Quantum Surface with Marked Boundary Points and Multiple SLE Driven by Dyson Model

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Plan

1. Introduction
   1.1 Log-gases on R
   1.2 Multiple SLE curves on H
   1.3 GFF on *D* and Quantum Surface (QS)
2. QS with Marked Boundary Points (MBPs)
3. Two Ways of Sampling QS-MBPs
4. Main Theorems
5. Proof of Theorem 4.1
6. Concluding Remarks

1. Introduction

1.1 Log-gases on R

- **For $N \in \mathbb{N} := \{1, 2, \dots\}$, consider a system of interacting Brownian motions on $\mathbb{R}$, $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(N)}) \in \mathbb{R}^N, t \geq 0$, following the SDEs,**

$$dX_t^{(i)} = \sqrt{\kappa} dB_t^{(i)} + F^{(i)}(\mathbf{X}_t) dt, \quad i = 1, \dots, N, \quad t \geq 0,$$

where $\{B_t^{(i)} : t \geq 0\}_{i=1}^N$ are mutually independent one-dimensional standard Brownian motions, and $\kappa > 0$. (Note that $\sqrt{\kappa} B_t \stackrel{\text{(law)}}{=} B_{\kappa t}, t \geq 0$.)

- **Example 1: Dyson model with parameter** $\beta > 0$
  Consider the case that

$$F^{(i)}(\mathbf{x}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{4}{x_i - x_j}, \quad i = 1, \dots, N.$$

A time change of the obtained SDEs ($\kappa t \to t$, $\mathbf{X}_{t/\kappa} \to \mathbf{X}_t$) **gives**

$$dX_t^{(i)} = dB_t^{(i)} + \frac{\beta}{2} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{dt}{X_t^{(i)} - X_t^{(j)}}, \quad i = 1, \dots, N, \quad t \geq 0, \quad \textbf{with} \quad \beta = \frac{8}{\kappa}.$$

- **Example 2: Bru–Wishart process with parameters (β, ν)**
  Consider the case that

$$F^{(i)}(\mathbf{x}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{4}{x_i - x_j} + \frac{4}{x_i + x_j} \right) + \frac{4 + \delta + \kappa/2}{x_i}, \quad i = 1, \dots, N.$$

A time change of the obtained SDEs ($\kappa t \to t$, $\mathbf{X}_{t/\kappa} \to \mathbf{X}_t$) gives

$$dX_t^{(i)} = dB_t^{(i)} + \left[\frac{2\nu + 1}{2} \frac{1}{X_t^{(i)}} + \frac{\beta}{2} \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{1}{X_t^{(i)} - X_t^{(j)}} + \frac{1}{X_t^{(i)} + X_t^{(j)}} \right) \right] dt,$$

$$t \geq 0, \quad i = 1, \dots, N, \quad \textbf{with} \quad \beta = \frac{8}{\kappa}, \quad \nu = \frac{4 + \delta}{\kappa}.$$

1.2 Multiple SLE curves on H

- Denote the upper half of complex plane as $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$.

Theorem 1.1 (Theorem 1.1 of [RS17]) *Let $\eta^{(i)} : (0, \infty) \to \mathbb{H}$, $i = 1, \ldots, N$, be non-colliding and non-self-intersecting curves in $\mathbb{H}$ anchored on $\mathbb{R}$. There exists a unique set of continuous driving functions $\mathbf{X}_t = (X_t^{(1)}, \ldots, X_t^{(N)}) \in \mathbb{R}^N$, $t \in [0, \infty)$, such that the family of conformal mappings (uniformization maps)*

$$g_t : \mathbb{H}_t^{\eta} := \mathbb{H} \Big\backslash \bigcup_{i=1}^{N} \eta^{(i)}(0, t] \to \mathbb{H}$$

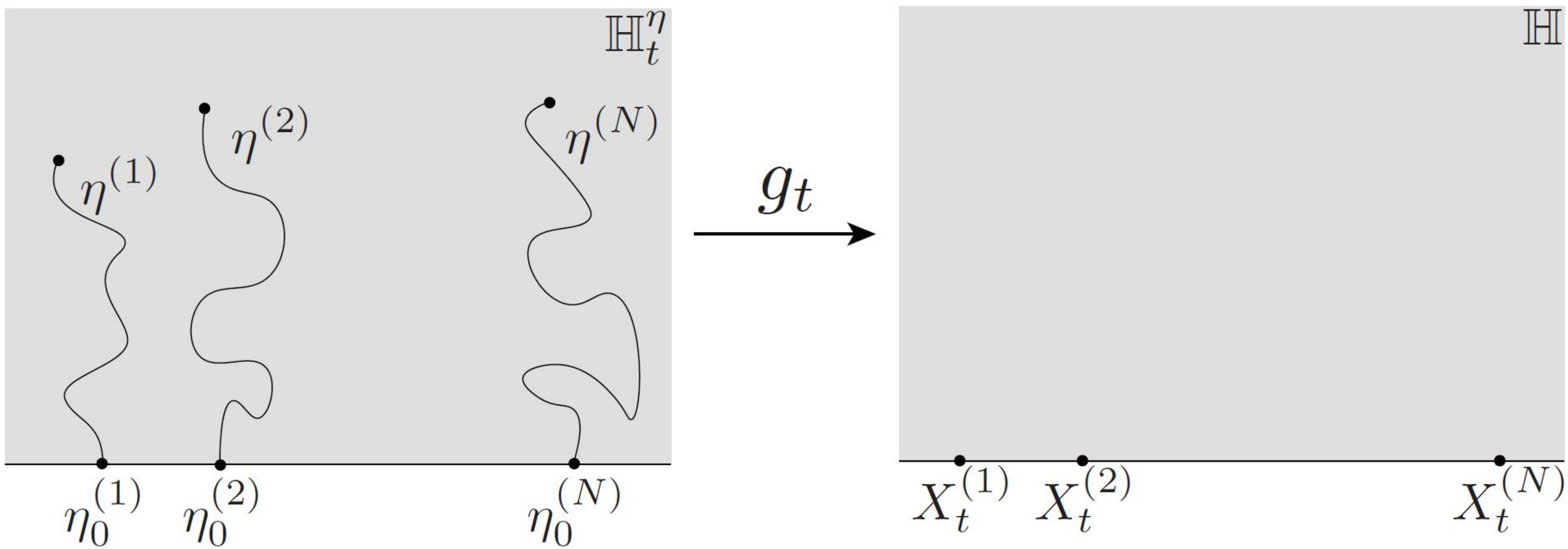
solves the multiple Loewner equation:

$$\frac{d}{dt} g_t(z) = \sum_{i=1}^{N} \frac{2}{g_t(z) - X_t^{(i)}}, \quad t \geq 0, \quad g_0(z) = z \in \mathbb{H},$$

i.e., $\{g_t\}_{t \geq 0}$ is the Loewner chain driven by $\{\mathbf{X}_t : t \geq 0\}$. Moreover, the driving functions are determined by

$$X_t^{(i)} = \lim_{\epsilon \to 0} g_t(\eta^{(i)}(t) + \epsilon), \quad i = 1, \ldots, N.$$

[RS17] D. Roth, S. Schleissinger : The Schramm-Loewner equation for multiple slits, J. Anal. Math. 131, 73–99 (2017).


$\mathbb{H}_t^\eta$
$\eta^{(1)}$
$\eta^{(2)}$
$\eta^{(N)}$
$\eta_0^{(1)}$
$\eta_0^{(2)}$
$\eta_0^{(N)}$
g_t
$\mathbb{H}$
$X_t^{(1)}$
$X_t^{(2)}$
$X_t^{(N)}$

- **There is a room for changing the model of uniformization maps. As a generalized multiple Loewner equation for N slits, we consider the following form**

$$\frac{d}{dt} g_t(z) = \Psi(g_t(z), \mathbf{X}_t), \quad t \geq 0, \quad g_0(z) = z,$$

where $\Psi(z, \mathbf{x})$ is a suitable functions of z and $\mathbf{x} = (x_1, \dots, x_N)$, and $\{\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(N)}) : t \geq 0\}$ is a set of driving processes.

- **The above uniformization map (the conformal map to $\mathbb{H}$) is obtained, when we take**

$$\Psi(z, \mathbf{x}) = \sum_{i=1}^{N} \frac{2}{z - x_i}, \quad z \in \mathbb{H}, \quad \mathbf{x} \in \mathbb{R}^N.$$

- **Let** $\mathbb{O} := \{z \in \mathbb{C} : \Re z > 0, \ \Im z > 0\}$ **be an orthant in** $\mathbb{C}$**. We adopt**

$$\Psi(z, \mathbf{x}) = \Psi_{\mathbb{O}}(z, \mathbf{x}) := \sum_{i=1}^{N} \left(\frac{2}{z - x_i} + \frac{2}{z + x_i} \right) + \frac{\delta}{z}, \quad z \in \mathbb{O}, \quad \mathbf{x} \in (\mathbb{R}_{>0})^N,$$

where $\delta \in \mathbb{R}$ **is a parameter and** $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$**.**

- **The associated Loewner equation becomes**

$$\frac{d}{dt} g_t(z) = \sum_{i=1}^{N} \left(\frac{2}{g_t(z) - X_t^{(i)}} + \frac{2}{g_t(z) + X_t^{(i)}} \right) + \frac{\delta}{g_t(z)}, \quad t \geq 0,$$
$$g_0(z) = z \in \mathbb{O},$$

driven by $\{\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(N)}) \in (\mathbb{R}_{>0})^N : t \geq 0\}$**.**
This is the multiple quadrant SLE which governs the N **non-colliding and non-self-intersecting slits** $\{\eta^{(i)} : (0, \infty) \to \mathbb{O}\}_{i=1}^N$ **anchored on** $\mathbb{R}_{>0}$**:** $\eta_0^{(i)} = X_0^{(i)} > 0$, $i = 1, \dots, N$

- **The multiple quadrant SLE** $g_t(\cdot)$, $t \geq 0$, **will give a uniformization map**

$$g_t : \mathbb{O}_t^\eta := \mathbb{O} \Big\backslash \bigcup^{N} \eta^{(i)}(0, t] \to \mathbb{O}.$$

1.3 GFF on *D* and Quantum Surface (QS)

- $D \subsetneq \mathbb{C}$: a simply connected domain.
- $C^\infty(\overline{D})$: the space of smooth functions on D that extend to the boundary.
- $W(D)$: the Hilbert space completion of $C^\infty(\overline{D})$ with respect to the Dirichlet inner product

$$(f,g)_\nabla = \frac{1}{2\pi}\int_D (\nabla f)(z)\cdot(\nabla g)(z)d\mu(z),$$

  where μ is the Lebesgue measure on $D \subset \mathbb{C}$; $d\mu(z) = dzd\overline{z}$.
- Then the Gaussian free field (GFF) with free boundary condition is defined as an isotopy

$$H_D : W(D) \to L^2(\Omega_D, \mathcal{F}_D, \mathbf{P}_D),$$

  where $L^2(\Omega_D, \mathcal{F}_D, \mathbf{P}_D)$ is a probability space such that each $H_D(\rho) := (H_D, \rho)_\nabla$, $\rho \in W(D)$, is a mean-zero Gaussian random variable.
- This can also be regarded as a random distribution $H_D : \Omega_D \to C^\infty(\overline{D})'$, where $C^\infty(\overline{D})'$ denotes the space of distributions with test functions in $C^\infty(\overline{D})$. (Note that each member of $W(D)$ makes sense only up to additive constants.)

- **For** $\rho \in C^\infty(\overline{\mathbb{H}})$**, we define**

$$(H_{\mathbb{H}}, \rho) := (H_{\mathbb{H}}, (-\Delta)^{-1}\rho)_{\nabla}.$$

- **Let** $\rho_1, \rho_2 \in C^\infty(\overline{\mathbb{H}})$ **be functions of zero total mass:** $\int_{\mathbb{H}} \rho_i(z) d\mu(z) = 0$**,** $i = 1, 2$**. Then** $(H_{\mathbb{H}}, \rho_i)$**,** $i = 1, 2$ **are mean-zero Gaussian variables with covariance**

$$\mathbf{E}[(H_{\mathbb{H}}, \rho_1)(H_{\mathbb{H}}, \rho_2)] = \int_{\mathbb{H}^2} \rho_1(z) G_{\mathbb{H}}(z, w) \rho_2(w) d\mu^{\otimes 2}(z, w),$$

where

$$G_{\mathbb{H}}(z, w) = -\log|z - w| - \log|z - \overline{w}|.$$

- **For each realization of GFF, $h(\cdot) = H_D(\cdot, \omega)$, $\omega \in \Omega_D$, let $h_\epsilon(z)$ be the mean value of h on the circle $\partial B_\epsilon(z)$ of radius ϵ centered at $z \in D$.**

- **Introduce a parameter $\gamma \in (0, 2]$.**

- **Then the area measure of the Liouville quantum gravity (LQG) is obtained by**

$$d\mu_h^\gamma(z) := \lim_{\epsilon \to 0} \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} d\mu(z), \quad z \in D.$$

  In a similar way, the boundary measure of LQG is given by

$$d\nu_h^\gamma(x) := \lim_{\epsilon \to 0} \epsilon^{\gamma^2/4} e^{\gamma h_\epsilon(x)/2} d\nu(x), \quad x \in \partial D,$$

  where ν is the Lebesgue measure on the boundary ∂D, while, in this case, $h_\epsilon(x)$ is the average over the semi-circle centered at $x \in \partial D$ of radius ϵ included in D.

- **Let $\tilde{D} \subsetneq \mathbb{C}$ be another simply connected domain, and $\psi : \tilde{D} \to D$ be a conformal map.**

- **Then an area measure is induced on $\tilde{D}$ by pulling back the measure μ_h^γ on D; $\psi^*\mu_h^\gamma(A) := \mu_h^\gamma(\psi(A))$ for a measurable set $A \subset \tilde{D}$.**

- **By changing integration variables, $\psi^*\mu_h^\gamma$ becomes**

$$\lim_{\epsilon \to 0} \int_{\psi(A)} \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} d\mu(z) = \lim_{\epsilon \to 0} \int_A (|\psi'(w)|\epsilon)^{\gamma^2/2} e^{\gamma(h\circ\psi)_\epsilon(w)} |\psi'(w)|^2 d\mu(w),$$

where $\psi'(w) = \dfrac{d\psi}{dw}(w)$. Note that, in the right hand side, the regularization parameter ϵ has to be rescaled by $|\psi'(w)|$.

- **This implies that if we introduce a distribution on $\tilde{D}$ by**

$$\tilde{h} = h \circ \psi + Q \log |\psi'| \quad \textbf{with} \quad Q = \left(\frac{\gamma^2}{2} + 2\right) \Big/ \gamma = \frac{2}{\gamma} + \frac{\gamma}{2},$$

then the corresponding area measure $\mu_{\tilde{h}}^\gamma$ agrees with the pulled-back measure $\psi^*\mu_h^\gamma$.

- **Motivated by the above observation, we make the following definition.**

Definition 1.2 (Quantum surface (QS)) *Let $\gamma \in (0,2]$. A γ-quantum surface is a collection of pairs (D, H_D) subject to the condition that, for all simply connected domains $D_1, D_2 \subsetneq \mathbb{C}$ and conformal map $\psi : D_1 \to D_2$, the following equality in probability law holds,*

$$H_{D_1} \overset{\text{(law)}}{=} H_{D_2} \circ \psi + Q \log |\psi'| \quad \textit{with} \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

- **See for more details,**

[Sch07] S. Sheffield: Gaussian free fields for mathematicians, Probab. Theory Relat. Fields, 139, 521–541 (2007).

[DS11] B. Duplantier, S. Sheffield, Liouville quantum gravity and KPZ, Invent. Math. 185, 333–393 (2011).

2. QS with Marked Boundary Points (MBPs)

- Hereafter, we set $D = \mathbb{H}$ with $\partial\mathbb{H} = \mathbb{R}$.

- Let $\alpha = (\alpha_1, \cdots, \alpha_N)$ be an N-tuple of real numbers.

- Let $\mathrm{Conf}_N^<(\mathbb{R}) := \{(x_1, \dots, x_N) \in \mathbb{R}^N | x_1 < \cdots < x_N\}$. And consider the probability space $(\mathrm{Conf}_N^<(\mathbb{R}), \mathcal{F}^{(N)}, \mathbf{P}^{(N)})$ for N-point process $\mathbf{X} = (X_1, \dots, X_N) \in \mathbb{R}^N$.

- For given realization $\mathbf{x} = (x_1, \dots, x_N) \in \mathrm{Conf}_N^<(\mathbb{R})$, we define a function on $\mathbb{H}$

$$u_{\mathbb{H}}^{\mathbf{x},\alpha}(z) = \sum_{i=1}^{N} \alpha_i \log|z - x_i|, \qquad z \in \mathbb{H}.$$

(Put (2D Coulomb) α_i-charges on the boundary $\mathbb{R} = \partial\mathbb{H}$, $i = 1, \dots, N$.)

- We consider the assignment

$$(H_{\mathbb{H}}^{\mathbf{X},\alpha}, \mathbf{X}) : \Omega_{\mathbb{H}} \times \mathrm{Conf}_N^<(\mathbb{R}) \ni (\omega, \mathbf{x}) \mapsto (H_{\mathbb{H}}(\omega) + u_{\mathbb{H}}^{\mathbf{x},\alpha}, \mathbf{x}).$$

- We consider an equivalent class induced by the conformal equivalence, which includes the above triplet $(\mathbb{H}, H^{\mathbf{X},\alpha}, \mathbf{X})$. This equivalence class is called a QS with marked boundary points (QS-MBPs) (of standard type).

3. Two Ways of Sampling QS-MBPs

Setting

- Let $0 < T < \infty$ and consider a time duration $t \in [0, T]$.
- Give an initial configuration of MBPs, $\mathbf{X}_0 = (X_0^{(1)}, \dots, X_0^{(N)}) \in \mathrm{Conf}_N^{<}(\mathbb{R})$.

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Sampling A

- Sample a GFF : $H_{\mathbb{H}}$.
- Sample a time-evolution of MBPs on $\mathbb{R}$ starting from given $\mathbf{X}_0$:

$$\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(N)}) \in \mathrm{Conf}_N^<(\mathbb{R}), \quad t \in [0, T].$$

- Using only the final MBPs $\mathbf{X}_T = (X_T^{(1)}, \dots, X_T^{(N)})$, obtain

$$H_{\mathbb{H}}^{\mathbf{X}_T, \alpha} = H_{\mathbb{H}} + u_{\mathbb{H}}^{\mathbf{X}_T, \alpha} := H_{\mathbb{H}} + \sum_{i=1}^{N} \alpha_i \log |\cdot - X_T^{(i)}|.$$

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independent

Sampling B

- **Sample a GFF :** $H_{\mathbb{H}}$.
- **Sample a time-evolution of MBPs on** $\mathbb{R}$ **starting from given** $\mathbf{X}_0$ **:**

$$\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(N)}) \in \mathrm{Conf}_N^{<}(\mathbb{R}), \quad t \in [0, T].$$

- **Generate multiple slits** $\{\eta_T^{(i)} = \eta^{(i)}(0,T]\}_{i=1}^N$ **by the multiple SLE** $g_t, t \in [0,t]$**, which is driven by** $\mathbf{X}_t, t \in [0,T]$**.**
- **Define** $H_{\mathbb{H}}\Big|_{\mathbb{H}_T^\eta}$ **by the restriction of** $H_{\mathbb{H}}$ **in** $\mathbb{H}_T^\eta := \mathbb{H} \Big\backslash \bigcup_{i=1}^N \eta_T^{(i)}$ **and put**

$$H_{\mathbb{H}}^{\mathbf{X}_0,\alpha}\Big|_{\mathbb{H}_T^\eta} := H_{\mathbb{H}}\Big|_{\mathbb{H}_T^\eta} + u_{\mathbb{H}}^{\mathbf{X}_0,\alpha}.$$

- **Then pull back by** g_T^{-1} **as**

$$g_T^{-1*} H_{\mathbb{H}}^{\mathbf{X}_0,\alpha}\Big|_{\mathbb{H}_T^\eta} := H_{\mathbb{H}}^{\mathbf{X}_0,\alpha}\Big|_{\mathbb{H}_T^\eta} \circ g_T^{-1} + Q \log |g_T^{-1\prime}|.$$

Sampling B

- Sample a GFF : $H_{\mathbb{H}}$.
- Sample a time-evolution of MBPs on $\mathbb{R}$ starting from given $\mathbf{X}_0$:

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$$H_{\mathbb{H}}^{\mathbf{X}_0, \alpha}\Big|_{\mathbb{H}_T^\eta} := H_{\mathbb{H}}\Big|_{\mathbb{H}_T^\eta} + u_{\mathbb{H}}^{\mathbf{X}_0, \alpha}.$$

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

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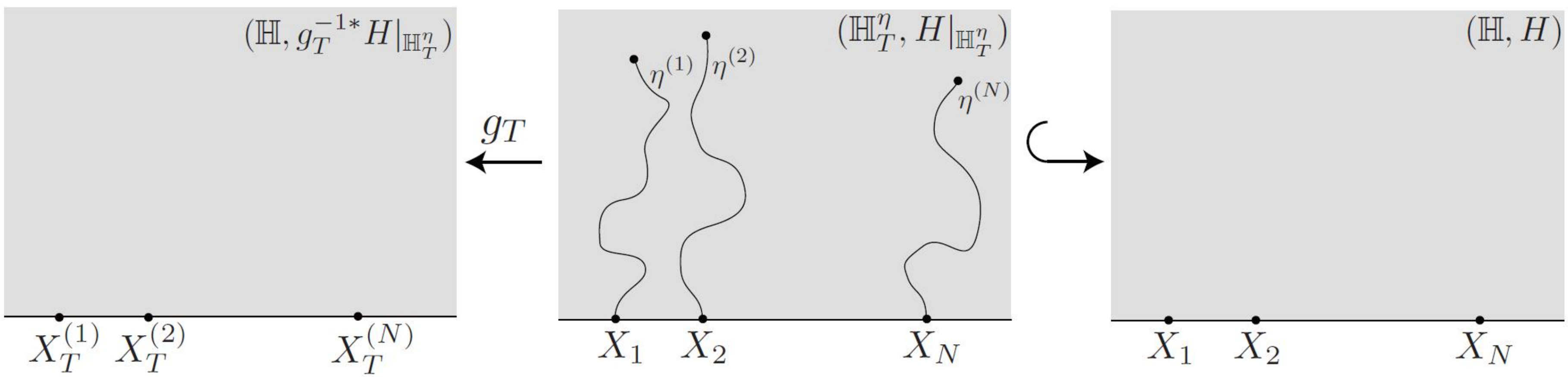
Coupling GFF and multiple SLE

$$g_T^{-1*} H_{\mathbb{H}}^{\mathbf{X}_0, \alpha}\Big|_{\mathbb{H}_T^\eta} := H_{\mathbb{H}}^{\mathbf{X}_0, \alpha}\Big|_{\mathbb{H}_T^\eta} \circ g_T^{-1} + Q \log |g_T^{-1\prime}|.$$


$(\mathbb{H}, g_T^{-1*} H|_{\mathbb{H}_T^\eta})$
g_T
$(\mathbb{H}_T^\eta, H|_{\mathbb{H}_T^\eta})$
$\eta^{(1)}$
$\eta^{(2)}$
$\eta^{(N)}$
$(\mathbb{H}, H)$
$X_T^{(1)}$ $X_T^{(2)}$ $X_T^{(N)}$
X_1 X_2 X_N
X_1 X_2 X_N

4. Main Theorems

Theorem 4.1 *The above two ways of sampling give the same result in distribution, that is,*

$$H_{\mathbb{H}}^{\mathbf{X}_T,\alpha} \stackrel{\text{(law)}}{=} g_T^{-1*} H_{\mathbb{H}}^{\mathbf{X}_0,\alpha}\Big|_{\mathbb{H}_T^\eta},$$

if the following three conditions are satisfied,

(i) $\kappa = \gamma^2,$

(ii) $(\alpha_1, \dots, \alpha_N) = \left(\frac{2}{\gamma}, \dots, \frac{2}{\gamma}\right),$

(iii) $F^{(i)}(\mathbf{x}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{4}{x_i - x_j}, \quad i = 1, \dots, N,$

i.e., $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(N)}) \in \mathrm{Conf}_N^<(\mathbb{R}), t \geq 0,$ *is the time change of the Dyson model with parameter* $\beta = \frac{8}{\kappa}.$

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if the following three conditions are satisfied,

(i) $\kappa = \gamma^2,$ **Relation between SLE and QS**

(ii) $(\alpha_1, \dots, \alpha_N) = \left(\frac{2}{\gamma}, \dots, \frac{2}{\gamma}\right),$ **Charges at MBPs**

(iii) $F^{(i)}(\mathbf{x}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{4}{x_i - x_j}, \quad i = 1, \dots, N,$ **System of Driving Process**

i.e., $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(N)}) \in \mathrm{Conf}_N^{<}(\mathbb{R}), t \geq 0,$ *is the time change of the Dyson model with parameter* $\beta = \frac{8}{\kappa}$.

Similar problem can be considered in the orthant in $\mathbb{C}$, $\mathbb{O} = \{z \in \mathbb{C} : \Re z > 0,\ \Im z > 0\}$, and we can prove the following.

Theorem 4.2 *The equivalence* $H_{\mathbb{O}}^{\mathbf{X}_T,\alpha} \overset{\text{(law)}}{=} g_T^{-1*} H_{\mathbb{O}}^{\mathbf{X}_0,\alpha}\Big|_{\mathbb{H}_T^\eta}$ *is established, if the following three conditions are satisfied,*

(i) $\kappa = \gamma^2$,

(ii) $(\alpha_1, \dots, \alpha_N) = \left(\frac{2}{\gamma}, \dots, \frac{2}{\gamma}\right)$,

(iii) $F^{(i)}(\mathbf{x}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{4}{x_i - x_j} + \frac{4}{x_i + x_j} \right) + \frac{4 + \delta + \kappa/2}{x_i}, \quad i = 1, \dots, N,$

i.e., $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(N)}) \in \mathrm{Conf}_N^{<}(\mathbb{R}_>)$, $t \geq 0$, *is the time change of the Bru–Wishart process with parameters* $\beta = \frac{8}{\kappa}, \nu = \frac{4+\delta}{\kappa}$.

5. Proof of Theorem 4.1

- **For a driving process $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(N)})$ of the multiple SLE, $g_t, t \in [0, T]$, let $\mathbf{Y}_{T;t} = (Y_{T;t}^{(1)}, \dots, Y_{T;t}^{(N)})$ with**

$$Y_{T;t}^{(i)} := X_{T-t}^{(i)}, \quad i = 1, \dots, N, \quad t \in [0, T].$$

- **The reverse flow of the multiple SLE is defined as the solution of**

$$\frac{d}{dt} f_t^T(z) = -\sum_{i=1}^{N} \frac{2}{f_t^T(z) - Y_{T;t}^{(i)}}, \quad t \in [0, T], \quad f_0^T(z) = z \in \mathbb{H}.$$

Lemma 5.1 *The equality $f_T^T = g_T^{-1}$ holds.*

- **Define**

$$\mathfrak{h}_t(z) := u_{\mathbb{H}}^{\mathbf{Y}_{T;t},\alpha}(f_t^T(z)) + Q\log|f_t^{T\prime}(z)|$$
$$:= \sum_{i=1}^{N} \alpha_i \log|f_t^T(z) - Y_{T;t}^{(i)}| + Q\log|f_t^{T\prime}(z)|, \quad t \in [0,T],$$

and put

$$\mathfrak{p}_t := \mathfrak{h}_t + H_{\mathbb{H}} \circ f_t^T, \quad t \in [0,T].$$

- **By definition, the equivalence**

$$H_{\mathbb{H}}^{\mathbf{X}_T,\alpha} \overset{(\mathrm{law})}{=} g_T^{-1*} H_{\mathbb{H}}^{\mathbf{X}_0,\alpha}\Big|_{\mathbb{H}_T^\eta},$$

is equal to

$$\boxed{\mathfrak{p}_0 = \mathfrak{p}_T.}$$

Lemma 5.2 *The stochastic process $\mathfrak{h}_t(z)$, $z \in \mathbb{H}$, $t \in [0,T]$ is a local martingale with increment*

$$d\mathfrak{h}_t(z) = -\sum_{i=1}^{N} \Re \frac{2}{f_t^T(z) - Y_{T;t}^{(i)}} dB_t^{(i)}, \quad z \in \mathbb{H}, \quad t \in [0,T],$$

if the three conditions of Theorem 4.1 are satisfied.

Proof **Assume $(\alpha_1, \ldots, \alpha_N) = (2/\gamma, \ldots, 2/\gamma)$.**

Note that $\mathfrak{h}_t(z)$ is the real part of $\mathfrak{h}_t^*(z) = \frac{2}{\gamma} \sum_{i=1}^{N} \log(f_t^T(z) - Y_{T;t}^{(i)}) + Q \log f_t^{T\prime}(z)$.

It is easy to verify the equality

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{1}{f_t^T(z) - Y_{T;t}^{(i)}} \frac{1}{Y_{T;t}^{(i)} - Y_{T;t}^{(j)}} = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{1}{(f_t^T(z) - Y_{T;t}^{(i)})(f_t^T(z) - Y_{T;t}^{(j)})}.$$

If we set $\kappa = \gamma^2$ and use the above equality, then Itô's formula gives

$$d\mathfrak{h}_t^*(z) = \frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \frac{1}{f_t^T(z) - Y_{T;t}^{(i)}} \left(F^{(i)}(\mathbf{Y}_{T;t}) - \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{4}{Y_{T;t}^{(i)} - Y_{T;t}^{(j)}} \right) dt$$
$$- \sum_{i=1}^{N} \frac{2dB_t^{(i)}}{f_t^T(z) - Y_{T;t}^{(i)}}, \quad t \in [0,T].$$

This proves the statement. ∎

- **In the following, we assume the three conditions of Theorem 4.1.**
- **The above lemma implies that, at each point $z \in \mathbb{H}$, the stochastic process $\{\mathfrak{h}_t(z) : t \in [0, T]\}$ can be regarded as a Brownian motion modulo time change.**
- **Moreover, the above lemma gives the cross variation between two points $z, w \in \mathbb{H}$ as**

$$d\langle \mathfrak{h}(z), \mathfrak{h}(w) \rangle_t = \sum_{i=1}^{N} \left(\Re \frac{2}{f_t^T(z) - Y_{T;t}^{(i)}} \right) \left(\Re \frac{2}{f_t^T(w) - Y_{T;t}^{(i)}} \right) dt, \quad t \in [0, T].$$

Lemma 5.3 *Define*

$$G_{\mathbb{H}_t^\eta}(z,w) := G_{\mathbb{H}}(f_t^T(z), f_t^T(w)), \quad t \in [0,T], \quad z,w \in \mathbb{H}.$$

Then

$$d\langle \mathfrak{h}(z), \mathfrak{h}(w)\rangle_t = -dG_{\mathbb{H}_t^\eta}(z,w), \quad t \in [0,T], \quad z,w \in \mathbb{H}.$$

Proof **This can be verified by direct computation. By definition, we have**

$$G_{\mathbb{H}_t^\eta}(z,w) = -\log|f_t^T(z) - f_t^T(w)| - \log|f_t^T(z) - \overline{f_t^T(w)}|.$$

Thus its increment is computed as

$$\begin{aligned}
dG_{\mathbb{H}_t^\eta}(z,w) &= -\Re\frac{df_t^T(z) - df_t^T(w)}{f_t^T(z) - f_t^T(w)} - \Re\frac{df_t^T(z) - d\overline{f_t^T(w)}}{f_t^T(z) - \overline{f_t^T(w)}} \\
&= -\sum_{i=1}^N \Re\frac{2dt}{(f_t^T(z) - Y_{T;t}^{(i)})(f_t^T(w) - Y_{T;t}^{(i)})} - \sum_{i=1}^N \Re\frac{2dt}{(f_t^T(z) - Y_{T;t}^{(i)})(\overline{f_t^T(w)} - Y_{T;t}^{(i)})} \\
&= -\sum_{i=1}^N \left(\Re\frac{2}{f_t^T(z) - Y_{T;t}^{(i)}}\right)\left(\Re\frac{2}{f_t^T(w) - Y_{T;t}^{(i)}}\right) dt
\end{aligned}$$

which is the same as $-d\langle \mathfrak{h}(z), \mathfrak{h}(w)\rangle_t$**,** $z, w \in \mathbb{H}$**.** ∎

Proof of Theorem 4.1

- For a test function $\rho \in C^\infty(\overline{\mathbb{H}})$ of zero-mass $\int_{\mathbb{H}} \rho(z) d\mu(z) = 0$, we have

$$d\langle(\mathfrak{h}, \rho), (\mathfrak{h}, \rho)\rangle_t = -dE_t(\rho),$$

  where

$$E_t(\rho) = \int_{\mathbb{H}^2} \rho(z) G_{\mathbb{H}_t^\eta}(z, w) \rho(w) d\mu^{\otimes 2}(z, w)$$

  is non-increasing in the time variable $t \in [0, T]$.

- This implies that $(\mathfrak{h}_t, \rho)$, $t \in [0, T]$, is a Brownian motion such that we can regard $-E_t(\rho)$ as time variable.

- Thus $(\mathfrak{h}_T, \rho)$ is normally distributed with mean $(\mathfrak{h}_0, \rho)$ and variance $-E_T(\rho) - (-E_0(\rho)) = -E_T(\rho) + E_0(\rho)$.

$$\mathfrak{p}_t := \mathfrak{h}_t + H_{\mathbb{H}} \circ f_t^T, \qquad t \in [0, T].$$

$$\mathfrak{p}_0 = \mathfrak{p}_T.$$

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Proof of Theorem 4.1

- For a test function $\rho \in C^\infty(\overline{\mathbb{H}})$ of zero-mass $\int_{\mathbb{H}} \rho(z) d\mu(z) = 0$, we have

$$d\langle(\mathfrak{h}, \rho), (\mathfrak{h}, \rho)\rangle_t = -dE_t(\rho),$$

  Dirichlet energy

  where

$$E_t(\rho) = \int_{\mathbb{H}^2} \rho(z) G_{\mathbb{H}_t^\eta}(z, w) \rho(w) d\mu^{\otimes 2}(z, w)$$

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$$\mathfrak{p}_t := \mathfrak{h}_t + H_{\mathbb{H}} \circ f_t^T, \qquad t \in [0, T].$$

- The random variable $(H_{\mathbb{H}} \circ f_T^T, \rho)$ is also normally distributed with mean zero and variance $E_T(\rho)$ by the conformal invariance of the GFF.

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- The random variable $(H_{\mathbb{H}} \circ f_T^T, \rho)$ is also normally distributed with mean zero and variance $E_T(\rho)$ by the conformal invariance of the GFF.

- Since the random variable $(H_{\mathbb{H}} \circ f_T^T, \rho)$ is conditionally independent of $(\mathfrak{h}_T, \rho)$, their sum

$$(\mathfrak{p}_T, \rho) := (H_{\mathbb{H}} \circ f_T^T + \mathfrak{h}_T, \rho)$$

is a normal random variable with mean $(\mathfrak{h}_0, \rho)$ and variance

$$\{-E_T(\rho) + E_0(\rho)\} + E_T(\rho) = E_0(\rho)$$

coinciding with $(\mathfrak{h}_0 + H_{\mathbb{H}}, \rho) = (\mathfrak{p}_0, \rho)$ in probability law.

- This implies $\mathfrak{p}_T \stackrel{\text{(law)}}{=} \mathfrak{p}_0$ as $C^\infty(\overline{\mathbb{H}})'$-valued random fields. The proof of Theorem 4.1 is complete. ∎

$$\mathfrak{p}_0 = \mathfrak{p}_T.$$

6. Concluding Remarks

- Theorem 4.1 is a multi-slit extension of the result by Sheffield [She16], in which the GFF/LQG is coupled with a single SLE curve (i.e., $N = 1$).
- In the case $N = 1$, the location of single MBP is irrelevant, since a shift does not change conformal equivalence. For general N-MBP system, time evolution of MBPs is essential;

$$H_{\mathbb{H}}^{\mathbf{X}_T, \alpha} \stackrel{\text{(law)}}{=} g_T^{-1*} H_{\mathbb{H}}^{\mathbf{X}_0, \alpha}\Big|_{\mathbb{H}_T^\eta},$$

[She16] S. Sheffield : Conformal weldings of random surfaces: SLE and the quantum gravity zipper, Ann. Probab. 44, 3474–3545 (2016).

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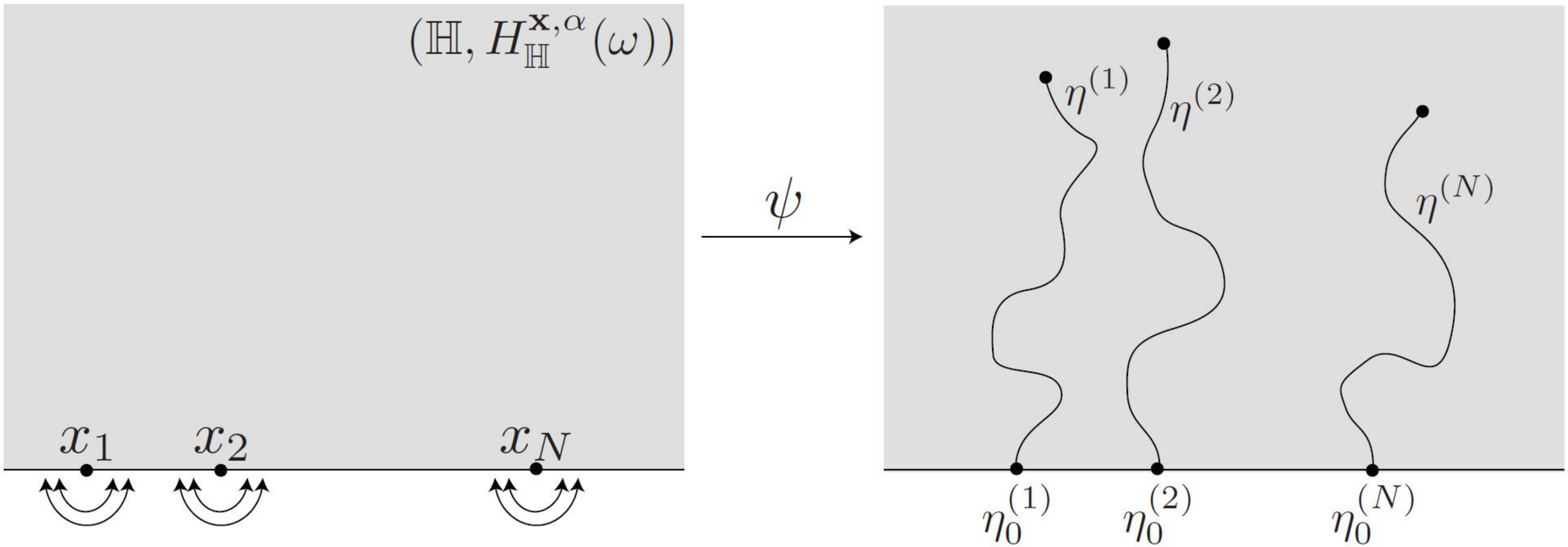
- **Our results solve the generalized conformal welding problems, whose original form (with $N=1$) was proposed and solved by Sheffield [She16]. The key is the following; from our construction, it is obvious the equalities,**

$$\nu^{\gamma}_{H_{\mathbb{H}}^{\mathbf{x}_0,\alpha}}\Big|_{\mathbb{H}^{\eta}_T}(\eta^{(i)}(0,t]_{\mathrm{L}}) = \nu^{\gamma}_{H_{\mathbb{H}}^{\mathbf{x}_0,\alpha}}\Big|_{\mathbb{H}^{\eta}_T}(\eta^{(i)}(0,t]_{\mathrm{R}}), \quad t \in [0,T], \quad i=1,\dots,N, \quad \text{a.s.},$$

where $\nu^{\gamma}_{H_{\mathbb{H}}^{\mathbf{x}_0,\alpha}}$ is the boundary measure of γ-QS-MBPs, and $\eta^{(i)}(0,t]_{\mathrm{L}}$ (resp. $\eta^{(i)}(0,t]_{\mathrm{R}}$) is the boundary segment lying on the left (resp. right) of the slit $\eta^{(i)}(0,t]$.

[She16] S. Sheffield : Conformal weldings of random surfaces: SLE and the quantum gravity zipper, Ann. Probab. 44, 3474–3545 (2016).



$$\nu^{\gamma}_{H_{\mathbb{H}}^{\mathbf{x}_0,\alpha}}\Big|_{\mathbb{H}_T^{\eta}}(\eta^{(i)}(0,t]_{\mathrm{L}}) = \nu^{\gamma}_{H_{\mathbb{H}}^{\mathbf{x}_0,\alpha}}\Big|_{\mathbb{H}_T^{\eta}}(\eta^{(i)}(0,t]_{\mathrm{R}}), \quad t \in [0,T], \quad i = 1,\ldots,N, \quad \text{a.s.}$$

- **In random matrix theory, the Dyson model and the Bru–Wishart process are considered as 'different systems' form each other. On the other hand, 'GFF on $\mathbb{H}$' and 'GFF on $\mathbb{O}$' are γ-equivalent in the sense,**

$$\left[\mathbb{O}, H_{\mathbb{O}}^{\mathbf{X},\alpha}, (\mathbf{X}, \infty)\right]_{\gamma} = \left[\mathbb{H}, H_{\mathbb{H}}^{\mathbf{X}^2,\alpha}, (\mathbf{X}^2, \infty)\right]_{\gamma},$$

where $\mathbf{X}^2 = ((X_1)^2, \dots, (X_N)^2)$. This suggests a new perspective of random matrix theory (and multiple SLE, GFF, LQG, ...).

- **The relation between parameters of the Dyson model and the multiple SLE is determined via the present coupling with QS with MBPs as**

$$\beta = \frac{8}{\kappa} \quad \Longleftrightarrow \quad \kappa = \frac{8}{\beta}.$$

In the (multiple) SLE it is well known that there occur transitions at

$$\kappa_{\mathrm{c}}^{(1)} = 4 \quad \text{and} \quad \kappa_{\mathrm{c}}^{(2)} = 8.$$

We know the colliding/noncolliding transition at $\beta_{\mathrm{c}}^{(2)} = \frac{8}{\kappa_{\mathrm{c}}^{(2)}} = 1$ in the Dyson model. What kind of transition will be observed at $\beta_{\mathrm{c}}^{(1)} = \frac{8}{\kappa_{\mathrm{c}}^{(1)}} = 2$?

Thank you very much for your attention.

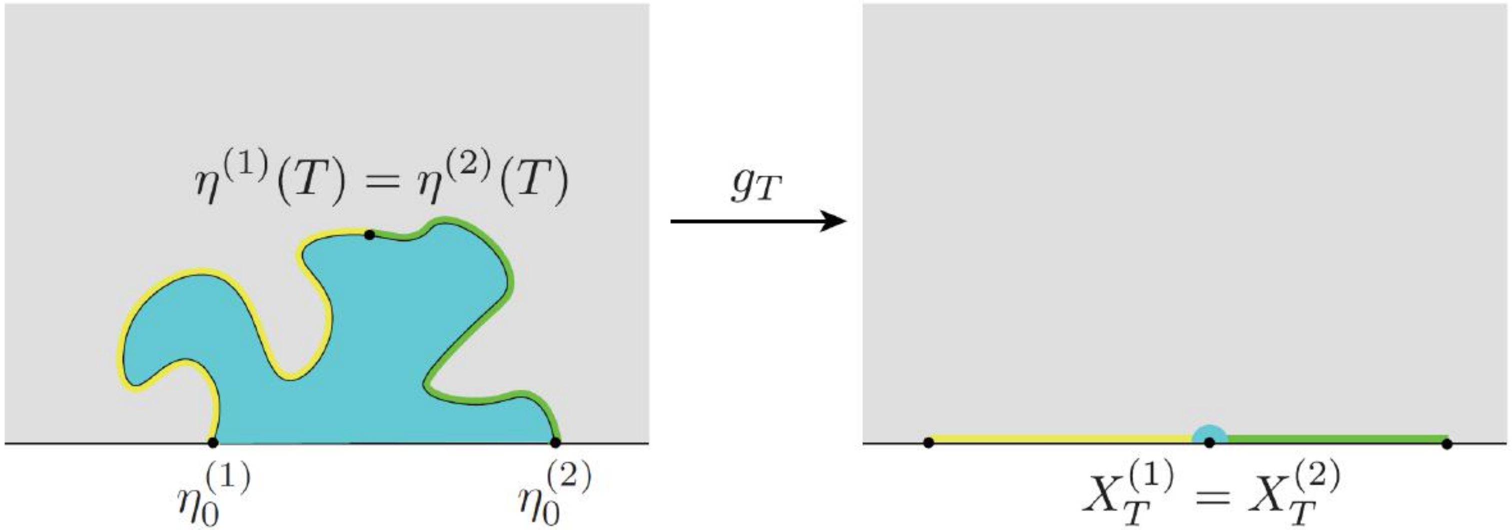


FIGURE 4.1. (I) Tips collide

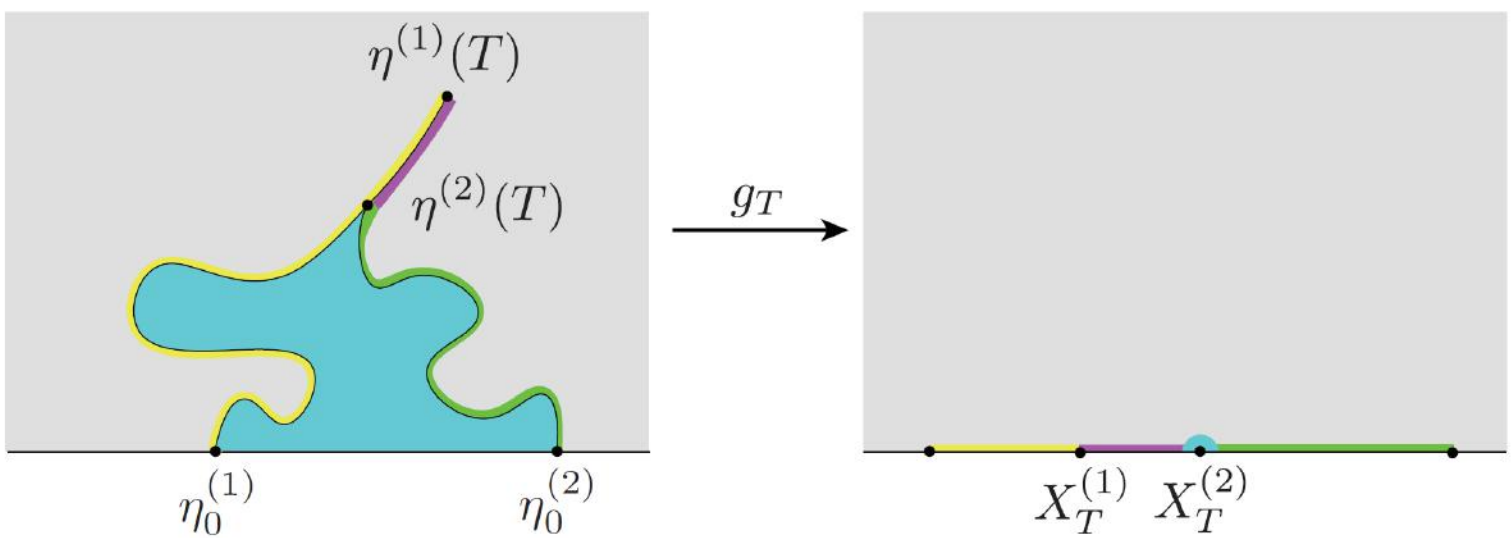


FIGURE 4.2. (II) A tip collides with an already existing slit

- **If two slits collide with each other, this event is classified into two cases.**

  (I) Two tips of slits collide with each other.

  (II) A tip collides with an already existing slit.

- **Since each of the driving processes is the image of a tip of a slit under the uniformization map, two of the driving processes collide with each other when the event (I) occurs.**

- **On the other hand, when the event (II) occurs, the driving processes are non-colliding even though the corresponding SLE slits are colliding.**

- **Following this argument, we could expect that the Dyson model will fall into three classes.**

  (A) When $\beta \geq 2$, the particles are non-colliding and the corresponding SLE slits are non-colliding and non-self-intersecting.

  (B) When $\beta \in [1, 2)$, the particles are non-colliding, but the event (II) almost surely occurs.

  (C) When $\beta \in (0, 1)$, the particles collide, and correspondingly, the event (I) almost surely occurs.

- **Though it is known that the colliding/non-colliding transition occurs at $\beta = 1$, the possible phenomenon that the characteristics of the Dyson model changes at $\beta = 2$ has not been well studied so far.**

- **It would be an interesting future direction to find a property that distinguishes the Dyson model of $\beta \in (1, 2)$ and that of $\beta \geq 2$.**