\}$ is the unique mild solution to the stochastic heat equation

$$\frac{\partial}{\partial t}\mathcal{Z} = \frac{1}{2}\Delta\mathcal{Z} + \beta\mathcal{Z}\dot{\mathcal{W}}, \quad \lim_{t\to 0}\mathcal{Z}(t,x)dx = \delta_0(dx),$$

where $\mathcal{W}$ is a time-space white noise and

$$\mathcal{Z}_\beta(t) = \int_{\mathbb{R}}\mathcal{Z}_\beta(t,x)dx.$$

Six vertex limit shape outside the free fermion point

Jan de Gier (University of Melbourne)

In collaboration with Kenyon and Watson we show how the surface tension can be computed for a non-free fermionic five vertex model. The limit shape boundary for any geometry can be given explicitly and consists of piecewise algebraic curves.

Fluctuations of the stochastic higher spin six vertex model

Tomohiro Sasamoto (Tokyo Institute of Technology)

The stochastic higher spin six vertex model is a stochastic version of a higher spin generalization of the well-known six vertex model. For this model, one can define a height function and its dynamics becomes that of the KPZ equation in a continuous limit. We study fluctuations of this model in its stationary state by relating it to the q-Whittaker measure and applying the techniques introduced in our previous paper to study stationary q-TASEP.

The presentation is based on a collaboration with T. Imamura and M. Mucciconi.

The invariant measure and flow associated to the Phi4-quantum field model on the three-dimensional torus

Seiichiro Kusuoka (Okayama University)

We consider the invariant measure and flow of the Phi4-model on the three-dimensional torus, which appears in the quantum field theory. By virtue of Hairer's breakthrough, such a nonlinear stochastic partial differential equation became solvable and is studied as a hot topic. In the talk, we apply the paracontrolled calculus and directly construct the global solution and the invariant measure by using the invariant measures of approximation equations and showing the tightness of associated processes. This is a joint work with Sergio Albeverio.

Thick points for simple random walk on two-dimensional lattice

Yoshihiro Abe (Gakushuin University)

I will consider thick points, which are sites frequently visited by the simple random walk on a square of side length N. As N goes to infinity, a point process corresponding to thick points converges in law to a random measure which appears in the study of intermediate level sets of the two-dimensional discrete Gaussian free field by Biskup and Louidor (2016+). This talk is based on work in progress with Marek Biskup (UCLA).

Gaussian fields, equilibrium potentials and Liouville random measures for Dirichlet forms I

Masatoshi Fukushima (Osaka University)

We consider the centered Gaussian field indexed by the extended Dirichlet space with covariance being the inner product for a given regular Dirichlet form. When the underling space E is a bounded domain of the complex plane $\mathbb{C}$ and the form is associated with the absorbing Brownian motion (ABM), D. Duplantier and S. Sheffield, 2011, employed the uniform probability measure on the shrinking circle and the associated Gaussian random variable to construct a Liouville random measure. Sheffield, 2016, also discusses the cases directly associated with the BM on $\mathbb{C}$ and the reflecting BM on the closure of the upper half-plane $\mathbb{H}$.

It is known that, in the logarithmic potential theory, there exists for a non-polar bounded Borel subset B of $\mathbb{C}$ a unique probability measure (called the equilibrium measure) concentrated on the boundary of B whose logarithmic potential (called the equilibrium potential) is constant (called the Robin constant) at every regular point of B. When B is a disk, the equilibrium measure coincides with the uniform probability measure on the circle.

We show that analogous concepts can be well introduced for general irreducible recurrent regular Dirichlet forms. But in this generality, we need to define those concepts depending on a choice of a certain compact subset F of E called an adimissibe set. We thereby try to construct Liouville random measures in planar cases by means of the equilibrium measure and the Robin constant in place of the uniform probability measure and the variance of the corresponding Gaussian variable, respectively, as will be explained in the second part of our talks in some details. An analogous consideration will be also made for general transient regular Dirichlet forms.

So far, several methods have been used in constructing Liouville measures in transient cases. Among them, the method due to N. Berestycki, 2017, based on the Cameron-Martin formulae for the Gaussian field works in recurrent cases as well, and we shall invoke it in our construction.

Gaussian fields, equilibrium potentials and Liouville random measures for Dirichlet forms II

Yoichi Oshima (Kumamoto University)

We consider a regular recurrent strongly local Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathbb{C};d\mathbf{x})$ and an associated symmetric diffusion process $\mathbb{M}=(X_t,\mathbb{P}_{\mathbf{x}})$ on $\mathbb{C}$. The transition function of $\mathbb{M}$ is assumed to have a positive jointly continuous density function $P(t,\mathbf{x},\mathbf{y})$. We choose the annulus $F=\overline{B(S+1)}\setminus B(S)$ as the admissible set and consider the family $\{R\mu\in\mathcal{F}_e:\mu\in\mathcal{M}_0\}$ of recurrent potentials relative to F. We try to construct a Liouville random measure on $B(S)$ for a given Radon measure σ on it.

For a disk $B(\mathbf{x},r)\subset B(S)$, let $\mu^{\mathbf{x},r}\in\mathcal{M}_0$ and $f(\mathbf{x},r)$ be its equilibrium measure and Robin constant relative to F, respectively. They admit simple probabilisitic expressions in terms of the diffusion $\mathbb{M}$. Let $\{X_u:u\in\mathcal{F}_e\}$ be the centered Gaussian field with $\mathbb{E}[X_uX_v]=\mathcal{E}(u,v)$ and define $Y^{\mathbf{x},\varepsilon}=X_{R\mu^{\mathbf{x},\varepsilon}}$. For a fixed $\gamma>0$. put

$$I_\varepsilon(\omega)=\int_A \exp\left[\gamma Y^{\mathbf{x},\varepsilon}\;-\;\frac{\gamma^2}{2}f(\mathbf{x},\varepsilon)\right]\sigma(d\mathbf{x}),\quad A\in\mathcal{B}(B(S)),$$

Under a certain condition on the growth rate of the Robin constant $f(\mathbf{x},r)$ as $r\downarrow 0$, the a.s. convergence of $I_\varepsilon(\omega)$ as $\varepsilon\downarrow 0$ will be derived. The possible range of the value γ will then be examined in the cases that $(\mathcal{E},\mathcal{F})$ equals $(\frac{1}{2}\mathbf{D},H^1(\mathbb{C}))$, $(\frac{1}{2}\mathbf{D},H^1(\mathbb{H}))$ and $(\mathbf{a},H^1(\mathbb{C}))$, where $\mathbf{a}$ corresponds to a uniformly elliptic partial differential operator of divergence form. The case that $(\mathcal{E},\mathcal{F})=(\frac{1}{2}\mathbf{D},H^1_0(D))$ for a domain $D\subset\mathbb{C}$ will be also examined under an analogous general consideration for transient Dirichlet forms.