Two-Dimensional Elliptic Determinantal Point Processes and Related Systems

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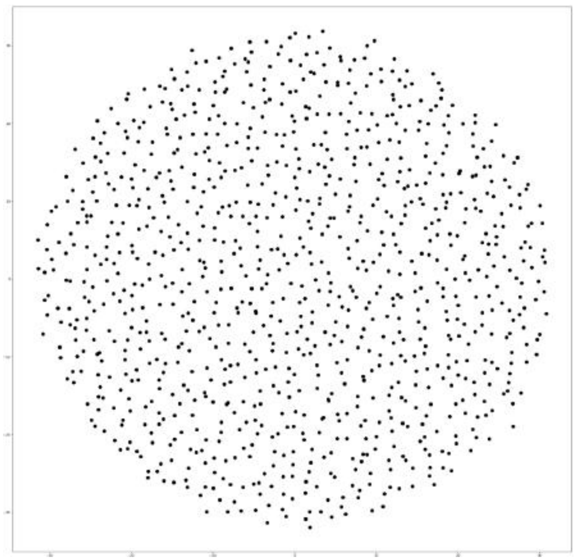
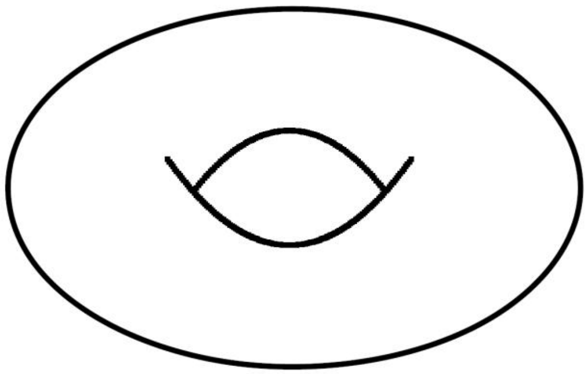




1. Introduction to Determinantal Point Processes (DPPs)

- Let S be a base space, which is locally compact Hausdorff space with countable base, and λ be a Radon measure on S.

- The configuration space over S is given by the set of nonnegative-integer-valued Radon measures;

$$\mathrm{Conf}(S) = \left\{ \xi = \sum_j \delta_{x_j} : x_j \in S,\ \xi(\Lambda) < \infty \text{ for all bounded set } \Lambda \subset S \right\}.$$

  $\mathrm{Conf}(S)$ is equipped with the topological Borel σ-fields with respect to the vague topology; we say $\xi_n, n \in \mathbb{N} := \{1, 2, \dots\}$ converges to ξ in the vague topology, if $\int_S f(x)\xi_n(dx) \to \int_S f(x)\xi(dx)$, $\forall f \in \mathcal{C}_c(S)$, where $\mathcal{C}_c(S)$ is the set of all continuous real-valued functions with compact support.

- A point process on S is a $\mathrm{Conf}(S)$-valued random variable $\Xi = \Xi(\cdot, \omega)$ on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. If $\Xi(\{x\}) \in \{0, 1\}$ for any point $x \in S$, then the point process is said to be simple.

- **Assume that $\Lambda_j, j = 1, \ldots, m$, $m \in \mathbb{N}$ are disjoint bounded sets in S.**
- **By definition,**

$$\Xi(\Lambda_j) = \text{number of points included in } \Lambda_j, j = 1, \ldots, m.$$

- **For $k_j \in \mathbb{N}_0 := \{0, 1, \ldots\}, j = 1, \ldots, m$ satisfying $\sum_{j=1}^m k_j = n \in \mathbb{N}_0$, we consider the following product of combinatorial numbers,**

$$\prod_{j=1}^{m} \binom{\Xi(\Lambda_j)}{k_j} := \prod_{j=1}^{m} \frac{\Xi(\Lambda_j)!}{k_j!(\Xi(\Lambda_j) - k_j)!}.$$

- **If its expectation is written as**

$$\mathbf{E}\left[\prod_{j=1}^{m} \binom{\Xi(\Lambda_j)}{k_j}\right] = \frac{1}{k_1! \cdots k_m!} \int_{\Lambda_1^{k_1} \times \cdots \times \Lambda_m^{k_m}} \rho^n(x_1, \ldots, x_n) \lambda^{\otimes n}(dx_1 \cdots dx_n),$$

where $\lambda^{\otimes n}$ denotes the n-product measure of λ, then $\rho^n(x_1, \ldots, x_n)$ is called the n-point correlation function with respect to the background measure λ.

- **Determinantal point process (DPP) is defined as follows [Sos00,ST03].**

Definition 1.1 *A simple point process Ξ on (S, λ) is said to be a determinantal point process (DPP) with correlation kernel $K : S \times S \to \mathbb{C}$ if it has correlation functions $\{\rho^n\}_{n \geq 1}$, and they are given by*

$$\rho^n(x_1, \dots, x_n) = \det_{1 \leq j,k \leq n} [K(x_j, x_k)] \quad \text{for every } n \in \mathbb{N}, \text{ and } x_1, \dots, x_n \in S.$$

The triplet $(\Xi, K, \lambda(dx))$ denotes the DPP, $\Xi \in \mathrm{Conf}(S)$, specified by the correlation kernel K with respect to the measure $\lambda(dx)$.

[Sos00] A. Soshnikov, Determinantal random point fields, Russian Math. Surveys 55 (2000) 923–975.

[ST03] T. Shirai and Y. Takahashi, Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point process, J. Funct. Anal. 205 (2003) 414–463.

- **If the integral operator $\mathcal{K}$ on $L^2(S, \lambda)$ with kernel K is of rank $N \in \mathbb{N}$, then the number of points is N a.s. If $N < \infty$ (resp. $N = \infty$), we call the system a finite DPP (resp. and infinite DPP).**

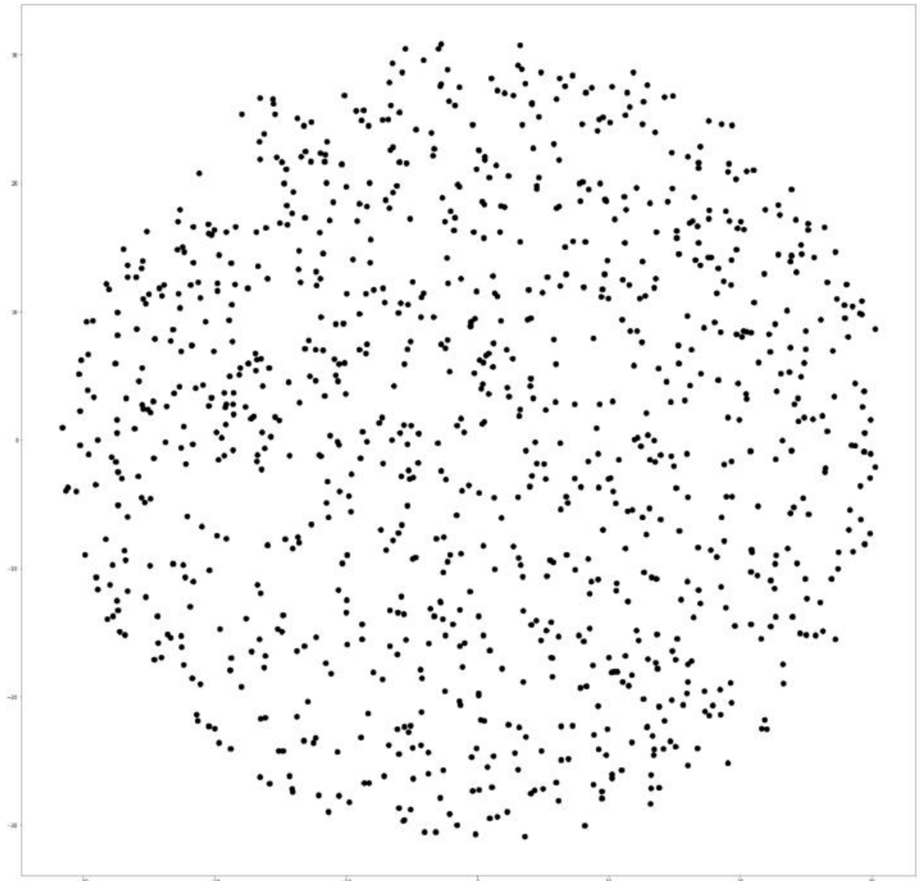
- **The density of points with respect to the background measure $\lambda(dx)$ is given by**

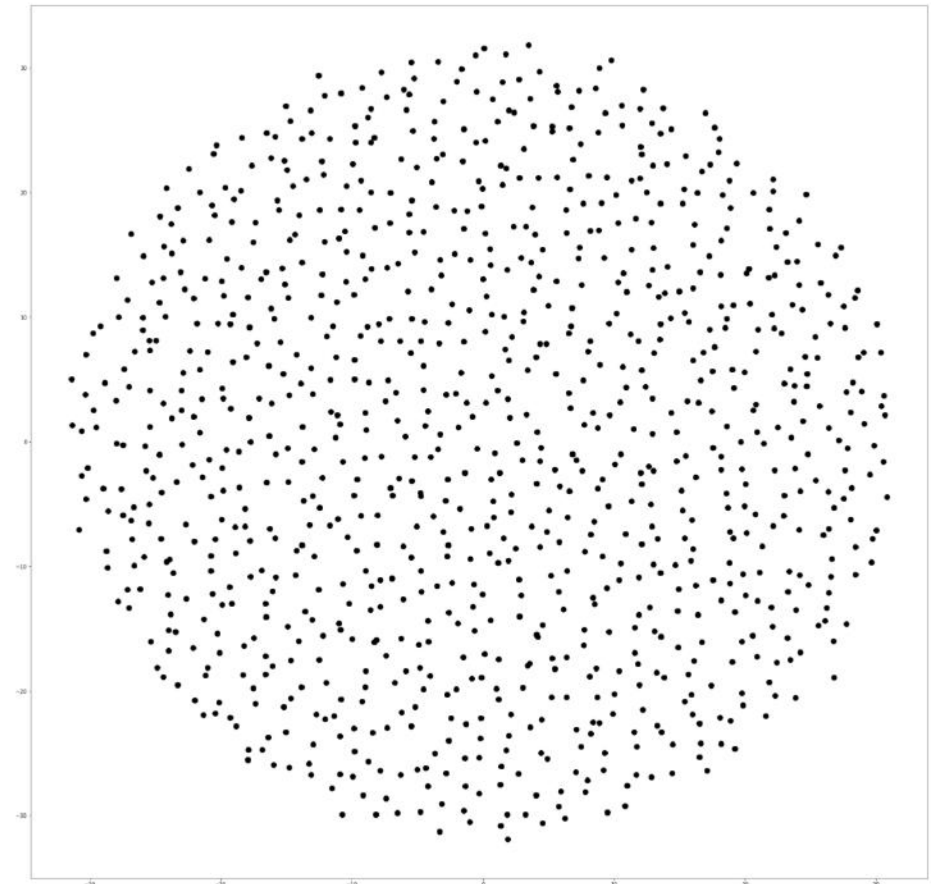
$$\rho(x) := \rho^1(x) = K(x, x).$$

- **The DPP is negatively correlated as shown by**

$$\begin{aligned} \rho^2(x, x') &= \det \begin{bmatrix} K(x, x) & K(x, x') \\ K(x', x) & K(x', x') \end{bmatrix} \\ &= K(x, x)K(x', x') - |K(x, x')|^2 \le \rho(x)\rho(x'), \quad x, x' \in S, \end{aligned}$$

provided that K is Hermitian.





Poisson point process **an example of DPP (Ginibre DPP)**

(Computer simulation by T. Matsui (Chuo U.))

- **Let $L^2(S, \lambda)$ be an L^2-space.**

- **For operators $\mathcal{A}, \mathcal{B}$ on $L^2(S, \lambda)$, we write $\mathcal{A} \geq O$ if $\langle \mathcal{A}f, f \rangle_{L^2(S,\lambda)} \geq 0$ for any $f \in L^2(S, \lambda)$, and $\mathcal{A} \geq \mathcal{B}$ if $\mathcal{A} - \mathcal{B} \geq O$.**

- **For a compact subset $\Lambda \subset S$, the projection from $L^2(S, \lambda)$ to the space of all functions vanishing outside Λ λ-a.e. is denoted by $\mathcal{P}_\Lambda$. $\mathcal{P}_\Lambda$ is the operation of multiplication of the indicator function $\mathbf{1}_\Lambda$ of the set Λ; $\mathbf{1}_\Lambda(x) = 1$ if $x \in \Lambda$, and $\mathbf{1}_\Lambda(x) = 0$ otherwise.**

- **We say that the bounded Hermitian operator $\mathcal{A}$ on $L^2(S, \lambda)$ is said to be of locally trace class, if the restriction of $\mathcal{A}$ to each compact subset Λ, $\mathcal{A}_\Lambda := \mathcal{P}_\Lambda \mathcal{A} \mathcal{P}_\Lambda$, is of trace class; $\operatorname{Tr} \mathcal{A}_\Lambda < \infty$.**

- **The totality of locally trace class operators on $L^2(S, \lambda)$ is denoted by $\mathcal{I}_{1,\text{loc}}(S, \lambda)$.**

- **Here we recall the existence theorem for DPPs.**
- **Let** (S, λ) **be a** σ**-finite measure space. We assume that** $\mathcal{K} \in \mathcal{I}_{1,\mathrm{loc}}(S, \lambda)$.
- **If, in addition,** $\mathcal{K} \geq O$**, then it admits a Hermitian integral kernel** $K(x, x')$ **such that [GY05]**

  (i) $\det_{1 \leq j,k \leq n}[K(x_j, x_k)] \geq 0$ **for** $\lambda^{\otimes n}$**-a.e.** $(x_1, \dots, x_n)$ **for every** $n \in \mathbb{N}$,

  (ii) $K_{x'} := K(\cdot, x') \in L^2(S, \lambda)$ **for** λ**-a.e.** x',

  (iii) $\mathrm{Tr}\, \mathcal{K}_\Lambda = \int_\Lambda K(x, x)\lambda(dx)$, $\Lambda \subset S$ **and**

$$\mathrm{Tr}\,(\mathcal{P}_\Lambda \mathcal{K}^n \mathcal{P}_\Lambda) = \int_\Lambda \langle K_{x'}, \mathcal{K}^{n-2} K_{x'} \rangle_{L^2(S,\lambda)} \lambda(dx'), \quad \forall n \in \mathbb{N}.$$

[GY05] H.-O. Georgii and H. J. Yoo, Conditional intensity and Gibbsianness of determinantal point processes, J. Stat. Phys. <u>118</u> (2005) 55–84.

Theorem 1.2 (Sos00,ST03) *Assume that* $\mathcal{K} \in \mathcal{I}_{1,\mathrm{loc}}(S, \lambda)$ *and* $O \leq \mathcal{K} \leq I$. *Then there exists a unique DPP on* S *such that the correlation function is given by*

$$\rho^n(x_1, \dots, x_n) = \det_{1 \leq j,k \leq n} [K(x_j, x_k)], \quad n \in \mathbb{N}, \quad x_1, \dots, x_n \in S.$$

[Sos00] A. Soshnikov, Determinantal random point fields, Russian Math. Surveys 55 (2000) 923–975.

[ST03] T. Shirai and Y. Takahashi, Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point process, J. Funct. Anal. 205 (2003) 414–463.

- If $\mathcal{K} \in \mathcal{I}_{1,\mathrm{loc}}(S,\lambda)$ is a projection onto a closed subspace $H \subset L^2(S,\lambda)$, one has the DPP associated with K and λ, or one may say the DPP associated with the subspace H.

- This situation often appears in the setting of reproducing kernel Hilbert space [Aro50]. Let $\mathcal{F} = \mathcal{F}(S)$ be

- a Hilbert space of complex functions on S with inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}}$. A function $K(x,x')$ on $S \times S$ is said to be a reproducing kernel of $\mathcal{F}$ if

  1. For every $x' \in S$, the function $K(\cdot, x')$ belongs to $\mathcal{F}$.
  2. The function $K(x,x')$ has reproducing kernel property: for any $f \in \mathcal{F}$,

$$f(x') = \langle f(\cdot), K(\cdot, x') \rangle_{\mathcal{F}}.$$

- A reproducing kernel of $\mathcal{F}$ is unique if exists, and a reproducing kernel of $\mathcal{F}$ exists if and only if the point evaluation map $\mathcal{F} \ni f \mapsto f(x) \in \mathbb{C}$ is bounded for every $x \in S$.

- The Moore-Aronszajn theorem states that if a kernel $K(\cdot,\cdot)$ on $S \times S$ is positive definite in the sense that for any $n \geq 1$, $x_1, \dots, x_n \in S$, the matrix $(K(x_j, x_k))_{j,k \in \{1,\dots,n\}}$ is positive definite, then there exists a unique Hilbert space H_K of functions with inner product in which $K(x,x')$ is a reproducing kernel [Aro50]. If H_K is realized in $L^2(S,\lambda)$ for some measure λ, the kernel $K(x,x')$ defines a projection onto H_K.

[Aro50] N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc., 68 (1950) 337–404.

- **In the present talk, we consider the case that**

$$\boxed{\mathcal{K}f = f \quad \textbf{for all } f \in (\ker \mathcal{K})^{\perp} \subset L^2(S, \lambda),}$$

where $(\ker \mathcal{K})^{\perp}$ denotes the orthogonal complement of the kernel of $\mathcal{K}$.

- **That is, $\mathcal{K}$ is an orthogonal projection.**
- **By definition, it is obvious that the condition $O \leq \mathcal{K} \leq I$ is satisfied.**

- **The purpose of the present talk is to propose a useful method to provide orthogonal projections $\mathcal{K}$ and DPPs whose correlation kernels are given by the Hermitian integral kernels $K(x, x'), x, x' \in S$ of $\mathcal{K}$.**

- **As examples of DPPs constructed by this method, seven kinds of DPPs on a torus are introduced using the R_N-theta functions of Rosengren and Schlosser [RS06] for the seven irreducible reduced affine root systems.**

- **In the bulk scaling limit, they are degenerated into the three types of Ginibre DPPs on a complex plane with an infinite number of points.**

[RS06] H. Rosengren and M. Schlosser, Elliptic determinant evaluations and the Macdonald identities for affine root systems, Compositio Math. 142 (2006) 937–961.

2. Partial Isometry and DPPs

- **First we recall the notion of partial isometries between Hilbert spaces.**
- **Let** $H_\ell, \ell = 1, 2$ **be separable Hilbert spaces with inner products** $\langle \cdot, \cdot \rangle_{H_\ell}$**. For a bounded linear operator** $\mathcal{W} : H_1 \to H_2$**, the adjoint of** $\mathcal{W}$ **is defined as the operator** $\mathcal{W}^* : H_2 \to H_1$**, such that**

$$\langle \mathcal{W} f, g \rangle_{H_2} = \langle f, \mathcal{W}^* g \rangle_{H_1} \quad \text{for all } f \in H_1 \text{ and } g \in H_2.$$

A linear operator $\mathcal{W}$ **is called an isometry if**

$$||\mathcal{W} f||_{H_2} = ||f||_{H_1} \quad \text{for all } f \in H_1.$$

- **For** $\mathcal{W}$ **its kernel is denoted as** $\ker \mathcal{W}$ **and the orthogonal complement of** $\ker \mathcal{W}$ **is written as** $(\ker \mathcal{W})^\perp$**.**
- **A linear operator** $\mathcal{W}$ **is called a partial isometry, if**

$$||\mathcal{W} f||_{H_2} = ||f||_{H_1} \quad \text{for all } f \in (\ker \mathcal{W})^\perp.$$

- **For the partial isometry $\mathcal{W}$, $(\ker \mathcal{W})^{\perp}$ is called the initial space and the range of $\mathcal{W}$, $\mathrm{ran}\mathcal{W}$, is called the final space.**

- **By the definition, $||\mathcal{W}f||^2_{H_2} = \langle \mathcal{W}f, \mathcal{W}f\rangle_{H_2} = \langle f, \mathcal{W}^*\mathcal{W}f\rangle_{H_1}$.**

- **This implies the following.**

Lemma 2.1 *The bounded linear operator $\mathcal{W}$ (resp. $\mathcal{W}^*$) is a partial isometry if and only if $\mathcal{W}^*\mathcal{W}$ (resp. $\mathcal{W}\mathcal{W}^*$) is the identity on $(\ker \mathcal{W})^{\perp}$ (resp. $(ker\mathcal{W}^*)^{\perp}$).*

- We put the first assumption.

Assumption 1 Both $\mathcal{W}$ and $\mathcal{W}^*$ are partial isometries.

- Under Assumption 1, the operator $\mathcal{W}^*\mathcal{W}$ (resp. $\mathcal{W}\mathcal{W}^*$) is the projection onto the initial space of $\mathcal{W}$ (resp. the final space of $\mathcal{W}$).

- Now we assume that H_1 and H_2 are realized as L^2-spaces, $L^2(S_1, \lambda_1)$ and $L^2(S_2, \lambda_2)$, respectively.

- We consider the case in which $\mathcal{W}$ admits an integral kernel $W : S_2 \times S_1 \to \mathbb{C}$ such that

$$(\mathcal{W}f)(y) = \int_{S_1} W(y,x) f(x) \lambda_1(dx), \quad f \in L^2(S_1, \lambda_1),$$

and then

$$(\mathcal{W}^* g)(x) = \int_{S_2} \overline{W(y,x)} g(y) \lambda_2(dy), \quad g \in L^2(S_2, \lambda_2).$$

- **We put the second assumption.**

Assumption 2 $\mathcal{W}^*\mathcal{W} \in \mathcal{I}_{1,\mathrm{loc}}(S_1, \lambda_1)$ **and** $\mathcal{W}\mathcal{W}^* \in \mathcal{I}_{1,\mathrm{loc}}(S_2, \lambda_2)$**.**

- **We have**

$$(\mathcal{W}^*\mathcal{W}f)(x) = \int_{S_1} K_{S_1}(x, x') f(x') \lambda_1(dx'), \quad f \in L^2(S_1, \lambda_1),$$

$$(\mathcal{W}\mathcal{W}^*g)(y) = \int_{S_2} K_{S_2}(y, y') g(y') \lambda_2(dy'), \quad g \in L^2(S_2, \lambda_2),$$

with the integral kernels,

$$K_{S_1}(x, x') = \int_{S_2} \overline{W(y, x)} W(y, x') \lambda_2(dy) = \langle W(\cdot, x'), W(\cdot, x) \rangle_{L^2(S_2, \lambda_2)},$$

$$K_{S_2}(y, y') = \int_{S_1} W(y, x) \overline{W(y', x)} \lambda_1(dx) = \langle W(y, \cdot), W(y', \cdot) \rangle_{L^2(S_1, \lambda_1)}.$$

- **We see that** $\overline{K_{S_1}(x', x)} = K_{S_1}(x, x')$ **and** $\overline{K_{S_2}(y', y)} = K_{S_2}(y, y')$**.**

- **The main theorem is the following.**

Theorem 2.2 *Under Assumptions 1 and 2, associated with* $\mathcal{W}^*\mathcal{W}$ *and* $\mathcal{W}\mathcal{W}^*$, *there exists a unique pair of DPPs;* $(\Xi_1, K_{S_1}, \lambda_1(dx))$ *on* S_1 *and* $(\Xi_2, K_{S_2}, \lambda_2(dy))$ *on* S_2. *The correlation kernels* $K_{S_\ell}, \ell = 1, 2$ *are Hermitian and given by*

$$K_{S_1}(x, x') = \int_{S_2} \overline{W(y, x)} W(y, x') \lambda_2(dy) = \langle W(\cdot, x'), W(\cdot, x) \rangle_{L^2(S_2, \lambda_2)},$$

$$K_{S_2}(y, y') = \int_{S_1} W(y, x) \overline{W(y', x)} \lambda_1(dx) = \langle W(y, \cdot), W(y', \cdot) \rangle_{L^2(S_1, \lambda_1)}.$$

3. Orthonormal Functions and Correlation Kernels

- **In addition to $L^2(S_\ell, \lambda_\ell)$, $\ell = 1, 2$, we introduce $L^2(\Gamma, \nu)$ as a parameter space for functions in $L^2(S_\ell, \lambda_\ell), \ell = 1, 2$.**

- **Assume that there are two families of measurable functions $\{\psi_1(x, \gamma) : x \in S_1, \gamma \in \Gamma\}$ and $\{\psi_2(y, \gamma) : y \in S_2, \gamma \in \Gamma\}$ such that two bounded operators $\mathcal{U}_\ell : L^2(S_\ell, \lambda_\ell) \to L^2(\Gamma, \nu)$ given by**

$$\widehat{f}(\gamma) = (\mathcal{U}_\ell f)(\gamma) := \int_{S_\ell} \overline{\psi_\ell(x, \gamma)} f(x) \lambda_\ell(dx), \quad \ell = 1, 2,$$

are well-defined. Then, their adjoints $\mathcal{U}_\ell^* : L^2(\Gamma, \nu) \to L^2(S_\ell, \lambda_\ell), \ell = 1, 2$ are given by

$$(\mathcal{U}_\ell^* F)(\cdot) = \int_\Gamma \psi_\ell(\cdot, \gamma) F(\gamma) \nu(d\gamma).$$

- **Now we define $\mathcal{W} : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$ by $\mathcal{W} = \mathcal{U}_2^* \mathcal{U}_1$, i.e.,**

$$(\mathcal{W} f)(y) = \int_\Gamma \psi_2(y, \gamma) \widehat{f}(\gamma) \nu(d\gamma).$$

- **We can see the following.**

Lemma 3.1 *If*

$$\mathcal{U}_\ell \mathcal{U}_\ell^* = I_\Gamma \quad \textit{for } \ell = 1, 2,$$

then both $\mathcal{W}$ *and* $\mathcal{W}^*$ *are partial isometries.*

Proof **It suffices to show that** $\mathcal{W}^*\mathcal{W}$ **is an orthogonal projection, or equivalently, it suffices to show** $(\mathcal{W}^*\mathcal{W})^2 = \mathcal{W}^*\mathcal{W}$ **since** $\mathcal{W}^*\mathcal{W}$ **is self-adjoint.**
By the assumption, we see that

$$\mathcal{W}^*\mathcal{W} = (\mathcal{U}_2^*\mathcal{U}_1)^*\mathcal{U}_2^*\mathcal{U}_1 = \mathcal{U}_1^*(\mathcal{U}_2\mathcal{U}_2^*)\mathcal{U}_1 = \mathcal{U}_1^*\mathcal{U}_1.$$

Hence, $(\mathcal{W}^*\mathcal{W})^2 = \mathcal{U}_1^*\mathcal{U}_1\mathcal{U}_1^*\mathcal{U}_1 = \mathcal{U}_1^*\mathcal{U}_1 = \mathcal{W}^*\mathcal{W}$.
By symmetry, the assertion for $\mathcal{W}^*$ **also follows.** ∎

- **We note from the proof that $\mathcal{W}^*\mathcal{W} = \mathcal{U}_1^*\mathcal{U}_1$ and $\mathcal{W}\mathcal{W}^* = \mathcal{U}_2^*\mathcal{U}_2$ so that $\mathcal{U}_\ell, \ell = 1, 2$ are partial isometries.**

Assumption 3 **We assume that $\mathcal{U}_\ell \mathcal{U}_\ell^* = I_\Gamma$ for $\ell = 1, 2$.**

Assumption 3 can be rephrased as the following orthonormality relations:

$$\langle \psi_\ell(\cdot, \gamma), \psi_\ell(\cdot, \gamma') \rangle_{L^2(S_\ell, \lambda_\ell)} \nu(d\gamma) = \delta(\gamma - \gamma') d\gamma, \quad \gamma, \gamma' \in \Gamma, \quad \ell = 1, 2.$$

We will use these relations below.

The following is immediately obtained as a corollary of Theorem 2.2.

Corollary 3.2 *Let $\mathcal{W} = \mathcal{U}_2^*\mathcal{U}_1$ as in the above. We assume Assumption 3 in addition to Assumption 2. Then, there exist a unique pair of DPPs;* $(\Xi_1, K_{S_1}, \lambda_1(dx))$ *on* S_1 *and* $(\Xi_2, K_{S_2}, \lambda_2(dy))$ *on* S_2*. Here the correlation kernels* $K_{S_\ell}, \ell = 1, 2$ *are given by*

$$K_{S_1}(x, x') = \int_\Gamma \psi_1(x, \gamma) \overline{\psi_1(x', \gamma)} \nu(d\gamma) = \langle \psi_1(x, \cdot), \psi_1(x', \cdot) \rangle_{L^2(\Gamma, \nu)},$$
$$K_{S_2}(y, y') = \int_\Gamma \psi_2(y, \gamma) \overline{\psi_2(y', \gamma)} \nu(d\gamma) = \langle \psi_2(y, \cdot), \psi_2(y', \cdot) \rangle_{L^2(\Gamma, \nu)}.$$

4. Seven Finite DPPs on a Torus

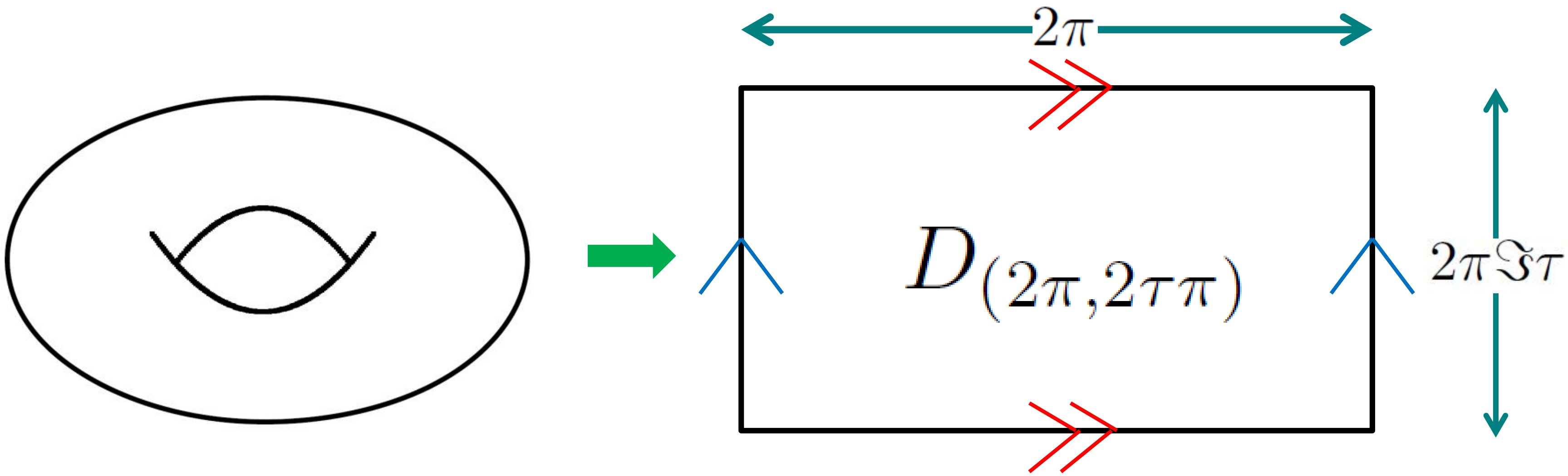
- **We will consider the finite DPPs on a surface of torus with double periodicity** $2\omega_1 = 2\pi$, $2\omega_3 = 2\tau\pi$ **with**

$$\tau = i\Im\tau \in \mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}, \quad i := \sqrt{-1}.$$

- **The surface of such a torus** $\mathbb{T}^2 = \mathbb{T}^2(2\pi, 2\tau\pi) := \mathbb{S}^1(2\pi) \times \mathbb{S}^1(2\pi\Im\tau)$ **can be identified with a rectangular domain in** $\mathbb{C}$**,**

$$D_{(2\pi,2\tau\pi)} = \{z \in \mathbb{C} : 0 \leq \Re z < 2\pi, 0 \leq \Im z < 2\pi\Im\tau\} \subset \mathbb{C} \quad \text{with double periodicity } (2\pi, 2\tau\pi).$$

So we first consider the systems on $D_{(2\pi,2\tau\pi)}$.



- **Let $S = \mathbb{C}$. For $x \in \mathbb{C}$, we write $x_{\mathrm{R}} := \Re x$, $x_{\mathrm{I}} := \Im x$.**
- **The background measure is given by**

$$\lambda(dx) = \mathbf{1}_{D_{(2\pi,2\tau\pi)}}(x)dx = \begin{cases} dx_{\mathrm{R}}dx_{\mathrm{I}}, & (x = x_{\mathrm{R}} + ix_{\mathrm{I}} \in D_{(2\pi,2\tau\pi)}), \\ 0, & (x \notin D_{(2\pi,2\tau\pi)}). \end{cases}$$

Let

$$z = e^{v\pi i}, \quad q = e^{\tau\pi i},$$

for $v \in \mathbb{C}$ **and** $\tau \in \mathbb{H}$. **The Jacobi theta functions are defined as follows,**

$$\vartheta_0(v;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} z^{2n} = 1 + 2\sum_{n=1}^{\infty} (-1)^n e^{\tau\pi i n^2} \cos(2n\pi v),$$

$$\vartheta_1(v;\tau) = i\sum_{n\in\mathbb{Z}} (-1)^n q^{(n-1/2)^2} z^{2n-1} = 2\sum_{n=1}^{\infty} (-1)^{n-1} e^{\tau\pi i(n-1/2)^2} \sin\{(2n-1)\pi v\},$$

$$\vartheta_2(v;\tau) = \sum_{n\in\mathbb{Z}} q^{(n-1/2)^2} z^{2n-1} = 2\sum_{n=1}^{\infty} e^{\tau\pi i(n-1/2)^2} \cos\{(2n-1)\pi v\},$$

$$\vartheta_3(v;\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} z^{2n} = 1 + 2\sum_{n=1}^{\infty} e^{\tau\pi i n^2} \cos(2n\pi v).$$

- **We define the following four types of functions;**

$$\begin{aligned}
\Theta^A(\sigma,z,\tau) &= e^{2\pi i\sigma z}\vartheta_2(\sigma\tau+z;\tau),\\
\Theta^B(\sigma,z,\tau) &= e^{2\pi i\sigma z}\vartheta_1(\sigma\tau+z;\tau) - e^{-2\pi i\sigma z}\vartheta_1(\sigma\tau-z;\tau),\\
\Theta^C(\sigma,z,\tau) &= e^{2\pi i\sigma z}\vartheta_2(\sigma\tau+z;\tau) - e^{-2\pi i\sigma z}\vartheta_2(\sigma\tau-z;\tau),\\
\Theta^D(\sigma,z,\tau) &= e^{2\pi i\sigma z}\vartheta_2(\sigma\tau+z;\tau) + e^{-2\pi i\sigma z}\vartheta_2(\sigma\tau-z;\tau),
\end{aligned}$$

for $\sigma \in \mathbb{R}, z \in \mathbb{C}, \tau \in \mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$**.**

- **We consider the seven types of irreducible reduced affine root systems** $R_N = A_{N-1}$, B_N, $B_N^\vee$, C_N, $C_N^\vee$, BC_N, D_N, $N \in \mathbb{N}$.

- **The following seven functions are essentially equal to the** R_N**-theta functions of Rosengren and Schlosser [RS06],**
for $N \in \mathbb{N}$,

$$\varphi_n^{R_N,(2\pi,2\tau\pi)}(x) = \frac{e^{-\mathcal{N}^{R_N} i x_{\mathrm{I}}^2/(4\tau\pi)}}{\sqrt{h_n^{R_N}(\tau)}} \Theta^{\sharp(R_N)}\left(\frac{J^{R_N}(n)}{\mathcal{N}^{R_N}}, \mathcal{N}^{R_N}\frac{x}{2\pi}, \mathcal{N}^{R_N}\tau\right), \quad n \in \{1, \dots, N\}.$$

[RS06] H. Rosengren and M. Schlosser, Elliptic determinant evaluations and the Macdonald identities for affine root systems, Compositio Math. 142 (2006) 937–961.

- **For $N \in \mathbb{N}$, let**

$$\varphi_n^{R_N,(2\pi,2\tau\pi)}(x) = \frac{e^{-\mathcal{N}^{R_N} i x_{\mathrm{I}}^2/(4\tau\pi)}}{\sqrt{h_n^{R_N}(\tau)}} \Theta^{\sharp(R_N)}\left(\frac{J^{R_N}(n)}{\mathcal{N}^{R_N}}, \mathcal{N}^{R_N}\frac{x}{2\pi}, \mathcal{N}^{R_N}\tau\right), \quad n \in \{1,\dots,N\},$$

where

$$\sharp(R_N) = \begin{cases} A, & \textbf{if } R_N = A_{N-1}, \\ B, & \textbf{if } R_N = B_N, B_N^\vee, \\ C, & \textbf{if } R_N = C_N, C_N^\vee, BC_N, \\ D, & \textbf{if } R_N = D_N, \end{cases} \qquad \mathcal{N}^{R_N} = \begin{cases} N, & R_N = A_{N-1}, \\ 2N-1, & R_N = B_N, \\ 2N, & R_N = B_N^\vee, C_N^\vee, \\ 2(N+1), & R_N = C_N, \\ 2N+1, & R_N = BC_N, \\ 2(N-1), & R_N = D_N, \end{cases}$$

$$J^{R_N}(n) = \begin{cases} n - 1/2, & R_N = A_{N-1}, C_N^\vee, \\ n-1, & R_N = B_N, B_N^\vee, D_N, \\ n, & R_N = C_N, BC_N, \end{cases}$$

and we set

$$h_n^{A_{N-1}}(\tau) = 4\pi^2 \sqrt{\frac{\Im\tau}{2\mathcal{N}^{A_{N-1}}}} e^{-2\tau\pi i J^{A_{N-1}}(n)^2/\mathcal{N}^{A_{N-1}}}, \quad n \in \{1,\dots,N^{A_{N-1}}\},$$

$$h_n^{R_N}(\tau) = 8\pi^2 \sqrt{\frac{\Im\tau}{2\mathcal{N}^{R_N}}} e^{-2\tau\pi i J^{R_N}(n)^2/\mathcal{N}^{R_N}}, \quad n \in \{1,\dots,N\}, \quad \textbf{for } R_N = C_N, C_N^\vee, BC_N,$$

$$h_n^{R_N}(\tau) = \begin{cases} 16\pi^2 \sqrt{\frac{\Im\tau}{2\mathcal{N}^{R_N}}} e^{-2\tau\pi i J^{R_N}(n)^2/\mathcal{N}^{R_N}}, & n = 1, \\ 8\pi^2 \sqrt{\frac{\Im\tau}{2\mathcal{N}^{R_N}}} e^{-2\tau\pi i J^{R_N}(n)^2/\mathcal{N}^{R_N}}, & n \in \{2,3,\dots,N\}, \end{cases} \qquad \textbf{for } R_N = B_N, B_N^\vee,$$

$$h_n^{D_N}(\tau) = \begin{cases} 16\pi^2 \sqrt{\frac{\Im\tau}{2\mathcal{N}^{R_N}}} e^{-2\tau\pi i J^{D_N}(n)^2/\mathcal{N}^{D_N}}, & n \in \{1,N\}, \\ 8\pi^2 \sqrt{\frac{\Im\tau}{2\mathcal{N}^{R_N}}} e^{-2\tau\pi i J^{D_N}(n)^2/\mathcal{N}^{D_N}}, & n \in \{2,3,\dots,N-1\}. \end{cases}$$

- **For $N \in \mathbb{N}$, let**

$$\varphi_n^{R_N,(2\pi,2\tau\pi)}(x) = \frac{e^{-\mathcal{N}^{R_N} i x_{\mathrm{I}}^2/(4\tau\pi)}}{\sqrt{h_n^{R_N}(\tau)}} \Theta^{\sharp(R_N)}\left(\frac{J^{R_N}(n)}{\mathcal{N}^{R_N}}, \mathcal{N}^{R_N}\frac{x}{2\pi}, \mathcal{N}^{R_N}\tau\right), \quad n \in \{1, \dots, N\}.$$

Then we can prove the following.

Lemma 4.1 *Let $N \in \mathbb{N}$. Then, if $n, m \in \Gamma := \{1, \dots, N\}$, the following orthonormality relations are established,*

$$\begin{aligned}
&\langle \varphi_n^{R_N,(2\pi,2\tau\pi)}, \varphi_m^{R_N,(2\pi,2\tau\pi)} \rangle_{L^2(\mathbb{C}, \mathbf{1}_{D_{(2\pi,2\tau\pi)}}(x)dx)} \\
&= \int_{D_{(2\pi,2\tau\pi)}} \varphi_n^{R_N,(2\pi,2\tau\pi)}(x) \varphi_m^{R_N,(2\pi,2\tau\pi)}(x) dx \\
&= \int_0^{2\pi} dx_{\mathrm{R}} \int_0^{2\tau\pi} dx_{\mathrm{I}} \frac{e^{-\mathcal{N}^{R_N} x_{\mathrm{I}}^2/(2\pi\Im\tau)}}{\sqrt{h_n^{R_N}(\tau) h_m^{R_N}(\tau)}} \\
&\quad \times \Theta^{\sharp(R_N)}\left(\frac{J^{R_N}(n)}{\mathcal{N}^{R_N}}, \mathcal{N}^{R_N}\frac{x}{2\pi}, \mathcal{N}^{R_N}\tau\right) \Theta^{\sharp(R_N)}\left(\frac{J^{R_N}(m)}{\mathcal{N}^{R_N}}, \mathcal{N}^{R_N}\frac{x}{2\pi}, \mathcal{N}^{R_N}\tau\right) \\
&= \delta_{nm},
\end{aligned}$$

for $R_N = A_{N-1}, B_N, B_N^{\vee}, C_N, C_N^{\vee}, BC_N, D_N$.

- **Then Corollary 3.2 (prepared for a pair of orthonormal functions) can be applied to the above seven sets of orthonormal functions with a discrete parameter space $\Gamma := \{1, \dots, N\}, N \in \mathbb{N}$.**

- **For $N \in \mathbb{N}$, let**

$$\varphi_n^{R_N,(2\pi,2\tau\pi)}(x) = \frac{e^{-\mathcal{N}^{R_N} i x_{\mathrm{I}}^2/(4\tau\pi)}}{\sqrt{h_n^{R_N}(\tau)}} \Theta^{\sharp(R_N)}\left(\frac{J^{R_N}(n)}{\mathcal{N}^{R_N}}, \mathcal{N}^{R_N}\frac{x}{2\pi}, \mathcal{N}^{R_N}\tau\right), \quad n \in \{1,\dots,N\}.$$

Then we can prove the following.

Lemma 4.1 *Let $N \in \mathbb{N}$. Then, if $n,m \in \Gamma := \{1,\dots,N\}$, the following orthonormality relations are established,*

$$\begin{aligned}
&\langle \varphi_n^{R_N,(2\pi,2\tau\pi)}, \varphi_m^{R_N,(2\pi,2\tau\pi)} \rangle_{L^2(\mathbb{C}, \mathbf{1}_{D_{(2\pi,2\tau\pi)}}(x)dx)} \\
&= \int_{D_{(2\pi,2\tau\pi)}} \varphi_n^{R_N,(2\pi,2\tau\pi)}(x)\varphi_m^{R_N,(2\pi,2\tau\pi)}(x)dx \\
&= \int_0^{2\pi} dx_{\mathrm{R}} \int_0^{2\tau\pi} dx_{\mathrm{I}} \frac{e^{-\mathcal{N}^{R_N} x_{\mathrm{I}}^2/(2\pi\Im\tau)}}{\sqrt{h_n^{R_N}(\tau)h_m^{R_N}(\tau)}} \\
&\qquad \times \Theta^{\sharp(R_N)}\left(\frac{J^{R_N}(n)}{\mathcal{N}^{R_N}}, \mathcal{N}^{R_N}\frac{x}{2\pi}, \mathcal{N}^{R_N}\tau\right) \Theta^{\sharp(R_N)}\left(\frac{J^{R_N}(m)}{\mathcal{N}^{R_N}}, \mathcal{N}^{R_N}\frac{x}{2\pi}, \mathcal{N}^{R_N}\tau\right) \\
&= \delta_{nm},
\end{aligned}$$

for $R_N = A_{N-1}, B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N$.

- **Then Corollary 3.2 (prepared for a pair of orthonormal functions) can be applied to the above seven sets of orthonormal functions with a discrete parameter space $\Gamma := \{1,\dots,N\}, N \in \mathbb{N}$.**

- **We obtain the seven types of DPPs with the correlation kernels,**

$$K^{R_N,(2\pi,2\tau\pi)}(x,x') = \sum_{n=1}^{N} \varphi_n^{R_N,(2\pi,2\tau\pi)}(x)\overline{\varphi_n^{R_N,(2\pi,2\tau\pi)}(x')},$$

with respect to the measure $\lambda(dx) = \mathbf{1}_{D_{(2\pi,2\tau\pi)}} dx$ **on** $\mathbb{C}$ **for** $R_N = A_{N-1}$, B_N, $B_N^{\vee}$, C_N, $C_N^{\vee}$, BC_N, D_N.

- **Using the quasi-periodicity of the Jacobi theta functions, we can show that the correlation kernels are quasi-double-periodic as,**

$$K^{R_N,(2\pi,2\tau\pi)}(x+2\pi,x') = K^{R_N,(2\pi,2\tau\pi)}(x,x'+2\pi)$$
$$= \begin{cases} (-1)^{\mathcal{N}^{A_{N-1}}} K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = A_{N-1}, \\ -K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = B_N, C_N^{\vee}, BC_N, \\ K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = B_N^{\vee}, C_N, D_N, \end{cases}$$

$$K^{R_N,(2\pi,2\tau\pi)}(x+2\tau\pi,x') = \begin{cases} e^{-\mathcal{N}^{R_N} i x_{\mathrm{R}}} K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = A_{N-1}, C_N, C_N^{\vee}, BC_N, D_N, \\ -e^{-\mathcal{N}^{R_N} i x_{\mathrm{R}}} K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = B_N, B_N^{\vee}, \end{cases}$$

$$K^{R_N,(2\pi,2\tau\pi)}(x,x'+2\tau\pi) = \begin{cases} e^{\mathcal{N}^{R_N} i x'_{\mathrm{R}}} K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = A_{N-1}, C_N, C_N^{\vee}, BC_N, D_N, \\ -e^{\mathcal{N}^{R_N} i x'_{\mathrm{R}}} K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = B_N, B_N^{\vee}. \end{cases}$$

- **The above implies the following double periodicity (up to an irrelevant gauge transformation),**

$$\mathcal{S}_{2\pi} K^{R_N,(2\pi,2\tau\pi)}(x,x') = \frac{e^{\mathcal{N}^{R_N} i x_{\mathrm{R}}}}{e^{\mathcal{N}^{R_N} i x'_{\mathrm{R}}}} \mathcal{S}_{2\tau\pi} K^{R_N,(2\pi,2\tau\pi)}(x,x')$$
$$= K^{R_N,(2\pi,2\tau\pi)}(x,x'), \quad x, x' \in D_{(2\pi,2\tau\pi)},$$

where $\mathcal{S}_u$ **denotes a shift by** u**: for** $u \in \mathbb{C}$**,** $\mathcal{S}_u \Xi := \sum_j \delta_{x_j+u}$**,**

$$\mathcal{S}_u K(x,x') = K(x+u, x'+u),$$

and $\mathcal{S}_u \lambda(dx) = \lambda(u+dx)$**.**

- **In other words, we have obtained the seven types of DPPs with a finite number of points** N **on a surface of torus** $\mathbb{T}^2(2\pi, 2\tau\pi)$**.**

- **Hence here we write them as** $\left(\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx\right)$**,** $R_N = A_{N-1}$**,** B_N**,** $B_N^\vee$**,** C_N**,** $C_N^\vee$**,** BC_N**,** D_N**.**

Using the Macdonald denominator formulas given by Rosengren and Schlosser [RS06], the probability densities for these finite DPPs with respect to the Lebesgue measures, $d\boldsymbol{x} = \prod_{j=1}^{N} dx_j$ are given as follows;

$$\mathbf{p}^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}(\boldsymbol{x}) = \frac{1}{Z^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}} \exp\left(-\frac{\mathcal{N}^{A_{N-1}}}{2\pi\Im\tau} \sum_{j=1}^{N} (x_j)_{\mathrm{I}}^2 \right) \times \begin{cases} \left| \vartheta_0\left(\sum_{k=1}^{N} \frac{x_k}{2\pi}; \tau \right) W^{A_{N-1}}\left(\frac{\boldsymbol{x}}{2\pi}; \tau \right) \right|^2, & \textbf{if } N \textbf{ is even}, \\ \left| \vartheta_3\left(\sum_{k=1}^{N} \frac{x_k}{2\pi}; \tau \right) W^{A_{N-1}}\left(\frac{\boldsymbol{x}}{2\pi}; \tau \right) \right|^2, & \textbf{if } N \textbf{ is odd}, \end{cases}$$

$$\mathbf{p}^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(x) = \frac{1}{Z^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}} \exp\left(-\frac{\mathcal{N}^{R_N}}{2\pi\Im\tau} \sum_{j=1}^{N} (x_j)_{\mathrm{I}}^2 \right) \left| W^{R_N}\left(\frac{\boldsymbol{x}}{2\pi}; \tau \right) \right|^2,$$

$$R_N = B_N, B_N^{\vee}, C_N, C_N^{\vee}, BC_N, D_N,$$

for $\boldsymbol{x} \in (\mathbb{T}^2(2\pi, 2\tau\pi))^N$, where W^{R_N} are the Macdonald denominators given as follows and $Z^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}$ are normalization constants.

- **For $\tau \in \mathbb{H}$, the Macdonald denominators are given as**

$$W^{A_{N-1}}(\boldsymbol{z};\tau) = \prod_{1 \le j < k \le N} \vartheta_1(z_k - z_j;\tau),$$

$$W^{B_N}(\boldsymbol{z};\tau) = \prod_{\ell=1}^{N} \vartheta_1(z_\ell;\tau) \prod_{1 \le j < k \le N} \Big\{\vartheta_1(z_k - z_j;\tau)\vartheta_1(z_k + z_j;\tau)\Big\},$$

$$W^{B_N^\vee}(\boldsymbol{z};\tau) = \prod_{\ell=1}^{N} \vartheta_1(2z_\ell;2\tau) \prod_{1 \le j < k \le N} \Big\{\vartheta_1(z_k - z_j;\tau)\vartheta_1(z_k + z_j;\tau)\Big\},$$

$$W^{C_N}(\boldsymbol{z};\tau) = \prod_{\ell=1}^{N} \vartheta_1(2z_\ell;\tau) \prod_{1 \le j < k \le N} \Big\{\vartheta_1(z_k - z_j;\tau)\vartheta_1(z_k + z_j;\tau)\Big\},$$

$$W^{C_N^\vee}(\boldsymbol{z};\tau) = \prod_{\ell=1}^{N} \vartheta_1\left(z_\ell;\frac{\tau}{2}\right) \prod_{1 \le j < k \le N} \Big\{\vartheta_1(z_k - z_j;\tau)\vartheta_1(z_k + z_j;\tau)\Big\},$$

$$W^{BC_N}(\boldsymbol{z};\tau) = \prod_{\ell=1}^{N} \Big\{\vartheta_1(z_\ell;\tau)\vartheta_0(2z_\ell;2\tau)\Big\} \prod_{1 \le j < k \le N} \Big\{\vartheta_1(z_k - z_j;\tau)\vartheta_1(z_k + z_j;\tau)\Big\},$$

$$W^{D_N}(\boldsymbol{z};\tau) = \prod_{1 \le j < k \le N} \Big\{\vartheta_1(z_k - z_j;\tau)\vartheta_1(z_k + z_j;\tau)\Big\}.$$

5. Symmetry and Bulk Scaling Limits

- **We can prove the following symmetry properties for the present DPPs on $\mathbb{T}^2(2\pi, 2\tau\pi)$.**

Proposition 5.1 (i) *The finite DPPs* $\left(\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx\right)$ *with* $\tau = i\Im\tau \in \mathbb{H}$ *have the following shift invariance,*

$$\mathcal{S}_{2\pi/N}(\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx) \stackrel{\text{(law)}}{=} (\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx),$$

$$\mathcal{S}_{2\tau\pi/N}(\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx) \stackrel{\text{(law)}}{=} (\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx),$$

$$\mathcal{S}_{\pi}(\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx) \stackrel{\text{(law)}}{=} (\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx), \quad R_N = B_N^{\vee}, C_N, D_N,$$

$$\mathcal{S}_{\tau\pi}(\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx) \stackrel{\text{(law)}}{=} (\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx), \quad R_N = C_N, C_N^{\vee}, BC_N, D_N.$$

(ii) *The densities of points* $\rho^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(x)$ *given by* $K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(x,x)$ *have the following zeros,*

$$\rho^{B_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(0) = 0,$$

$$\rho^{B_N^{\vee}}_{\mathbb{T}^2(2\pi,2\tau\pi)}(0) = \rho^{B_N^{\vee}}_{\mathbb{T}^2(2\pi,2\tau\pi)}(\pi) = 0,$$

$$\rho^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(0) = \rho^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(\tau\pi) = 0, \quad R_N = C_N^{\vee}, BC_N,$$

$$\rho^{C_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(0) = \rho^{C_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(\pi) = \rho^{C_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(\tau\pi) = 0.$$

- **We note that the periodicities $2\pi/N \in \mathbb{R}$ and $2\tau\pi/N \in \mathbb{H}$ of $(\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx)$ shown by Proposition 5.1 (i) become zeros as $N \to \infty$.**

- **Hence, as the $N \to \infty$ limit of $(\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx)$, it is expected to obtain a uniform system of infinite number of points on $\mathbb{C}$.**

- **We introduce the following operation.**

  (Dilatation) **For $c > 0$, we set $c \circ \Xi := \sum_j \delta_{cx_j}$**

$$c \circ K(x, x') := K\left(\frac{x}{c}, \frac{x'}{c}\right), \quad x, x' \in cS,$$

  and $c \circ \lambda(dx) := \lambda(dx/c)$. We define $c \circ (\Xi, K, \lambda(dx)) := (c \circ \Xi, c \circ K, c \circ \lambda(dx))$.

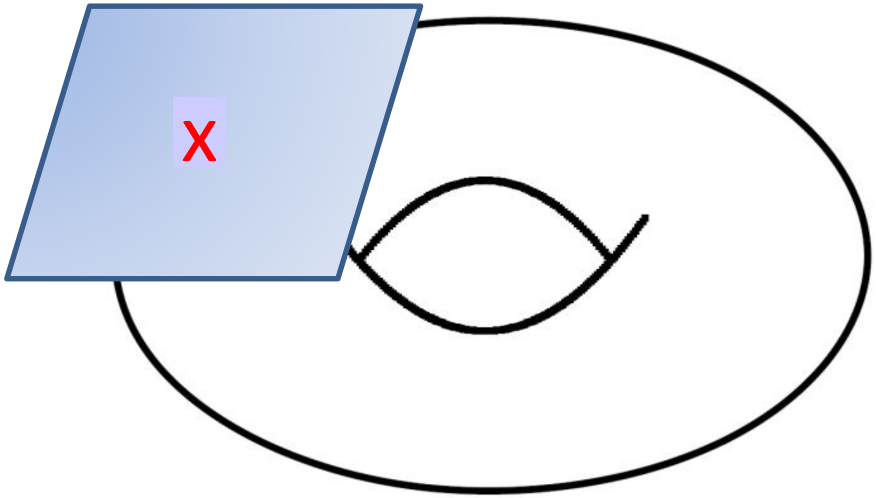
Proposition 5.2 *The following weak convergence is established,*

$$\frac{1}{2}\sqrt{\frac{N}{\pi \Im \tau}} \circ \left(\Xi, K_{\mathbb{T}^2(2\pi,2\tau\pi)}^{A_{N-1}}, dx\right) \overset{N\to\infty}{\Longrightarrow} \left(\Xi, K_{\rm Ginibre}^{A}, \lambda_{{\rm N}(0,1;\mathbb{C})}(dx)\right),$$

$$\sqrt{\frac{N}{2\pi \Im \tau}} \circ \left(\Xi, K_{\mathbb{T}^2(2\pi,2\tau\pi)}^{R_N}, dx\right) \overset{N\to\infty}{\Longrightarrow} \left(\Xi, K_{\rm Ginibre}^{C}, \lambda_{{\rm N}(0,1;\mathbb{C})}(dx)\right), \quad R_N = B_N, B_N^{\vee}, C_N, C_N^{\vee}, BC_N,$$

$$\sqrt{\frac{N}{2\pi \Im \tau}} \circ \left(\Xi, K_{\mathbb{T}^2(2\pi,2\tau\pi)}^{D_N}, dx\right) \overset{N\to\infty}{\Longrightarrow} \left(\Xi, K_{\rm Ginibre}^{D}, \lambda_{{\rm N}(0,1;\mathbb{C})}(dx)\right),$$

where the limit point processes are the three types of Ginibre DPPs given below.



Three types of Ginibre DPPs

- **The background measure is the complex normal distribution,**

$$\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) := \frac{1}{\pi} e^{-|x|^2} dx_{\mathrm{R}} dx_{\mathrm{I}}.$$

- **The correlation kernels are given by**

$$\begin{aligned}
K^{A}_{\mathrm{Ginibre}}(x, x') &= e^{x\overline{x'}}, \\
K^{C}_{\mathrm{Ginibre}}(x, x') &= \sinh(x\overline{x'}), \\
K^{D}_{\mathrm{Ginibre}}(x, x') &= \cosh(x\overline{x'}), \quad x, x' \in \mathbb{C}.
\end{aligned}$$

$$K^A_{\mathrm{Ginibre}}(x,x') = e^{x\overline{x'}}, \quad K^C_{\mathrm{Ginibre}}(x,x') = \sinh(x\overline{x'}), \quad K^D_{\mathrm{Ginibre}}(x,x') = \cosh(x\overline{x'}), \quad x, x' \in \mathbb{C},$$

$$\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) := \frac{1}{\pi} e^{-|x|^2} dx_{\mathrm{R}} dx_{\mathrm{I}}.$$

- **The DPP, $(\Xi, K^A_{\mathrm{Ginibre}}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx))$ describes the eigenvalue distribution of the Gaussian random complex matrix in the bulk scaling limit, which is called the complex Ginibre ensemble. This is uniform on $\mathbb{C}$ with the density**

$$\rho_{\mathrm{Ginibre}}(x)dx = K^A_{\mathrm{Ginibre}}(x,x)\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) = \frac{1}{\pi} dx_{\mathrm{R}} dx_{\mathrm{I}}, \quad x \in \mathbb{C}.$$

- **On the other hands, the Ginibre DPPs of types C and D are rotationally symmetric around the origin, but non-uniform on $\mathbb{C}$. The density profiles are given by**

$$\rho^C_{\mathrm{Ginibre}}(x)dx = K^C_{\mathrm{Ginibre}}(x,x)\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) = \frac{1}{2\pi}(1 - e^{-2|x|^2}) dx_{\mathrm{R}} dx_{\mathrm{I}}, \quad x \in \mathbb{C},$$
$$\rho^D_{\mathrm{Ginibre}}(x)dx = K^D_{\mathrm{Ginibre}}(x,x)\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) = \frac{1}{2\pi}(1 + e^{-2|x|^2}) dx_{\mathrm{R}} dx_{\mathrm{I}}, \quad x \in \mathbb{C}.$$

6. Concluding Remarks

- Among the present seven types of finite DPPs on a torus, the three types (A_{N-1}, C_N, D_N) were extended to the 2D exactly solvable one-component plasma models in [K19].

- Relationship to the Gaussian free field on a torus was also discussed for these three plasma models in [K19].

[K19] M. Katori, Two-dimensional elliptic determinantal point processes and related systems, Commun. Math. Phys. (2019). https://doi.org/10.1007/s00220-019-03351-5.

- **In [KS19+], we have demonstrated that the class of DPPs obtained by our method is large enough to study universal structures in a variety of DPPs by showing plenty of examples of DPPs in one-, two-, and higher-dimensional spaces S.**

- **There we have shown that the family of DPPs given by our method is a generalization of the class of DPPs called the Weyl–Heisenberg ensembles studied by Abreu *et al.***
  For $d \in \mathbb{N}$, let
$$S_1 = \mathbb{C}^d, \quad S_2 = \Gamma = \mathbb{R}^d,$$
  with the Lebesgue measures $\lambda_1(dx) = dx_{\rm R}dx_{\rm I}$, $\lambda_2(dy) = dy$, where $x = x_{\rm R} + ix_{\rm I}$ with $x_{\rm R}, x_{\rm I} \in \mathbb{R}^d$. Provided that $||G||^2_{L^2(\mathbb{R}^2, dx_{\rm R})} = 1$, we have
$$\begin{aligned}
(\mathcal{W}_{\rm WH}f)(y) &= \int_{\mathbb{C}^d} \overline{G(y-x_{\rm R})} e^{-2\pi i y\cdot x_{\rm I}} f(x_{\rm R}+ix_{\rm I}) dx_{\rm R}dx_{\rm I}, \quad f \in L^2(\mathbb{C}^d, dx_{\rm R}dx_{\rm I}), \\
(\mathcal{W}^*_{\rm WH}g)(x) &= \int_{\mathbb{R}^d} G(y-x_{\rm R}) e^{2\pi i y\cdot x_{\rm I}} g(y) dy, \quad g \in L^2(\mathbb{R}^d, dy), \\
K_{\rm WH}(x,x') &= \int_{\mathbb{R}^d} G(y-x_{\rm R}) \overline{G(y-x'_{\rm R})} e^{2\pi i y\cdot(x_{\rm I}-x'_{\rm I})} dy, \qquad (6.1)
\end{aligned}$$
  for $(x,x') = (x_{\rm R}+ix_{\rm I}, x'_{\rm R}+ix'_{\rm I}) \in \mathbb{C}^d \times \mathbb{C}^d$. The second formula in (6.1) is regarded as the short-time Fourier transform of $g \in L^2(\mathbb{R}^d, dy)$ with respect to a window function $G \in L^2(\mathbb{R}^d, dx_{\rm R})$ [Gröchenig 2001]. The formulas (6.1) define the Weyl–Heisenberg ensemble of DPP, $(\Xi, K_{\rm WH}, dx_{\rm R}dx_{\rm I})$, studied by Abreu *et al.*

[KS19+] M. Katori, T. Shirai, Partial isometries, duality, and determinantal point processes, arXiv: math.PR/1903.04945.

- **With $L^2(S,\lambda)$ and $L^2(\Gamma,\nu)$, we can consider the system of biorthonormal functions, which consists of a pair of distinct families of measurable functions $\{\psi(x,\gamma) : x \in S, \gamma \in \Gamma\}$ and $\{\varphi(x,\gamma) : x \in S, \gamma \in \Gamma\}$ satisfying the biorthonormality relations**

$$\langle \psi(\cdot,\gamma), \varphi(\cdot,\gamma') \rangle_{L^2(S,\lambda)} \nu(d\gamma) = \delta(\gamma - \gamma') d\gamma, \quad \gamma, \gamma' \in \Gamma. \tag{6.2}$$

If the integral kernel defined by

$$K^{\mathrm{bi}}(x,x') = \int_\Gamma \psi(x,\gamma) \overline{\varphi(x',\gamma)} \nu(d\gamma), \quad x, x' \in S, \tag{6.3}$$

is of finite rank, we can construct a finite DPP on S whose correlation kernel is given by (6.3) following a standard method of random matrix theory. By the biorthonormality (6.2), it is easy to verify that K^{bi} is a projection kernel, but it is not necessarily an orthogonal projection. This observation means that such a DPP is not constructed by the method reported here. Generalization of the present framework in order to cover such DPPs associated with biorthonormal systems is required. Moreover, the dynamical extensions of DPPs called determinantal processes shall be studied in the context of the present talk.

Thank you very much for your attention.

- **M. Katori, Two-dimensional elliptic determinantal point processes and related systems, Commun. Math. Phys. (2019). https://doi.org/10.1007/s00220-019-03351-5.**
- **M. Katori, T. Shirai, Partial isometries, duality, and determinantal point processes,** arXiv: math.PR/1903.04945.