Three-parametric Marcenko-Pastur Density

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1.Original and two-parametric Marcenko-Pastur density

Consider $M \times N (M \geq N)$ random matrices $K = (K_{jk})$ such that the elements are complex, i.i.d and normally distributed with zero mean and variance 1.This setting is described as

$$\Re K_{jk} \sim N(0,1/2), \quad \Im K_{jk} \sim N(0,1/2), \quad j = 1, \dots, M, \quad \mathrm{k} = 1, \dots, N. \quad (1.1)$$

We consider a statistical ensemble of $N \times N$ Hermitian random matrices L defined by

$$L = K^{\dagger} K. \quad (1.2)$$

We denote the eigenvalues of L as X_j , $j = 1, \dots, N$.In the scaling limit, $N \to \infty$, $M \to \infty$ with $N/M = r \in (0,1]$,the empirical distribution of $\{X_j/M\}$ converges to a deterministic measure. The limit measure has a finite support in $\mathbb{R}$ and it is explicitly given as a function of the parameter $\mathbf{r}$.

$$\rho(x;r) = \frac{\sqrt{(x-x_L(x;r))(x_R(x;r)-x)}}{2\pi r x} \mathbf{1}_{(x_L(r),x_R(r))}(x), \quad x_L(r) \coloneqq (1-\sqrt{r})^2, \quad x_R(r) \coloneqq (1+\sqrt{r})^2. \quad (1.3)$$

A dynamical extension of eigenvalue distribution of Wishart-matrix ensemble is realized by the solution of the following system of stochastic differential equations(SDEs),

$$dX_j^N(t) = 2\sqrt{X_j^N(t)}dB_j(t) + 2(\nu+1)dt + 4X_j^N(t)\sum_{\substack{1\leq k\leq N,\\ k\neq j}} \frac{1}{X_j^N(t)-X_k^N(t)}, \quad j = 1, \dots, N, \quad t \geq 0, \quad (1.4)$$

where $\nu = M - N$ and $B_j(t)$, $t \geq 0$, $1 \leq j \leq N$ are independent one-dimensional standard Brownian motions. For $\rho_\xi(x;r,t)$, $t \geq 0$ with initial distribution ξ, we define the Green function (the resolvent) $G_\xi(z;r,t)$ by the Stieltjes transform of ρ_ξ. Then we can prove that this solves the following nonlinear partial differential equation (PDE),

$$\frac{\partial G_\xi}{\partial t} = -\frac{\partial G_\xi}{\partial z} + r\{\frac{\partial G_\xi}{\partial z} - 2zG_\xi \frac{\partial G_\xi}{\partial z} - {G_\xi}^2\}, \quad t \geq 0. \quad (1.5)$$

Under the initial condition that all particles are concentrated on the origin , Green function $G_{\delta_0}(z;r,t)$ is given by the solution of the equation,

$$z = \frac{1}{G_{\delta_0}(z)} + \frac{t}{1-rtG_{\delta_0}(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad r \in (0,1], \quad t \geq 0. \quad (1.6)$$

The probability density on the time-dependent extension is obtained by solving the equation and using the Sokhotski-Plemelj theorem [1],

$$\rho(x;r,t) = \frac{\sqrt{(x-x_L(x;r,t))(x_R(x;r,t)-x)}}{2\pi r x} \mathbf{1}_{(x_L(r,t),x_R(r,t))}(x), \quad x_L(r,t) \coloneqq (1-\sqrt{r})^2 t, \quad x_R(r,t) \coloneqq (1+\sqrt{r})^2 t. \quad (1.7)$$

2.Three-parametric Marcenko-Pastur density

PDE(1.5),with starting from $\boldsymbol{a} \geq \mathbf{0}$, the Green function, is obtained by the solution of the equation,[2]

$$z = \frac{1}{G_{\delta_0}(z)} + \frac{t}{1-rtG_{\delta_0}(z)} + \frac{a}{(1-rtG_{\delta_0}(z))^2}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \ r \in (0,1], \ t \geq 0 , \ a \geq 0.$$

We write the probability density via the Sokhotski-Plemelj theorem as ρ_{δ_a} and call it the ***three-parametric Marcenko-Pastur density***. ρ_{δ_a} is given by the following explicit formula $\rho(x;r,t,a)$. In Figure1, the original MP density and the three-parametric MP density are represented by curves. These histograms are generated finite random matrices have correspond parameters.

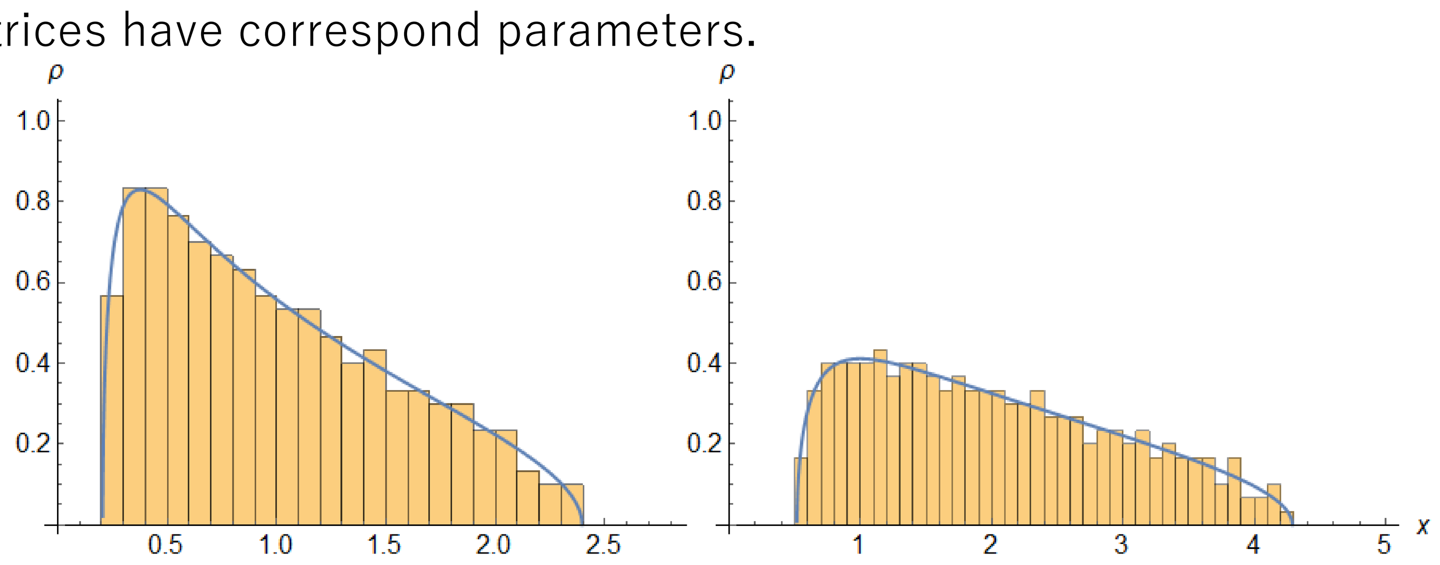

Figure1:Histograms of eigenvalues of matrices L given by random rectangular matrices K with size 1000×300. K's elements are randomly generalized following complex normal distribution, with mean 0(right),and $\sqrt{300}\delta_{ij}$(left).MP density are shown by curves.

3Main Theorem

The ***three-parametric Marcenko-Pastur density*** $\rho(x;r,t,a)$ is given by,

$$\rho(x;r,t,a) = \frac{\sqrt{(x-f_L(x;r,t,a))(f_R(x;r,t,a)-x)}}{2\pi r x t} \mathbf{1}_{(x_L(r,t,a),x_R(r,t,a))}(x),$$

with,

$$f_L(x;r,t,a) = \left(\sqrt{\frac{t-a+\sqrt{-4a\varphi+(t-a)^2}}{2}} - \sqrt{\varphi + x + \frac{t-a+\sqrt{-4a\varphi+(t-a)^2}}{2}}\right)^2,$$

$$f_R(x;r,t,a) = \left(\sqrt{\frac{t-a+\sqrt{-4a\varphi+(t-a)^2}}{2}} + \sqrt{\varphi + x + \frac{t-a+\sqrt{-4a\varphi+(t-a)^2}}{2}}\right)^2,$$

where,

$$\varphi(x;\ r,t,a) \coloneqq -\frac{2}{3}\{x-(r-1)t\} - \frac{2^{\frac{1}{3}}}{3}\frac{x^2+\{3a-(2r+1)t\}x+t^2(r-1)^2}{g^{\frac{1}{3}}} - \frac{g^{\frac{1}{3}}}{3\times 2^{\frac{1}{3}}},$$

$$g(x;r,t,a) \coloneqq -2x^3 + 3\{(2\mathrm{r}+1)\mathrm{t}+6\mathrm{a}\}x^2 - 3\big[(r-1)\{(2r+1)t-3a\}t - \sqrt{-3S}\big)\big]x + 2(r-1)^3t^3,$$

$$S(x;r,t,a) \coloneqq 4ax^3 - \{8a^2+4a(3r+2)t-t^2\}x^2 + 2[2a^3-2a^2(5r-2)t + a\{r(6r-1)+1\}t^2 - (r+1)t^3]x + (r-1)^2t^2\{a^2-a(4r-2)t+t^2\}.$$

$x_L(r,t,a), x_R(r,t,a)$:Let x_1, x_2, x_3 are real solutions of $S(x;r,t,a) = 0$, and $x_1 \leq x_2 \leq x_3$,Define $x_L(r,t,a) \coloneqq x_2, x_R(r,t,a) \coloneqq x_3$.

4.Proposition

(1)If and only if $r = 1$,the domain $\{(x_L(r,t,a), x_R(r,t,a)) : t \geq 0\}$ touches the $x = 0$.

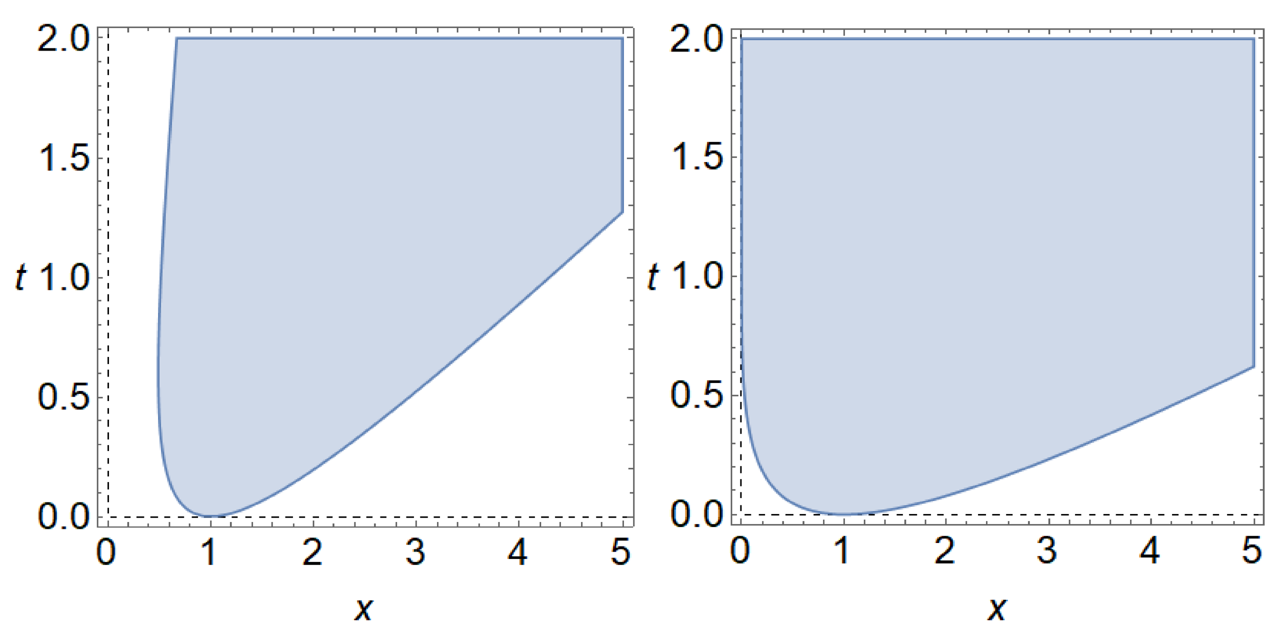

Figure2:Time evolution of the support of the three-parametric Marcenko-Pastur Density starting from 1. Left figure : $r = 0.3$, Right figure : $r = 1$.

(2)There is a critical time $t_c(a) = a$ such that time domain touches $x = 0$. More specifically , while $0 \leq t \leq t_{c(a)}$,

$$x_L(r,t,a) \simeq \frac{4}{27a^2}(t_c(a)-t)^3, \text{as } t \nearrow t_c(a)$$

(3)When , the three-parametric Marcenko-Pastur density shows the following dynamic critical phenomena at $\mathrm{t} = t_c(a)$.

(i)For $0 < t < t_c(a)$,

$$\rho(x;1,t,a) \simeq \frac{9a}{4\pi}(t_c(a)-t)^{-5/2}\left(x - x_L(1,t,a)\right)^{1/2}$$

as $x \searrow x_L(1,t,a)$, $\mathrm{t} \nearrow t_c(a)$.

(ii)For $t = t_c(a)$,

$$\rho(x;1,t_c(a),a) \simeq \frac{\sqrt{3}}{2\pi}a^{-2/3}x^{-1/3}$$

as $x \searrow 0$.

(iii)For $t_c(a) < t$,

$$\rho(x;1,t,a) \simeq \frac{1}{\pi t_c(a)}(t-t_c(a))^{1/2}x^{-1/2}$$

as $x \searrow 0$, $\mathrm{t} \nearrow t_c(a)$.

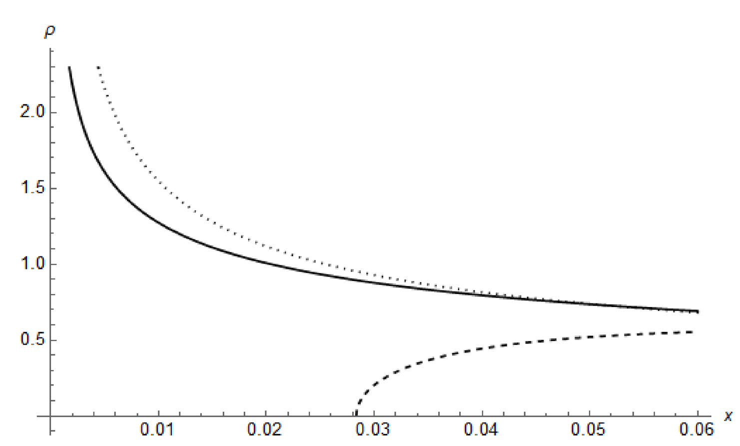

Figure3:For (3) in $r = 1, a = 1$. The dashed curve : (i) at $t = 0.5t_c(1)$. The solid curve : (ii) at the critical time $t = t_c(1)$. The dotted curve : (iii) at $t = 1.5t_c(1)$.

In Figure3,the dashed curve denotes the emergence of ρ at $x = x_L > 0$ with the critical exponent 1/2 at subcritical time. The solid curve $\sim x^{-1/3}$ is weaker than the dotted curve $x^{-1/2}$.

5.References

[1]Blaizot, J.-P., Nowak, M. A., Warchoł, P.: Universal shocks in the Wishart random-matrix ensemble. Phys. Rev. E **87**, 052134/1–10 (2013)
[2] Blaizot, J.-P., Nowak, M. A., Warchoł, P.: Universal shocks in the Wishart random-matrix ensemble. II. Nontrivial initial conditions. Phys. Rev. E **89**, 042130/1–7 (2014)
[3]Endo, T., Katori, M.: Three-Parametric Marcenko-Pastur Density. arXiv: math. PR/1907.07413