Extensions of the Chaundy-Bullard Identity

星 奈津子
中央大学大学院理工学研究科
物理学専攻修士2年

共同研究者：香取眞理（中大），T. H. Koornwinder（アムステルダム大），
M. J. Schlosser（ウィーン大）

無限粒子系、確率場の諸問題XVI　2021年10月17日

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1. Chaundy-Bullard恒等式の紹介

Chaundy-Bullard恒等式[Chaundy T. W., Bullard J. E.(1960)]

非負の整数m,nに対して次が成り立つ.

$$1 = (1-x)^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k + x^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} (1-x)^k.$$

複数の証明方法のまとめ[Koornwinder T. H., Schlosser M. J.(2008)]

- 数学的帰納法
- 微分を繰り返す方法
- 母関数を使う方法
- 格子経路法
- ベータ積分に分ける方法

など.

Pochhammer記号(n,k∈{0,1,2,⋯})

$$(n)_k = \begin{cases} n(n+1)\cdots(n+k-1), & k \in \{1,2,\ldots\}, \\ 1, & k=0 \end{cases}$$

を用いて

$$1 = (1-x)^{n+1} \sum_{k=0}^{m} \frac{(n+1)_k}{k!} x^k + x^{m+1} \sum_{k=0}^{n} \frac{(m+1)_k}{k!} (1-x)^k$$

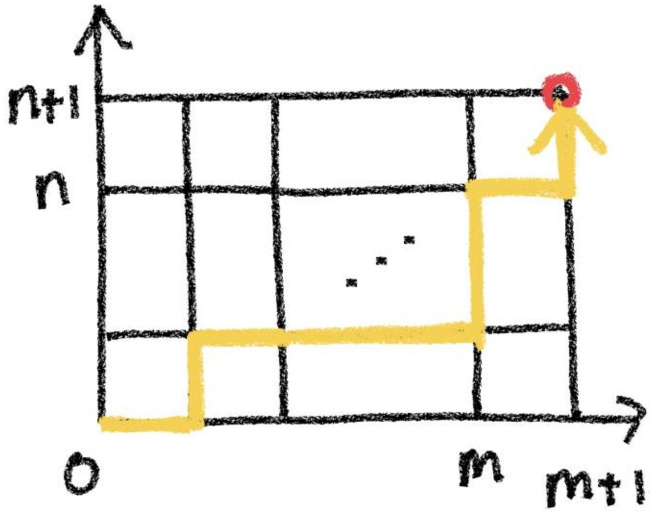
と書くこともできる.

格子経路法による証明

$m, n \in \{0,1,2,\cdots\}$とする.正方格子上の矩形領域

$\Lambda_{m+1,n+1} := \{(i,j) : i \in \{0,1,\cdots,m+1\}, j \in \{0,1,\cdots,n+1\}\}$

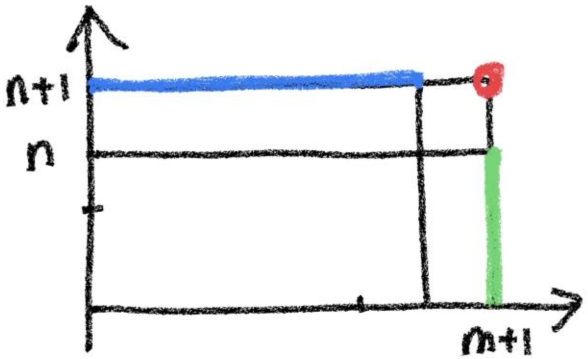
の中を右か上に進む経路を考える.



$\Pi_{m+1,n+1} : (0,0) \to (m+1, n+1)$ の経路の集合.

経路πはステップ $s := \begin{cases} (i,j) \to (i+1,j) & : s_{\rightarrow} \\ (i,j) \to (i,j+1) & : s_{\uparrow} \end{cases}$ の集合.

各ステップに重みw(s)を課す.

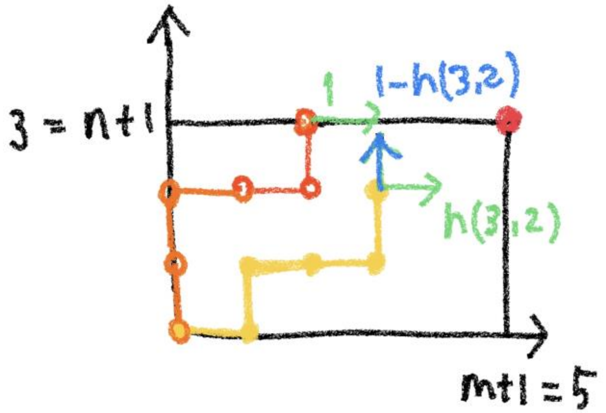


$$w(s_{\rightarrow}) := \begin{cases} h(i,j), & 0 \le i \le m,\ 0 \le j \le n \\ 1, & \underline{0 \le i \le m,\ j = n+1} \end{cases}$$

$$w(s_{\uparrow}) := \begin{cases} 1 - h(i,j), & 0 \le i \le m,\ 0 \le j \le n \\ 1, & \underline{i = m+1,\ 0 \le j \le n} \end{cases}$$

経路の重み　$w(\pi) := \prod_{s\in\pi} w(s)$ を考える.　　π_τ : τ ステップ目までの経路, $s_\tau(\pi_\tau)$: π_τ の τ ステップ目

例　m=4, n=2, τ =5



とりうる6ステップ目についての重みの和は

黄： $h(3,2) + (1-h(3,2)) = 1.$
橙： 1.

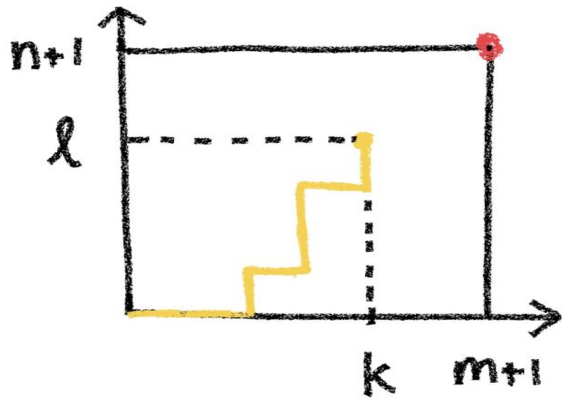
のように, どのような π_5 に対しても
必ず1になる.

$$\sum_{\pi_\tau:\pi_{\tau-1}} w(s_\tau(\pi_\tau)) = 1.$$

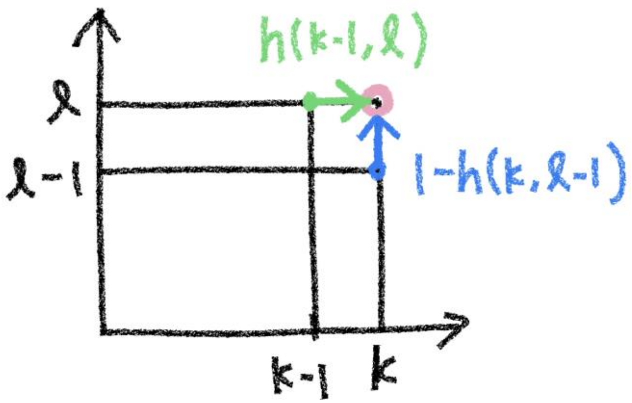
$$\sum_{\pi_\tau} w(\pi_\tau) = \sum_{\pi_{\tau-1}} \left(w(\pi_{\tau-1}) \sum_{\pi_\tau:\pi_{\tau-1}} w(s_\tau(\pi_\tau)) \right) = \sum_{\pi_{\tau-1}} w(\pi_{\tau-1}) = \cdots = 1 \text{ より } \sum_{\pi\in\Pi_{m+1,n+1}} w(\pi) = 1.$$

点 (k,ℓ) に到達するすべての経路の重みの和を

$$A(k,\ell) := \begin{cases} \displaystyle\sum_{\pi:(0,0)\to(k,\ell)} w(\pi), & (k,\ell) \in \Lambda_{m,n}\backslash\{(0,0)\}, \\ 1, & (k,\ell) = (0,0) \end{cases}$$

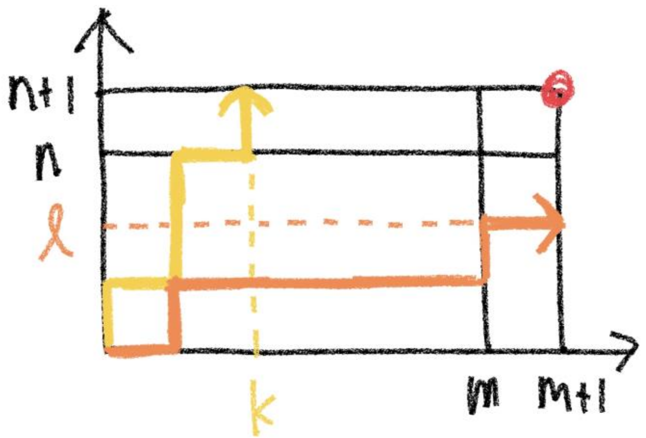


と書くことにする.



$$A(k,\ell) = \underline{h(k-1,\ell)A(k-1,\ell)} + \underline{(1-h(k,\ell-1))A(k,\ell-1)}$$

であるから



$$\begin{aligned} A(m+1,n+1) &= 1 \times A(m,n+1) + 1 \times A(m+1,n) \\ &= \sum_{k=0}^{m}(1-h(k,n))A(k,n) + \sum_{\ell=0}^{n} h(m,\ell)A(m,\ell) \\ &= \sum_{\pi \in \Pi_{m+1,n+1}} w(\pi) = 1. \end{aligned}$$

$$\sum_{k=0}^{m}(1-h(k,n))A(k,n) + \sum_{\ell=0}^{n} h(m,\ell)A(m,\ell) = 1 \ .$$

$$1=\sum_{k=0}^{m}(1-h(k,n))A(k,n)+\sum_{\ell=0}^{n}h(m,\ell)A(m,\ell) \qquad \cdots①$$

$$1=(1-x)^{n+1}\sum_{k=0}^{m}\binom{n+k}{k}x^k+x^{m+1}\sum_{\ell=0}^{n}\binom{m+\ell}{\ell}(1-x)^\ell. \qquad \cdots②$$ ：証明したい恒等式

各項が一致するようにh(i,j)を設定する.

$h(i,j)=x$ としたとき, $A(k,\ell)=h(k-1,\ell)A(k-1,\ell)+(1-h(k,\ell-1))A(k,\ell-1)$ より

$$A(k,\ell)=xA(k-1,\ell)+(1-x)A(k,\ell-1),\quad \begin{cases}A(k,0)=x^k,\\ A(0,\ell)=(1-x)^\ell.\end{cases}$$

これを解くと $A(k,\ell)=\binom{k+\ell}{k}x^k(1-x)^\ell.$

①に代入して

$$\begin{aligned}1&=\sum_{k=0}^{m}(1-x)\binom{k+n}{k}x^k(1-x)^n+\sum_{\ell=0}^{n}x\binom{m+\ell}{\ell}x^m(1-x)^\ell\\&=(1-x)^{n+1}\sum_{k=0}^{m}\binom{n+k}{k}x^k+x^{m+1}\sum_{\ell=0}^{n}\binom{m+\ell}{\ell}(1-x)^\ell.\end{aligned}$$ ：②

2. q-拡張・楕円関数拡張

q-Pochhammer記号　パラメータ　$q \in \mathbb{C},\ 0 < |q| < 1$　(ベースと呼ぶ)を導入して，

$x \in \mathbb{C}$ に対して $(x;q)_\infty := \prod_{\ell=0}^{\infty}(1 - xq^\ell)$,

$$(x;q)_k := \frac{(x;q)_\infty}{(xq^k;q)_\infty},$$ 特に $k \in \{1, 2, \cdots\}$ では $\prod_{\ell=0}^{k-1}(1 - xq^\ell)$.

変形ヤコビテータ関数　$p \in \mathbb{C},\ 0 < |p| < 1$ (ノームと呼ぶ)，$x \in \mathbb{C}\backslash\{0\}$ に対して

$$\theta(x;p) := \left(x, \frac{p}{x}; p\right)_\infty = \prod_{\ell=0}^{\infty}(1 - xp^\ell)(1 - \frac{p^{\ell+1}}{x}) \xrightarrow{p \to 0} 1 - x.$$

Theta-shifted factorial または (q,p)-shifted factorial

$$(x;q,p)_k = \prod_{\ell=0}^{k-1}\theta(xq^\ell;p),\ k \in \{1, 2, \cdots\} \xrightarrow{p \to 0} \prod_{\ell=0}^{k-1}(1 - xq^\ell) = (x;q)_k,\ k \in \{1, 2, \cdots\}.$$

q-拡張　ベース $q \in \mathbb{C},\ 0 < |q| < 1$ を導入

$$1 = (x;q)_{n+1} \sum_{k=0}^{m} \left[\begin{array}{c} n+k \\ k \end{array} \right]_q x^k + x^{m+1} \sum_{k=0}^{n} \left[\begin{array}{c} m+k \\ k \end{array} \right]_q q^k (x;q)_k .$$

ただし $\left[\begin{array}{c} n+k \\ k \end{array} \right]_q = \dfrac{(q;q)_{n+k}}{(q;q)_k (q;q)_n}$ ： q-2項係数

(a,b;q)-拡張 first kind　パラメータ $a, b \in \mathbb{C}\backslash\{0\}$ を導入

$$1 = \frac{(bx;q)_{n+1}}{(b/a;q)_{n+1}} \sum_{k=0}^{m} \frac{(q^{n+1}, ax;q)_k}{(q, aq/b;q)_k} q^k + \frac{(ax;q)_{m+1}}{(a/b;q)_{m+1}} \sum_{k=0}^{n} \frac{(q^{m+1}, bx;q)_k}{(q, bq/a;q)_k} q^k .$$

(a,b;q)-拡張 second kind

$$1 = \frac{(bx, b/x;q)_{n+1}}{(ab, b/a;q)_{n+1}} \sum_{k=0}^{m} \frac{(q^{n+1}, ax, a/x;q)_k}{(q, aq/b, abq^{1+n};q)_k} q^k + \frac{(ax, a/x;q)_{m+1}}{(ab, a/b;q)_{m+1}} \sum_{k=0}^{n} \frac{(q^{m+1}, bx, b/x;q)_k}{(q, bq/a, abq^{1+m};q)_k} q^k ,$$

(a,b,c;q)-拡張　パラメータ $c \in \mathbb{C} \backslash \{0\}$ を導入

$$1=\frac{(ac,c/a,bx,b/x;q)_{n+1}}{(ab,b/a,cx,c/x;q)_{n+1}}\sum_{k=0}^{m}\frac{(1-acq^{n+2k})(acq^n,bcq^n,c/b,q^{n+1},ax,a/x;q)_k}{(1-acq^n)(q,aq/b,abq^{1+n},ac,cq^{n+1}/x,cxq^{n+1};q)_k}q^k$$
$$+\frac{(bc,c/b,ax,a/x;q)_{m+1}}{(ab,a/b,cx,c/x;q)_{m+1}}\sum_{k=0}^{n}\frac{(1-bcq^{m+2k})(bcq^m,acq^m,c/a,q^{m+1},bx,b/x;q)_k}{(1-bcq^m)(q,bq/a,abq^{1+m},bc,cq^{m+1}/x,cxq^{m+1};q)_k}q^k.$$

楕円関数拡張　ノーム $p \in \mathbb{C},\ 0<|p|<1$ を導入

$$1=\frac{(ac,c/a,bx,b/x;q,p)_{n+1}}{(ab,b/a,cx,c/x;q,p)_{n+1}}\sum_{k=0}^{m}\frac{\theta(acq^{n+2k};p)(acq^n,bcq^n,c/b,q^{n+1},ax,a/x;q,p)_k}{\theta(acq^n;p)(q,aq/b,abq^{1+n},ac,cq^{n+1}/x,cxq^{n+1};q,p)_k}q^k$$
$$+\frac{(bc,c/b,ax,a/x;q,p)_{m+1}}{(ab,a/b,cx,c/x;q,p)_{m+1}}\sum_{k=0}^{n}\frac{\theta(bcq^{m+2k};p)(bcq^m,acq^m,c/a,q^{m+1},bx,b/x;q,p)_k}{\theta(bcq^m;p)(q,bq/a,abq^{1+m},bc,cq^{m+1}/x,cxq^{m+1};q,p)_k}q^k.$$

q-拡張

$$h(i,j)=xq^j \quad \text{とすると} \quad A(k,\ell)=\begin{bmatrix} k+\ell \\ k \end{bmatrix}_q x^k(x;q)_\ell.$$

(a,b;q)-拡張 first kind

$$h(i,j)=\frac{1-axq^i}{1-(a/b)q^{i-j}},\ 1-h(i,j)=\frac{1-bxq^j}{1-(b/a)q^{j-i}},\ A(k,\ell)=\left(1-\frac{a}{b}q^{k-\ell}\right)\frac{(ax;q)_k(q^{k+1};q)_\ell(bx;q)_\ell}{(a/b;q)_{k+1}(q;q)_\ell(q(b/a);q)_\ell}q^\ell$$
$$=\left(1-\frac{b}{a}q^{\ell-k}\right)\frac{(bx;q)_\ell(q^{\ell+1};q)_k(ax;q)_k}{(b/a;q)_{\ell+1}(q;q)_k(q(a/b);q)_k}q^k.$$

(a,b;q)-拡張 second kind

$$h(i,j)=\frac{(1-axq^i)(1-(a/x)q^i)}{(1-abq^{i+j})(1-(a/b)q^{i-j})},\ 1-h(i,j)=\frac{(1-bxq^j)(1-(b/x)q^j)}{(1-abq^{i+j})(1-(b/a)q^{j-i})},$$

$$A(k,\ell)=\left(1-\frac{a}{b}q^{k-\ell}\right)\frac{(ax,a/x;q)_k(q^{k+1};q)_\ell(bx,b/x;q)_\ell}{(a/b;q)_{k+1}(q;q)_\ell(q(b/a);q)_\ell(ab;q)_{k+\ell}}q^\ell,$$
$$=\left(1-\frac{b}{a}q^{\ell-k}\right)\frac{(bx,b/x;q)_\ell(q^{\ell+1};q)_k(ax,a/x;q)_k}{(b/a;q)_{\ell+1}(q;q)_k(q(a/b);q)_k(ab;q)_{\ell+k}}q^k.$$

(a,b,c;q)-拡張

$$h(i,j) = \frac{(1-bcq^{i+2j})(1-(c/b)q^i)(1-axq^i)(1-(a/x)q^i)}{(1-abq^{i+j})(1-(a/b)q^{i-j})(1-cxq^{i+j})(1-(c/x)q^{i+j})},$$

$$1-h(i,j) = \frac{(1-acq^{2i+j})(1-(c/a)q^j)(1-bxq^j)(1-(b/x)q^j)}{(1-abq^{i+j})(1-(b/a)q^{j-i})(1-cxq^{i+j})(1-(c/x)q^{i+j})},$$

$$\begin{aligned} A(k,\ell) &= \left(1-\frac{a}{b}q^{k-\ell}\right)\frac{(bcq^\ell, c/b, ax, a/x;q)_k(q^{k+1};q)_\ell(acq^k, c/a, bx, b/x;q)_\ell}{(a/b;q)_{k+1}(q;q)_\ell(q(b/a);q)_\ell(ab, cx, c/x;q)_{k+\ell}}q^\ell \\ &= \left(1-\frac{b}{a}q^{\ell-k}\right)\frac{(acq^k, c/a, bx, b/x;q)_\ell(q^{\ell+1};q)_k(bcq^\ell, c/b, ax, a/x;q)_k}{(b/a;q)_{\ell+1}(q;q)_k(q(a/b);q)_k(ab, cx, c/x;q)_{\ell+k}}q^k. \end{aligned}$$

楕円関数拡張

$$h(i,j) = \frac{\theta(bcq^{i+2j}, (c/b)q^i, axq^i, (a/x)q^i; p)}{\theta(abq^{i+j}, (a/b)q^{i-j}, cxq^{i+j}, (c/x)q^{i+j}; p)},$$

$$1 - h(i,j) = \frac{\theta(acq^{2i+j}, (c/a)q^j, bxq^j, (b/x)q^j; p)}{\theta(abq^{i+j}, (b/a)q^{j-i}, cxq^{i+j}, (c/x)q^{i+j}; p)},$$

$$\begin{aligned} A(k,\ell) &= \theta\left(\frac{a}{b}q^{k-\ell}; p\right) \frac{(bcq^\ell, c/b, ax, a/x; q,p)_k (q^{k+1}, acq^k, c/a, bx, b/x; q,p)_\ell}{(a/b; q,p)_{k+1} (q, q(b/a); q,p)_\ell (ab, cx, c/x; q,p)_{k+\ell}} q^\ell, \\ &= \theta\left(\frac{b}{a}q^{\ell-k}; p\right) \frac{(acq^k, c/a, bx, b/x; q,p)_\ell (q^{\ell+1}, bcq^\ell, c/b, ax, a/x; q,p)_k}{(b/a; q,p)_{\ell+1} (q, q(a/b); q,p)_k (ab, cx, c/x; q,p)_{\ell+k}} q^k. \end{aligned}$$

テータ関数の性質

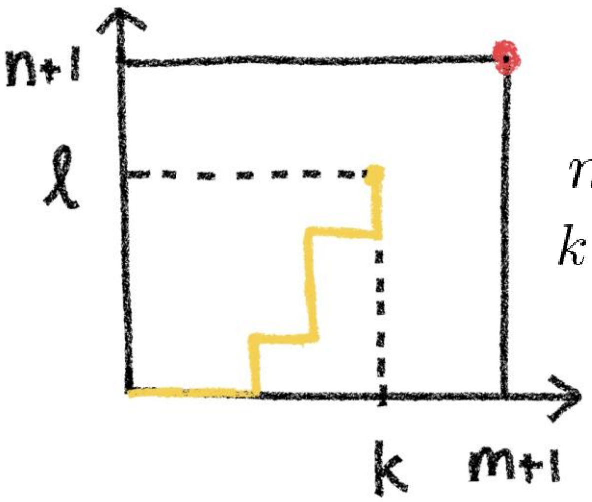
反転公式 $\theta\left(\frac{1}{z}; p\right) = -\frac{1}{z}\theta(z;p)$

と

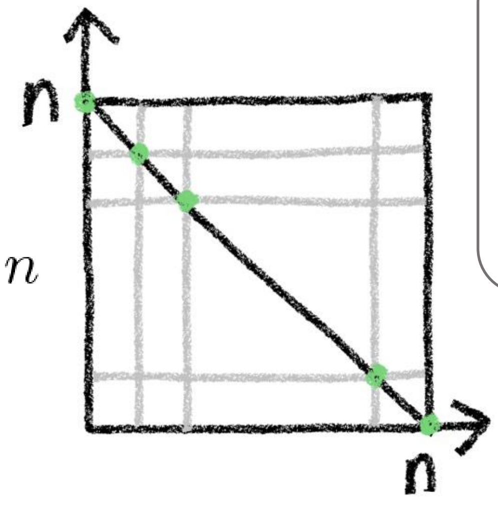
擬周期性 $\theta(pz;p) = -\frac{1}{z}\theta(z;p)$

Riemann-Weierstrassの加法定理 $\theta\left(xy, \frac{x}{y}, uv, \frac{u}{v}; p\right) - \theta\left(xv, \frac{x}{v}, uy, \frac{u}{y}; p\right) = \frac{u}{y}\theta\left(yv, \frac{y}{v}, xu, \frac{x}{u}; p\right)$ を用いた.

3. 確率解釈$(0 < h(i,j) < 1)$と数値計算



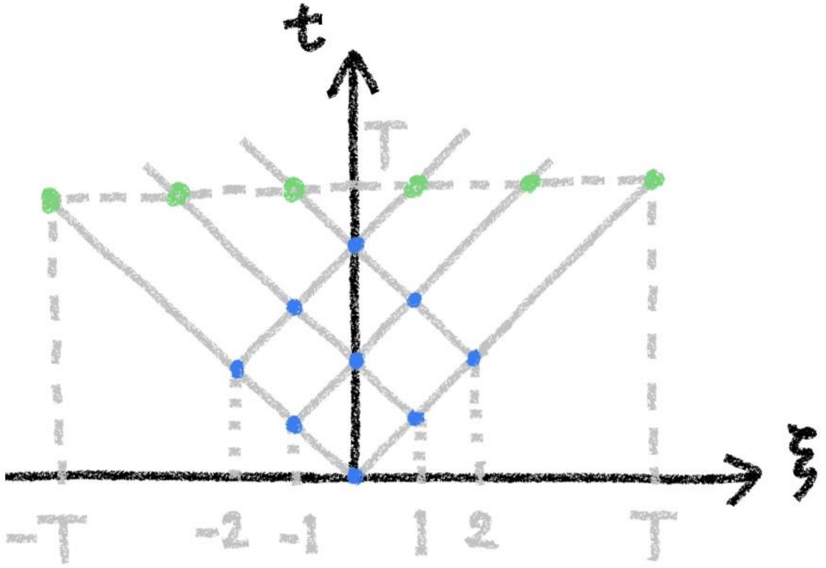
$$m = n$$
$$k + \ell = n$$



変数変換

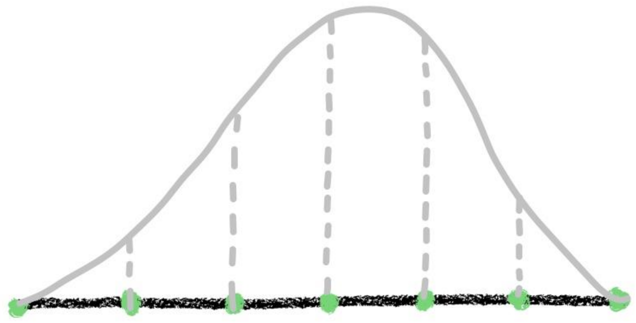
$$i = \frac{t+\xi}{2}$$
$$j = \frac{t-\xi}{2}$$

$$n = T$$



$h(i,j)$が確率なら,$A(k,\ell)$は
$k+\ell$ステップ後に
点(k,ℓ)にいる確率

nステップ進んだとき,
斜めの線上の格子点にいる



$t = T$ 上の
$\xi = -T, -T+2, \cdots, T-2, T$
の点が, 左図の格子点と対応

<u>ランダムウォーク</u>とみなせる
$(h \in \mathbb{R},\ 0 < h < 1)$

重みを確率解釈するための条件は

q-拡張 $0 < q < 1$, $0 < x < 1$

(a,b;q)-拡張 first kind (1) $a > 0,\ b < 0,\ \frac{1}{b} < x < \frac{1}{a}$

(2) (1)でa⇔bとしたもの.

(a,b;q)-拡張 second kind (1) $0 < a < 1,\ -1 < b < 0,\ a < x < \frac{1}{a}$

(2) $0 < a < 1,\ -1 < b < 0,\ \frac{1}{b} < x < b$

(3) (1), (2)でa⇔bとしたもの.

(a,b,c;q)-拡張 (1) $0 < c < a < 1,\ -1 < b < 0,\ a < x < \frac{1}{a}$

(2) $0 < a < 1,\ -1 < b < c < 0,\ a < x < \frac{1}{a}$

(3) $0 < c < a < 1,\ -1 < b < 0,\ \frac{1}{b} < x < b$

(4) $0 < a < 1,\ -1 < b < c < 0,\ \frac{1}{b} < x < b$

(5) (1)～(4)でそれぞれa⇔bとしたもの.

楕円関数拡張　$0 \leq p < x < 1$ のとき $\theta(x;p) > 0,\ 1/\theta(x;p) > 0$ となることから

(1)　$0 < c < a < 1, \quad -1 < b < 0, \quad a < x < \dfrac{1}{a}, \quad p < acq^{3T}.$

(2)　$0 < a < 1, \quad -1 < b < c < 0, \quad a < x < \dfrac{1}{a}, \quad p < \min\left\{bcq^{3T}, axq^{T}, \dfrac{a}{x}q^{T}\right\}.$

(3)　$0 < c < a < 1, \quad -1 < b < 0, \quad \dfrac{1}{b} < x < b, \quad p < \min\left\{acq^{3T}, bxq^{T}, \dfrac{b}{x}q^{T}\right\}.$

(4)　$0 < a < 1, \quad -1 < b < c < 0, \quad \dfrac{1}{b} < x < b, \quad p < bcq^{3T}.$

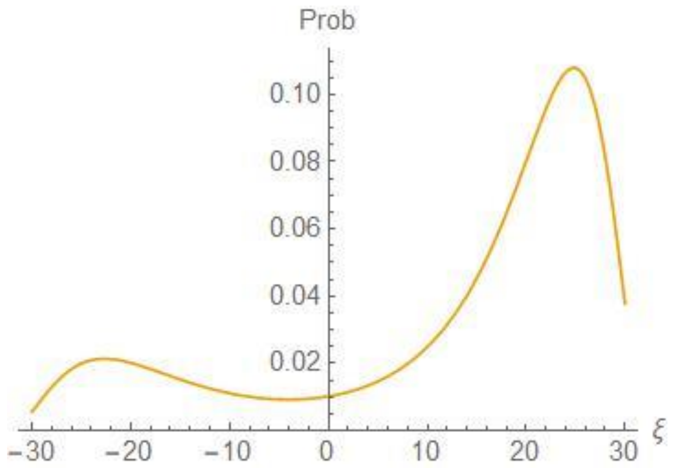
(5)　(1)～(4)でそれぞれa↔bとしたもの.

以上の条件を満たすようにパラメータの値を決め, それぞれの$A(k,\ell)$に代入して
分布をプロットしていく.

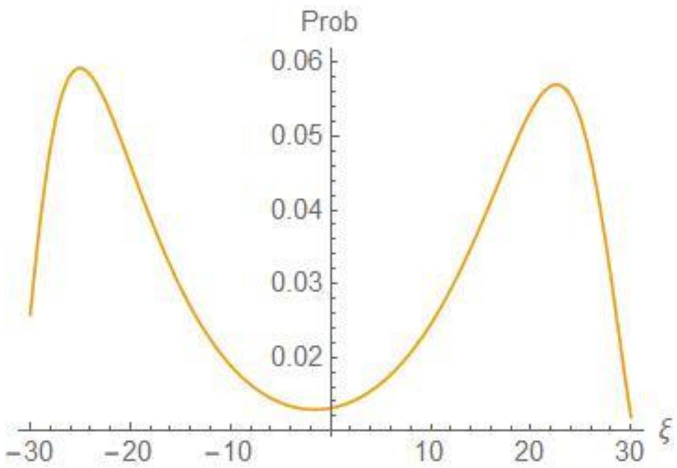
(a,b;q)-拡張 first kind の分布

$$a>0,\ b<0,\ \frac{1}{b}<x<\frac{1}{a}$$

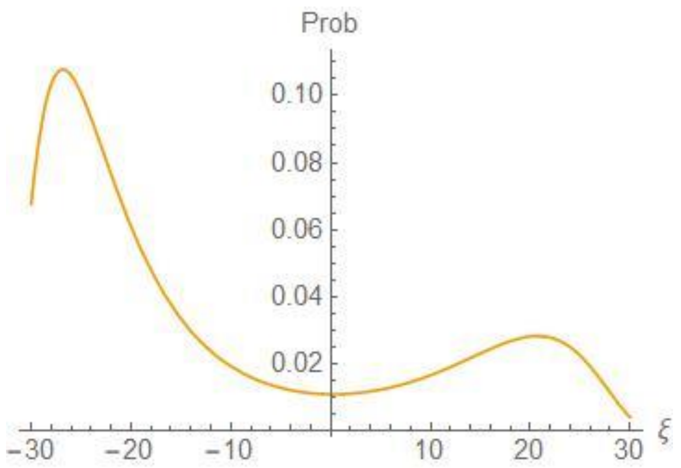
$$\mathbf{P}(\Xi_t^{(1)}=\xi)=\left(1-\frac{a}{b}q^{\xi}\right)\frac{(ax;q)_{(t+\xi)/2}(q^{(t+\xi+2)/2},bx;q)_{(t-\xi)/2}}{(a/b;q)_{(t+\xi+2)/2}(q,q(b/a);q)_{(t-\xi)/2}}q^{(t-\xi)/2}.$$



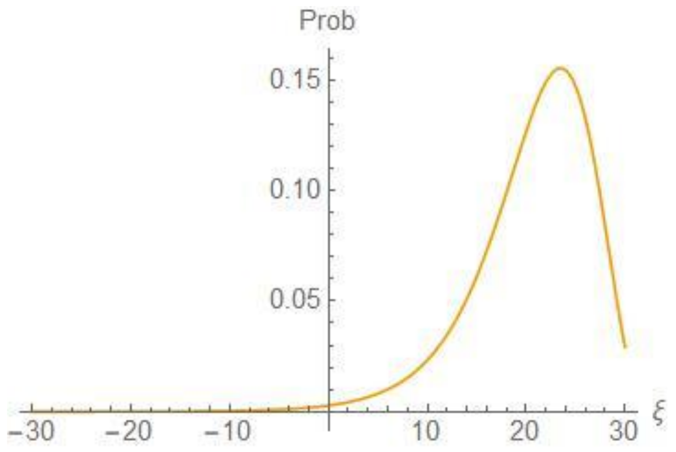
q=0.8, x=0.5, (a,b)=(0.4,-0.8)



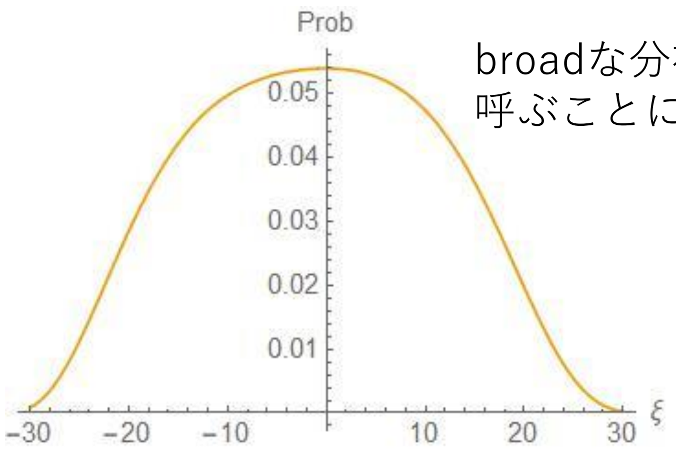
q=0.8, x=0.5, (a,b)=(0.55,-0.8)



q=0.8, x=0.5, (a,b)=(0.7,-0.8)

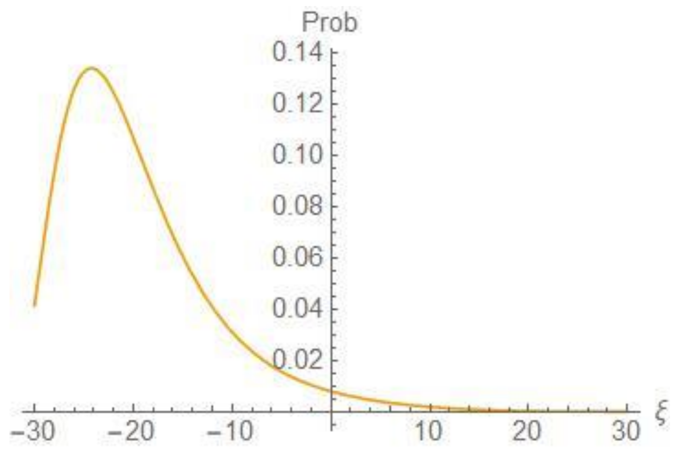


q=0.9, x=0.5, (a,b)=(0.2,-0.7)



broadな分布と
呼ぶことにする

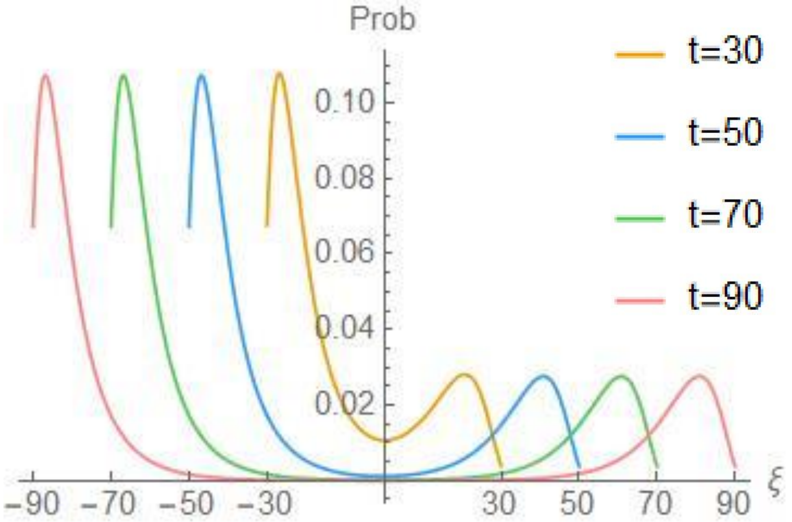
q=0.9, x=0.5, (a,b)=(0.5,-0.7)



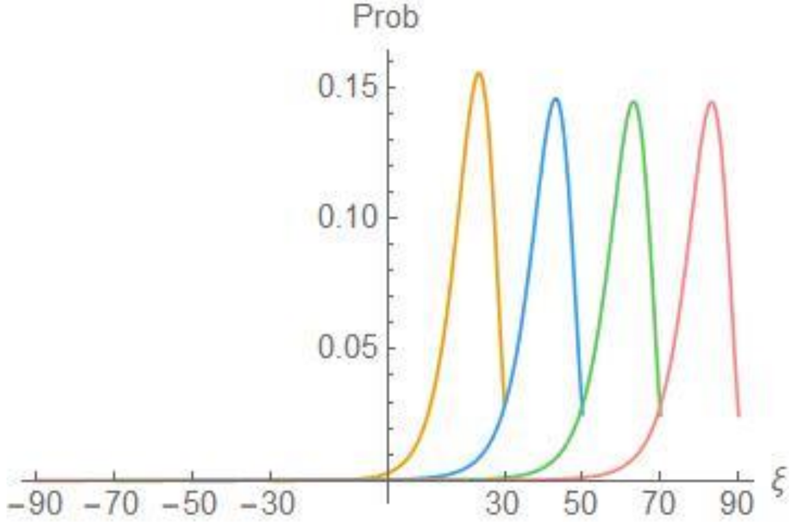
q=0.9, x=0.5, (a,b)=(0.9,-0.7)

(a,b;q)-拡張 first kind の分布

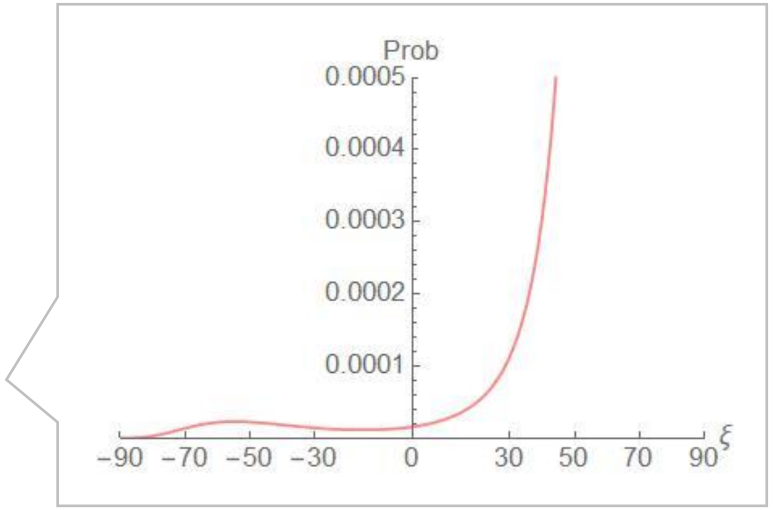
$$\mathbf{P}(\Xi_t^{(1)}=\xi)=\left(1-\frac{a}{b}q^{\xi}\right)\frac{(ax;q)_{(t+\xi)/2}(q^{(t+\xi+2)/2},bx;q)_{(t-\xi)/2}}{(a/b;q)_{(t+\xi+2)/2}(q,q(b/a);q)_{(t-\xi)/2}}q^{(t-\xi)/2},\ a>0,\ b<0,\ \frac{1}{b}<x<\frac{1}{a}.$$



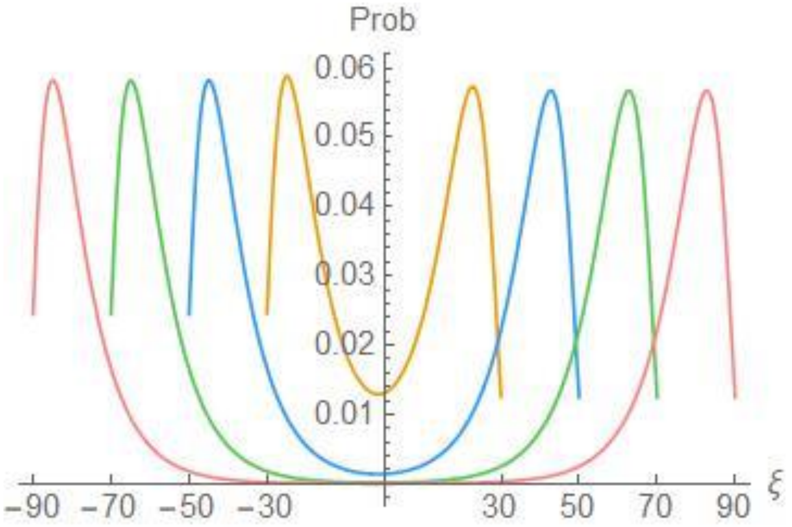
q=0.8, x=0.5, (a,b)=(0.7,-0.8)



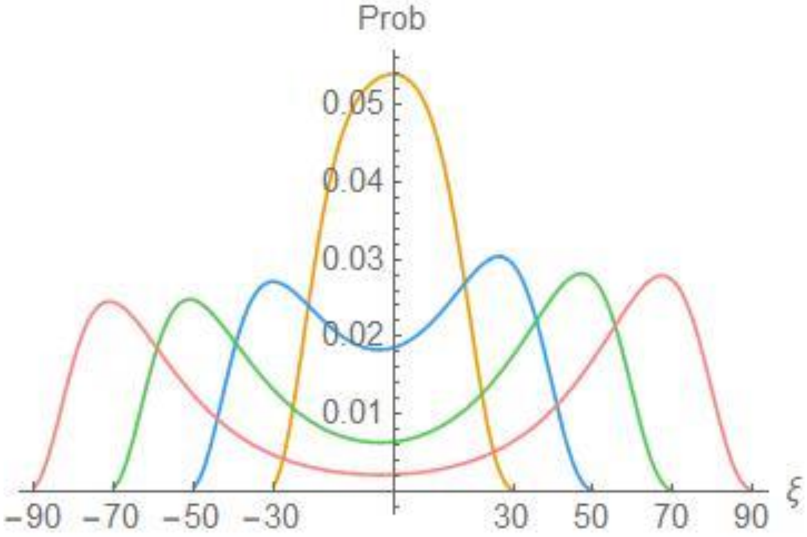
q=0.9, x=0.5, (a,b)=(0.2,-0.7)



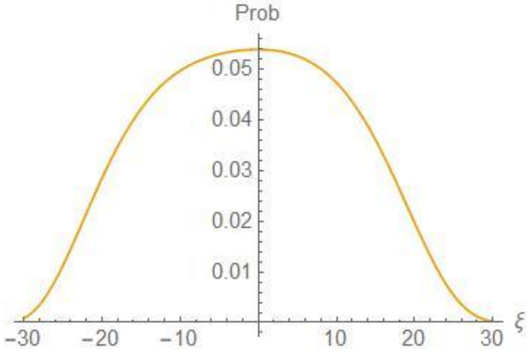
差分を計算し, 正負の入れ替わりが
２回以上あるものを複峰としている



q=0.8, x=0.5, (a,b)=(0.5,-0.7)



q=0.9, x=0.5, (a,b)=(0.5,-0.7)



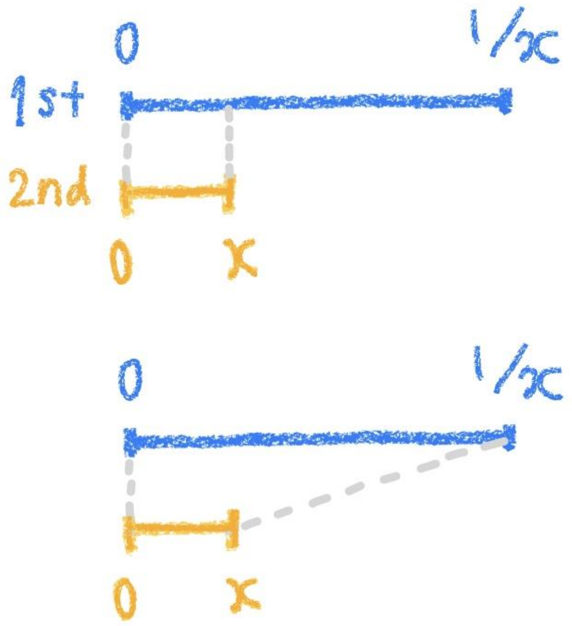
時間を進めると複峰になる

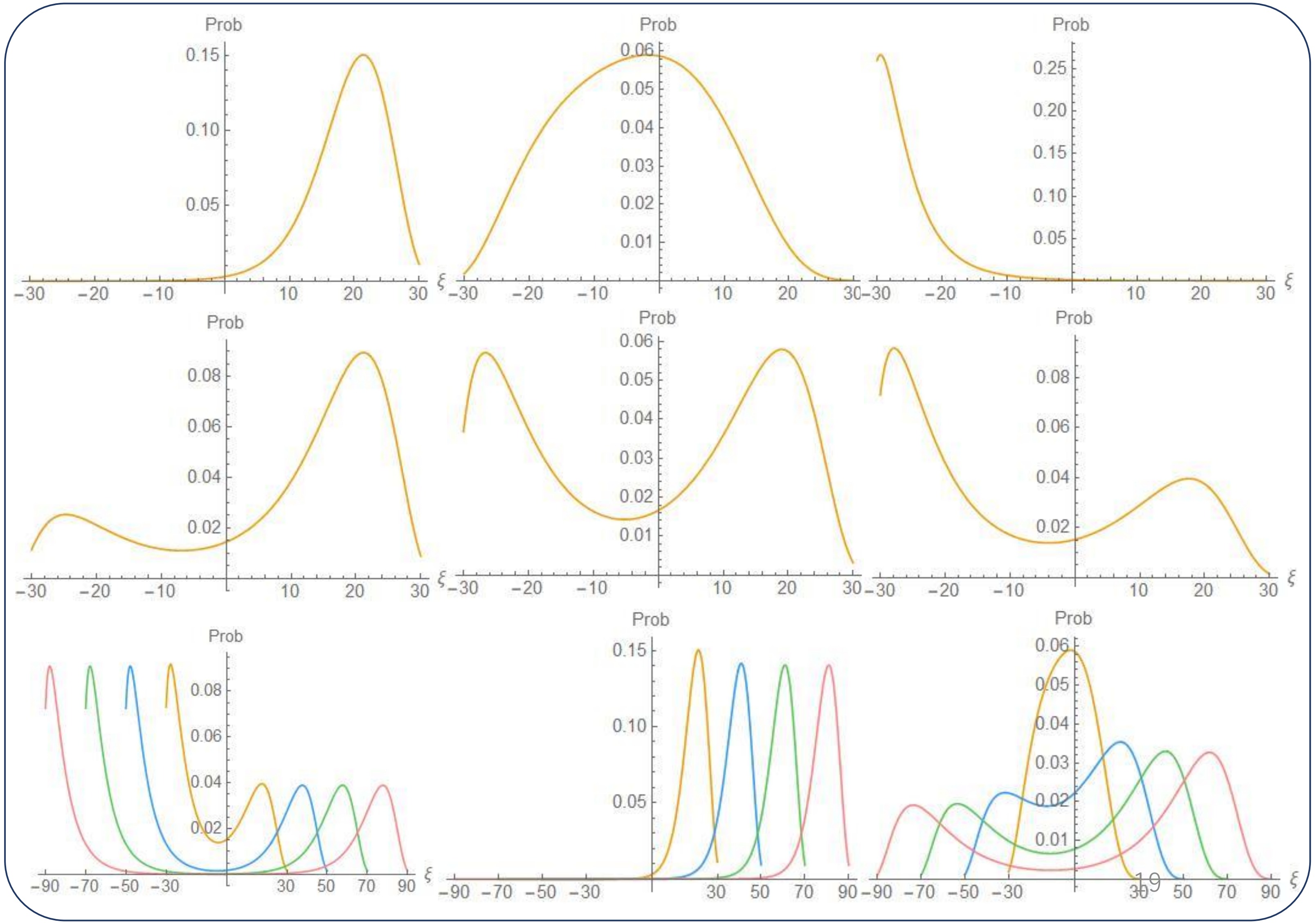
(a,b;q)-拡張 second kind の分布

$$\mathbf{P}(\Xi_t^{(2)} = \xi) = \left(1 - \frac{a}{b}q^{\xi}\right) \frac{(ax, a/x; q)_{(t+\xi)/2}(q^{(t+\xi+2)/2}, bx, b/x; q)_{(t-\xi)/2}}{(a/b; q)_{(t+\xi+2)/2}(q, q(b/a); q)_{(t-\xi)/2}(ab; q)_t} q^{(t-\xi)/2}.$$

$$0 < a < 1,\ -1 < b < 0,\quad a < x < \frac{1}{a}$$

- 単峰(とbroad)と複峰がある
- aを大きくすると右から左へシフトする
- 時間経過で単峰が複峰になる

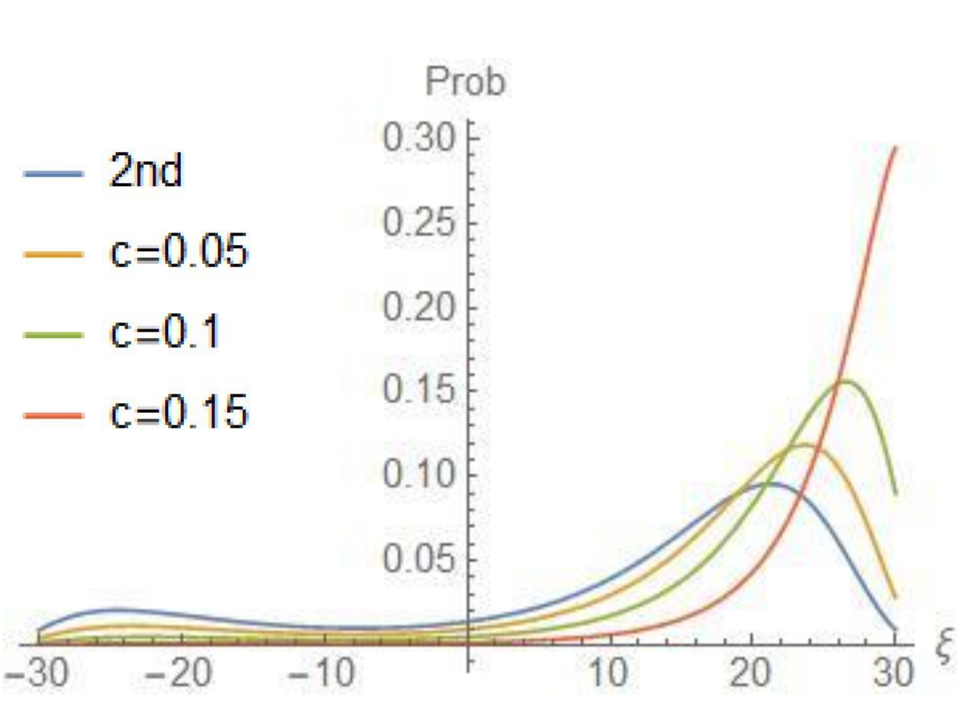




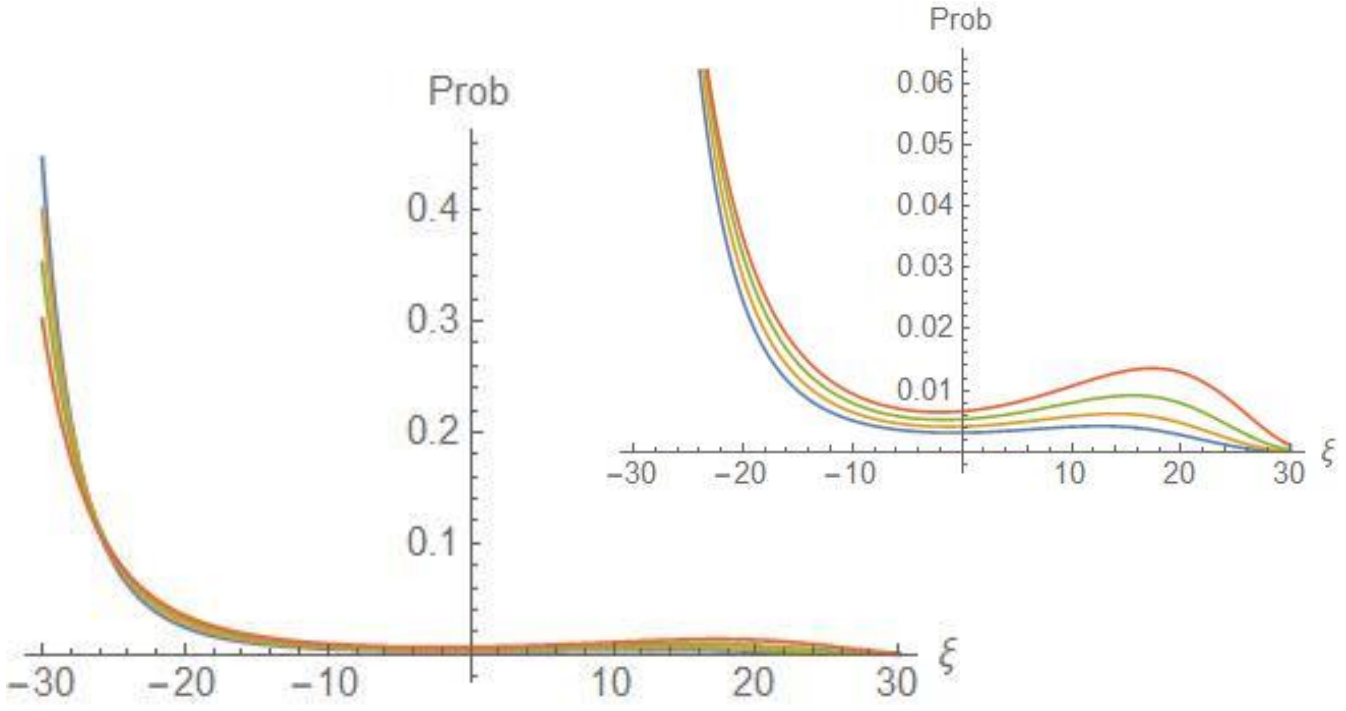
(a,b,c;q)-拡張 の分布

$$0 < c < a < 1,\ -1 < b < 0,\ a < x < \frac{1}{a}$$

$$\mathbf{P}(\Xi_t^{(3)} = \xi) = \left(1 - \frac{a}{b}q^{\xi}\right) \frac{(bcq^{(t-\xi)/2}, c/b, ax, a/x; q)_{(t+\xi)/2}(q^{(t+\xi+2)/2}, acq^{(t+\xi)/2}, c/a, bx, b/x; q)_{(t-\xi)/2}}{(a/b;q)_{(t+\xi+2)/2}(q, q(b/a);q)_{(t-\xi)/2}(ab, cx, c/x; q)_t} q^{(t-\xi)/2}.$$



q=0.8, x=0.5, (a,b)=(0.2,-0.9)

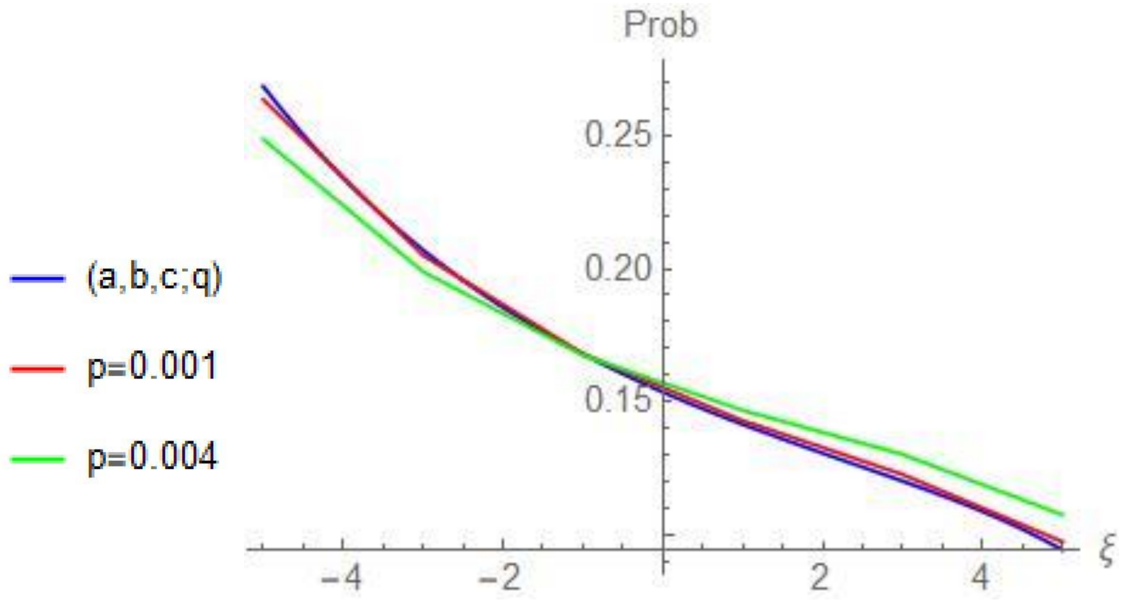


q=0.8, x=0.5, (a,b)=(0.4,-0.9)

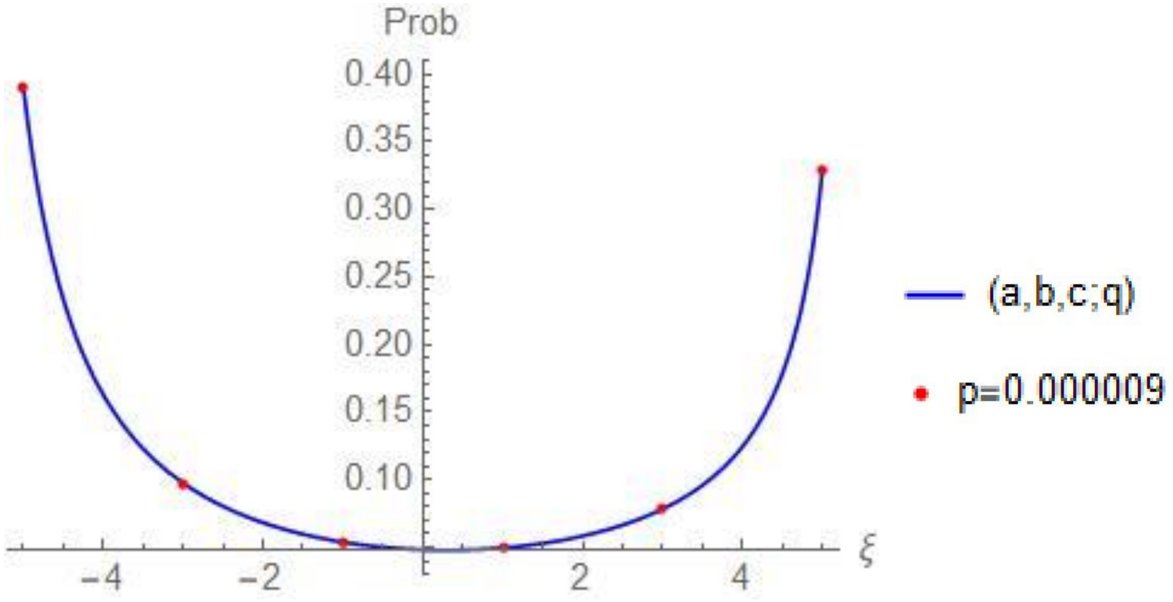
楕円関数拡張 の分布

$$0 < c < a < 1,\ -1 < b < 0,\ a < x < \frac{1}{a},\ p < acq^{3T}$$

$$\mathbf{P}(\Xi_t^T = \xi) = \theta\left(\frac{a}{b}q^{\xi};p\right) \frac{(bcq^{(t-\xi)/2}, c/b, ax, a/x; q, p)_{(t+\xi)/2}(q^{(t+\xi+2)/2}, acq^{(t+\xi)/2}, c/a, bx, b/x; q, p)_{(t-\xi)/2}}{(a/b; q, p)_{(t+\xi+2)/2}(q, q(b/a); q, p)_{(t-\xi)/2}(ab, cx, c/x; q, p)_t} q^{(t-\xi)/2}.$$



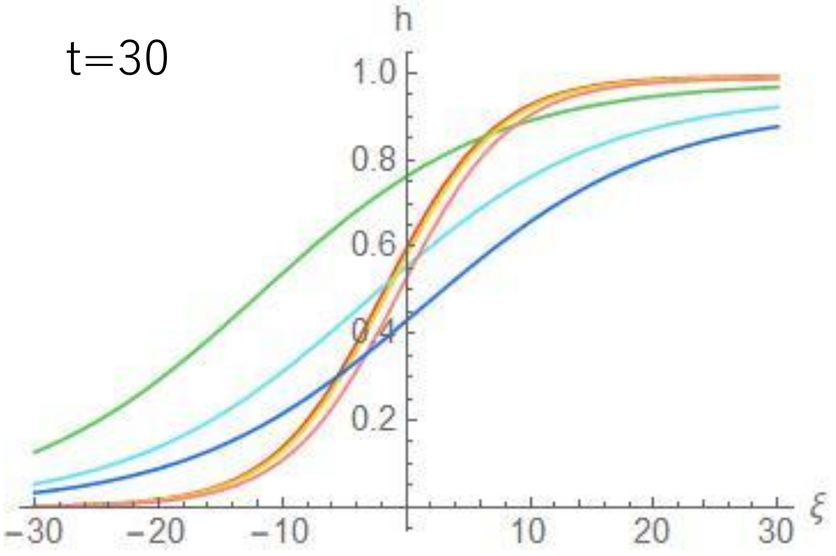
q=0.8, x=0.5, (a,b)=(0.4,-0.5), c=0.3



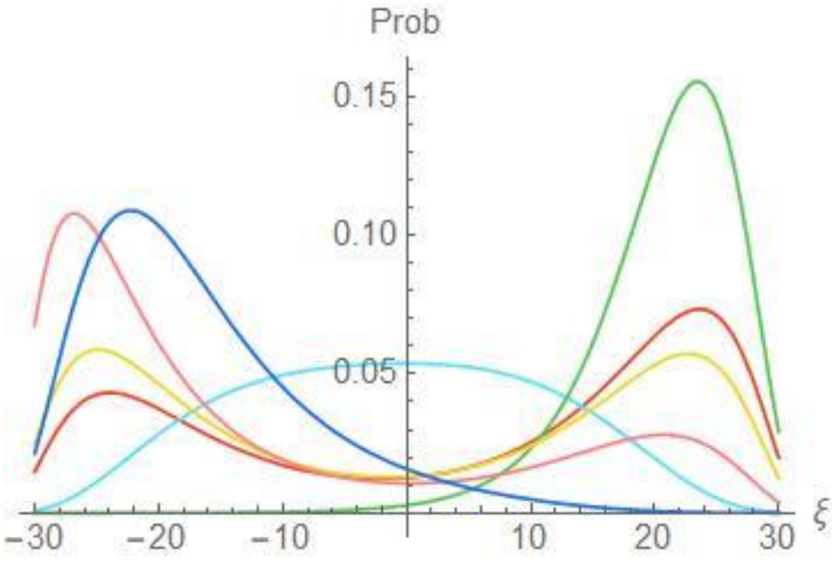
q=0.5, x=0.8, (a,b)=(0.6,-0.5), c=0.5

(a,b;q)-拡張 first kind の $h(\xi,t;x;a,b;q)=\dfrac{1-axq^{(t+\xi)/2}}{1-\frac{a}{b}q^{\xi}}$ のグラフ

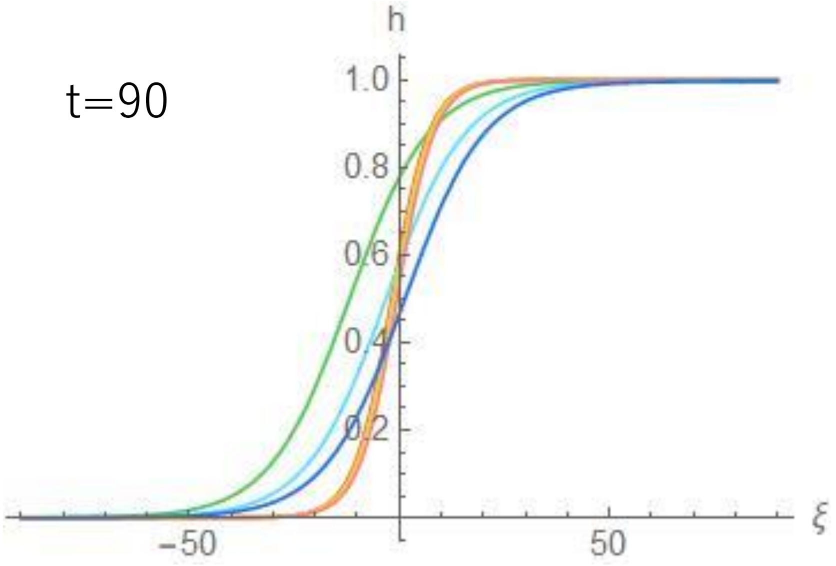
t=30

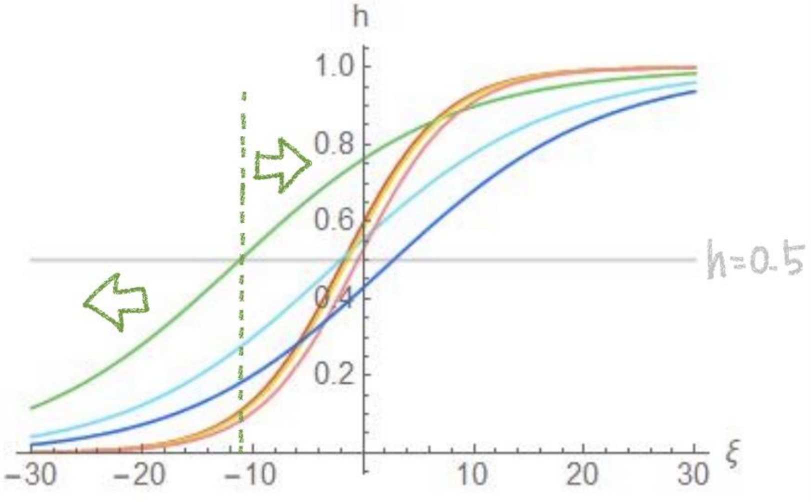


	q	(a, b)
右複	0.8	(0.4, −0.6)
broad	0.9	(0.5, −0.7)
右単	0.9	(0.2, −0.7)
対称	0.8	(0.5, −0.7)
左複	0.8	(0.7, −0.8)
左単	0.9	(0.8, −0.7)

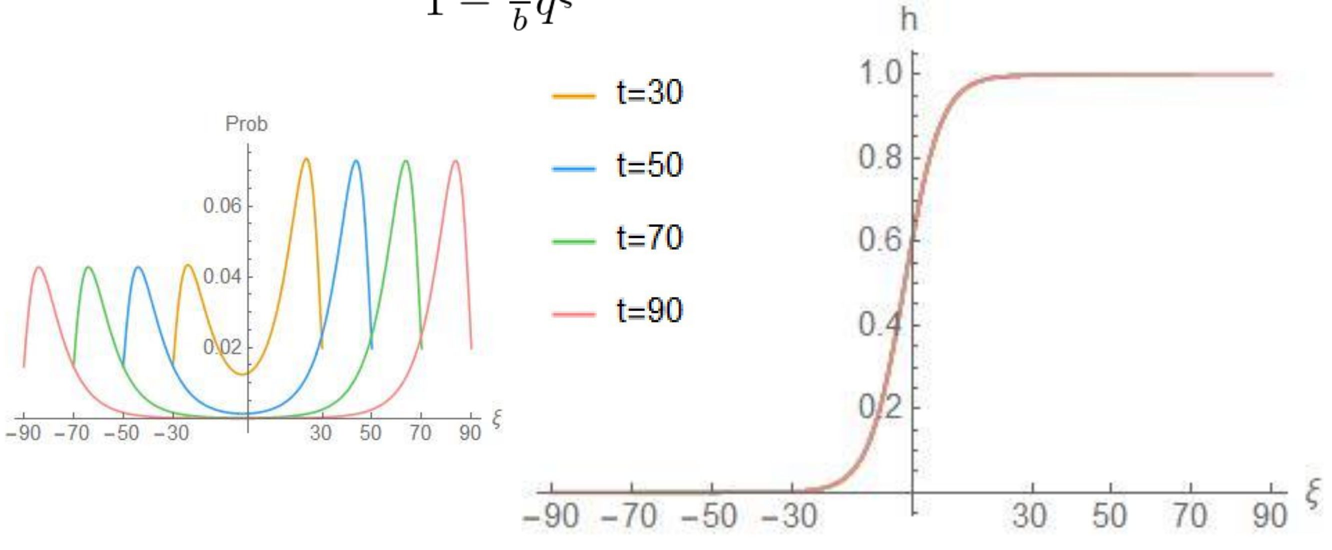


t=90

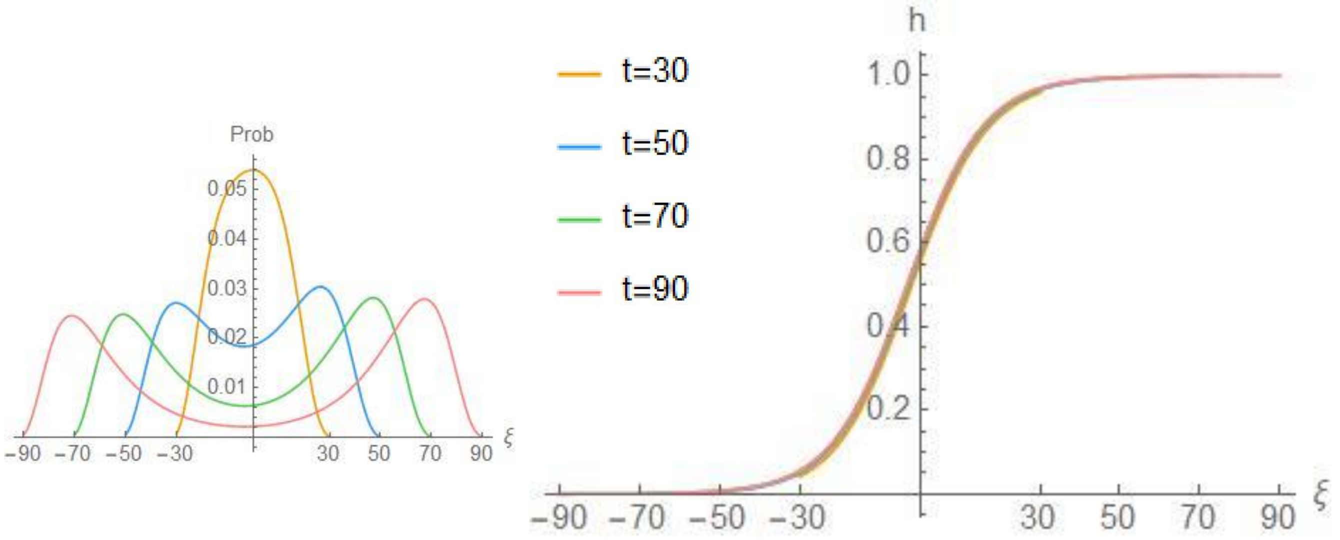




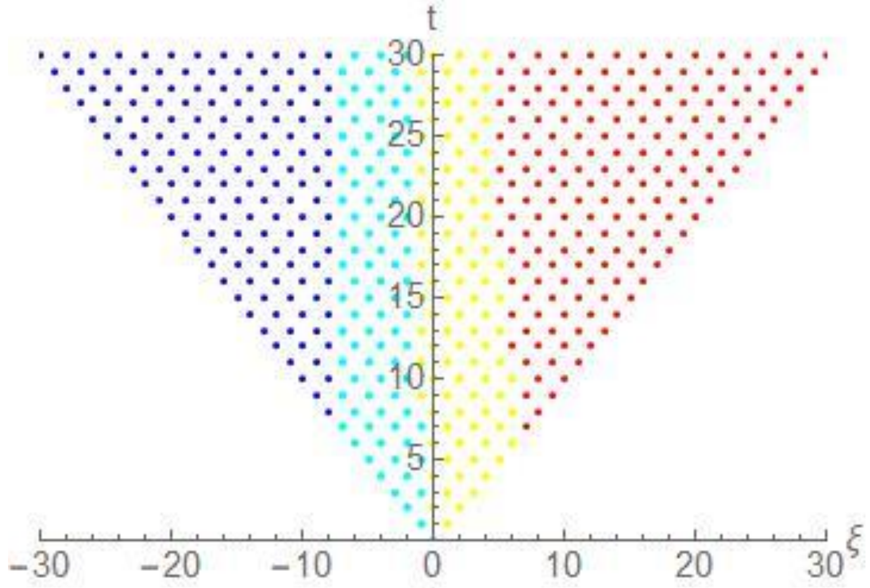
$$h(\xi, t; x; a, b; q) = \frac{1 - axq^{(t+\xi)/2}}{1 - \frac{a}{b}q^{\xi}}$$

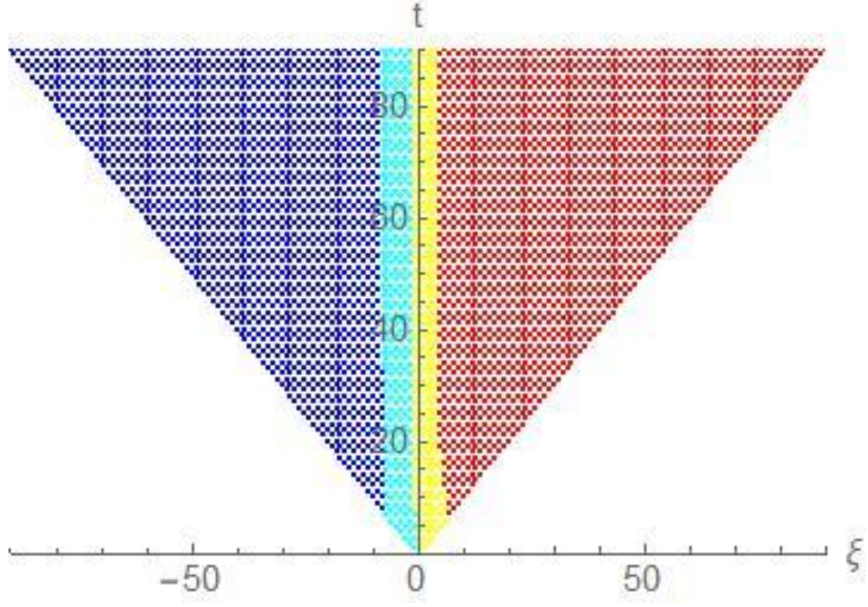


q=0.8, x=0.5, (a,b)=(0.4,-0.6)



q=0.9, x=0.5, (a,b)=(0.5,-0.7)

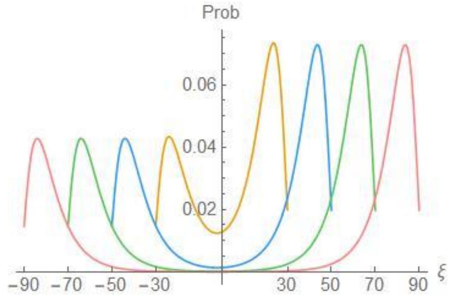


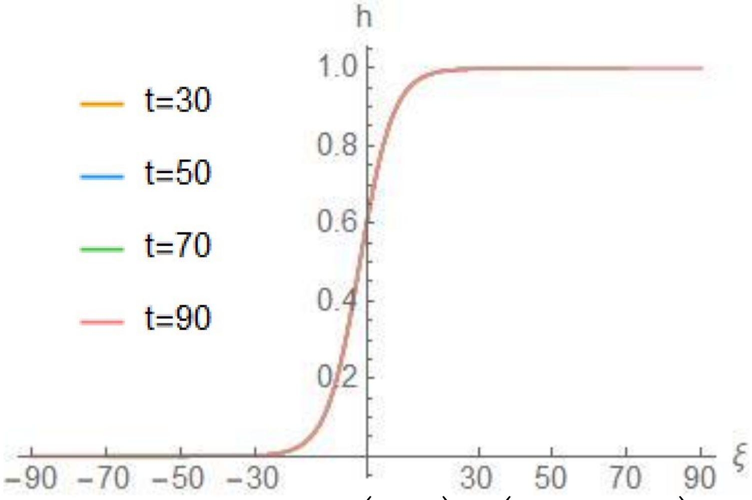


$0 \leq$ (青) < 0.2, $0.2 \leq$ (水色) < 0.5,

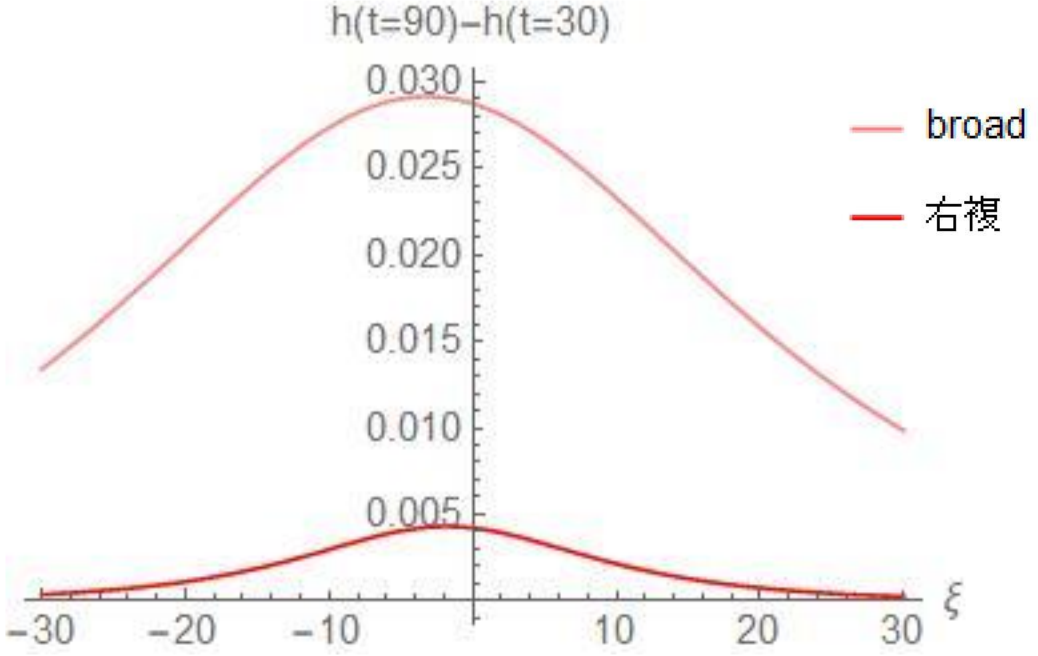
$0.5 \leq$ (黄色) < 0.8, $0.8 \leq$ (赤) ≤ 1.

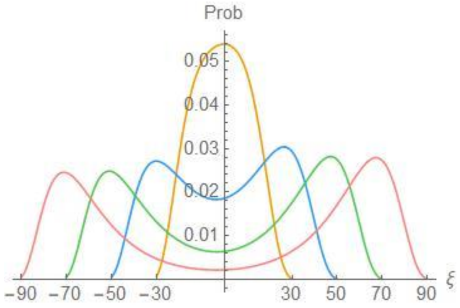
t=90のときとt=30のとき の $h(\xi, t; x; a, b; q) = \dfrac{1 - axq^{(t+\xi)/2}}{1 - \frac{a}{b}q^{\xi}}$ の差をプロット

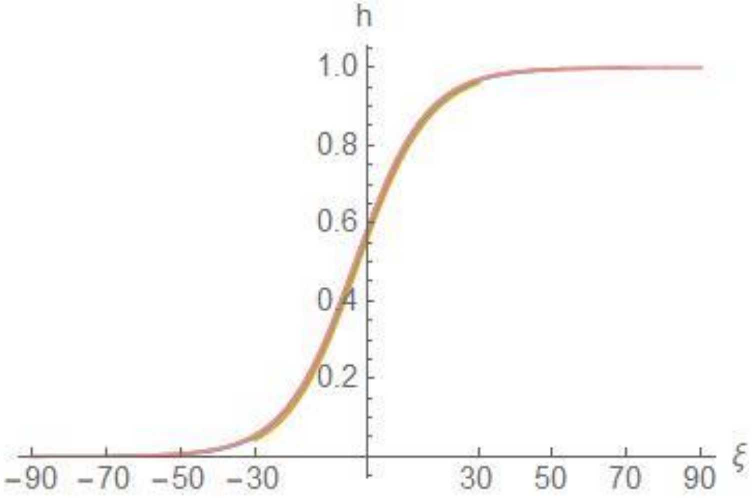




q=0.8, x=0.5, (a,b)=(0.4,-0.6)





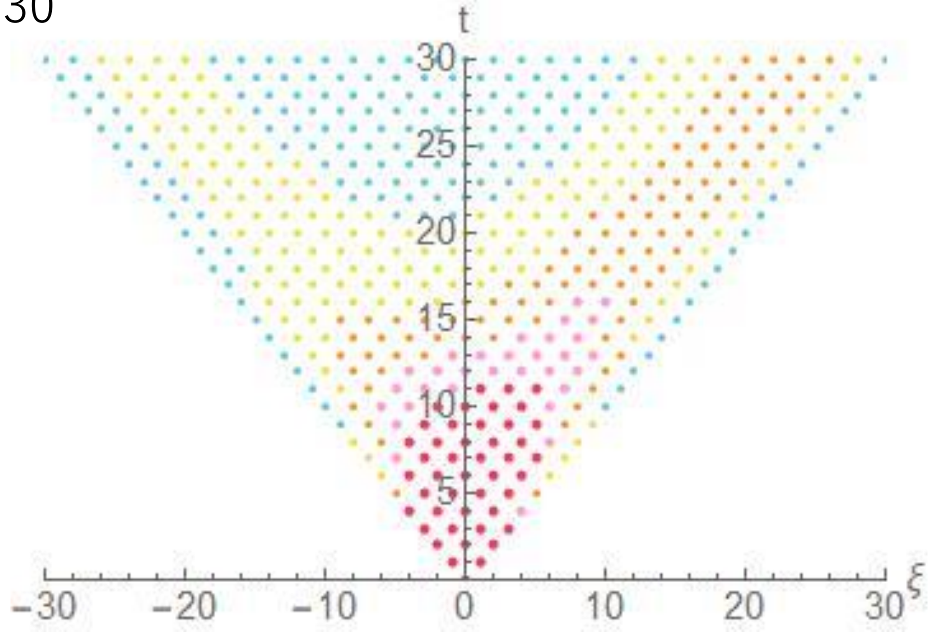


q=0.9, x=0.5, (a,b)=(0.5,-0.7)

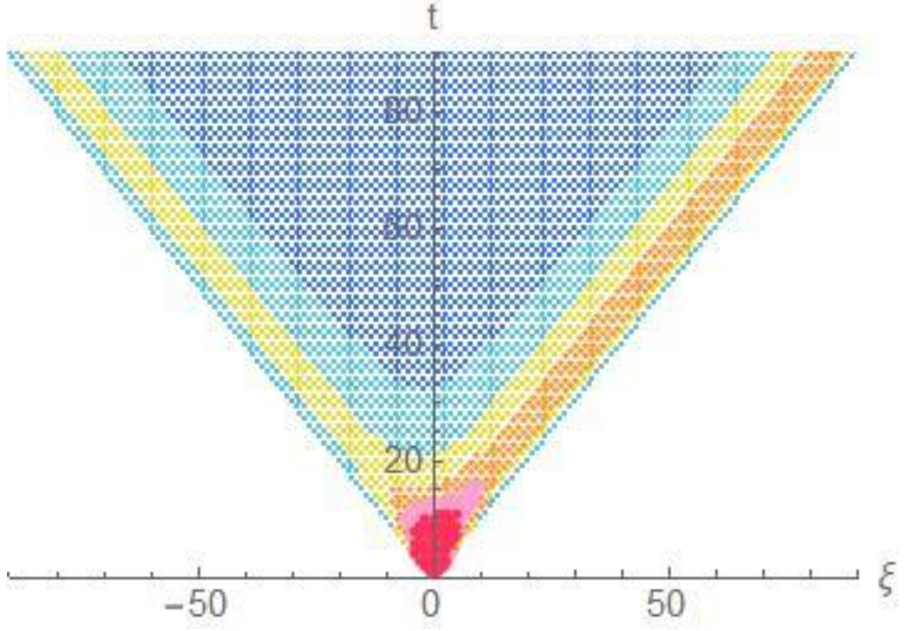
(a,b;q)-拡張 first kind の分布

$$\mathbf{P}(\Xi_t^{(1)} = \xi) = \left(1 - \frac{a}{b}q^{\xi}\right) \frac{(ax;q)_{(t+\xi)/2}(q^{(t+\xi+2)/2}, bx;q)_{(t-\xi)/2}}{(a/b;q)_{(t+\xi+2)/2}(q, q(b/a);q)_{(t-\xi)/2}} q^{(t-\xi)/2}$$

T=30



T=90

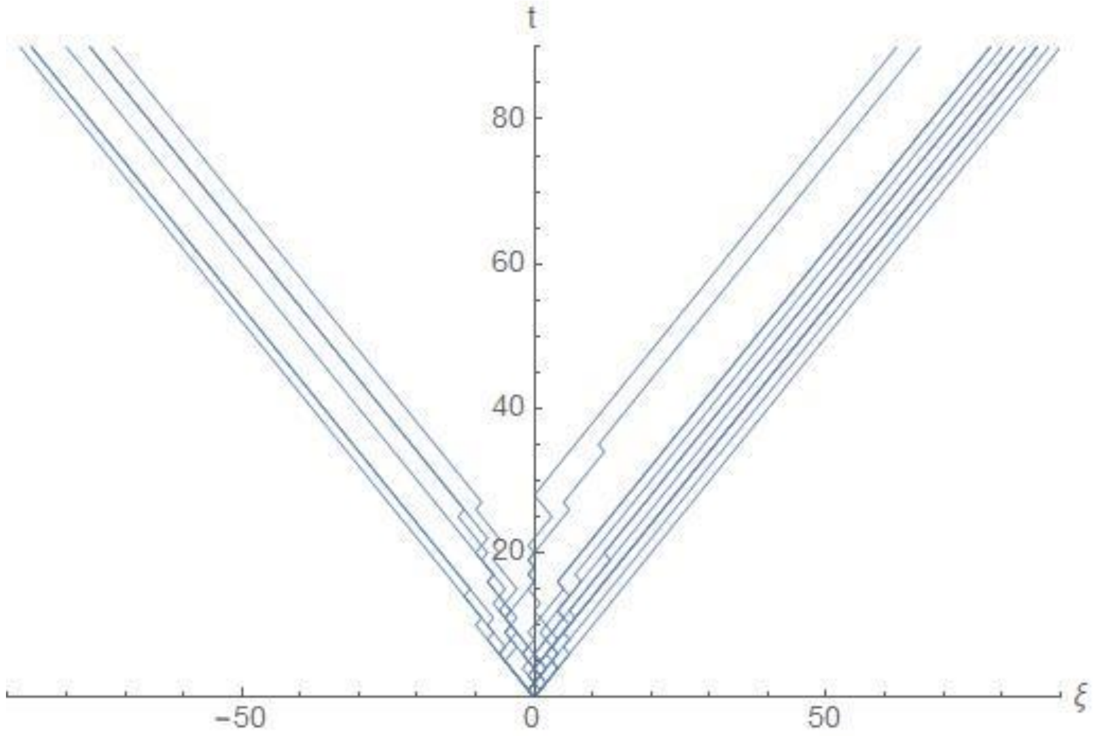


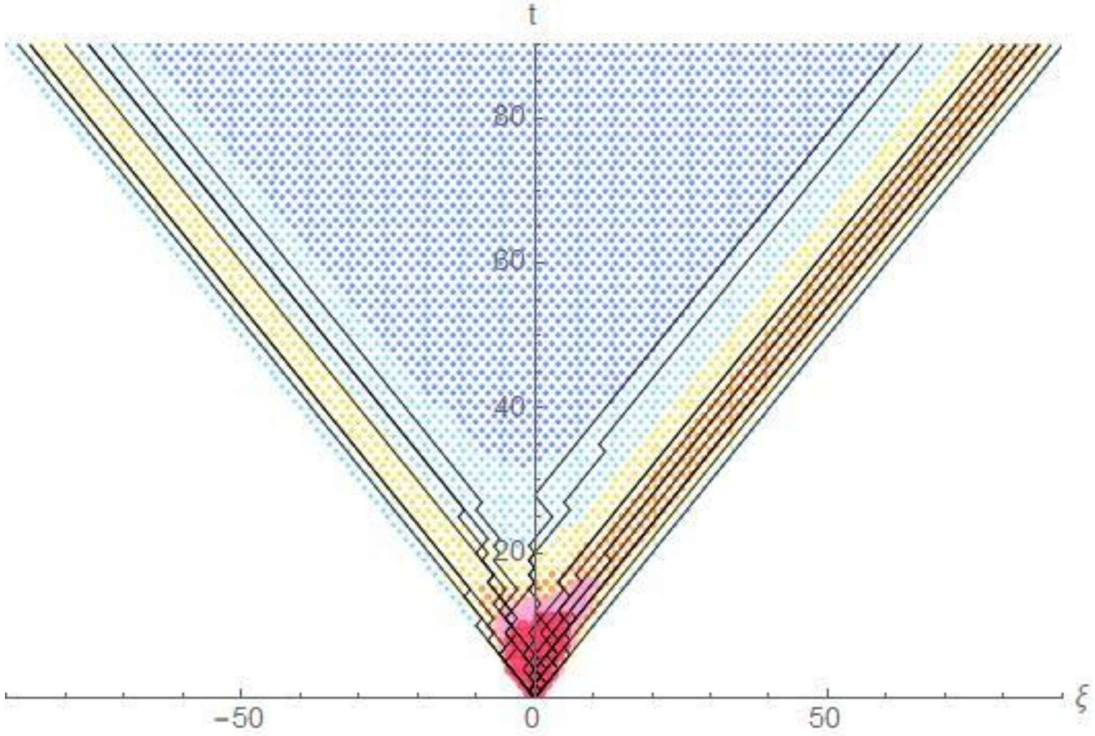
q=0.8, x=0.5, (a,b)=(0.4,-0.6)

$0 \leq$ (青) < 0.01, $0.01 \leq$ (水色) < 0.035, $0.035 \leq$ (黄色) < 0.06,

$0.06 \leq$ (橙色) < 0.085, $0.085 \leq$ (ピンク) < 0.11, $0.11 \leq$ (赤) ≤ 1.

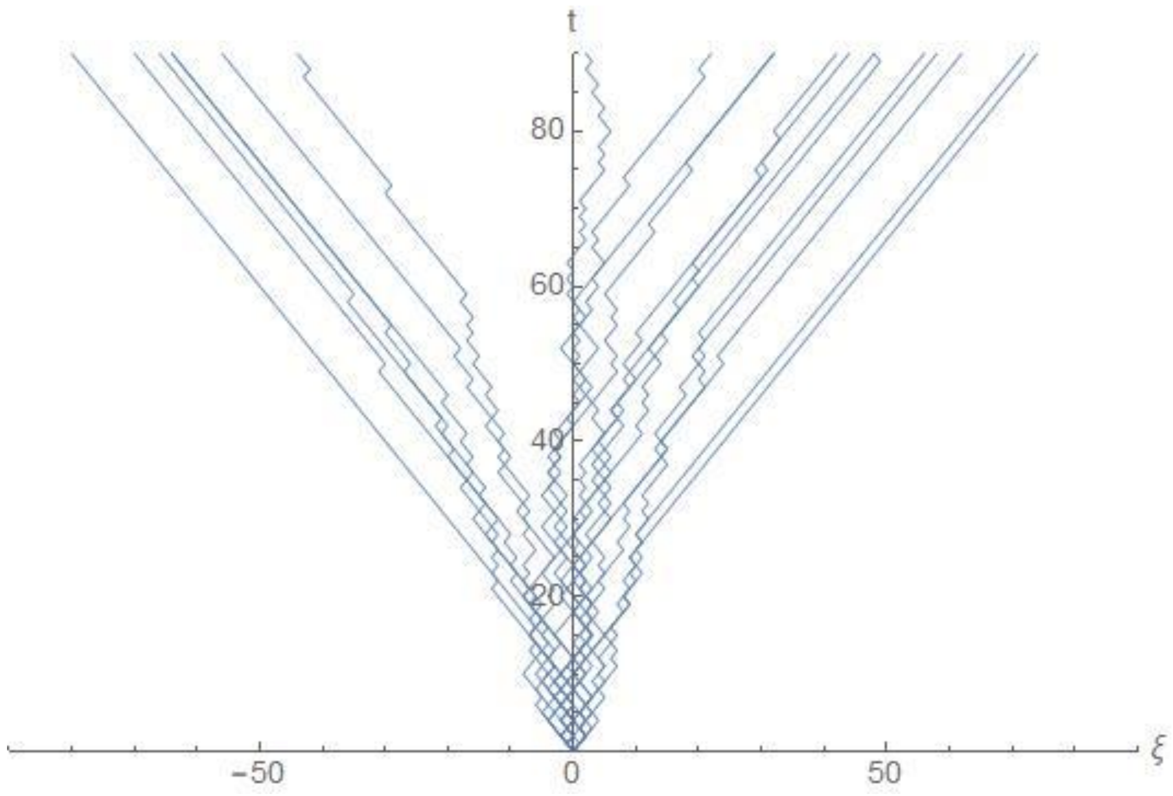
(a,b;q)-拡張 first kindランダムウォークのシミュレーション

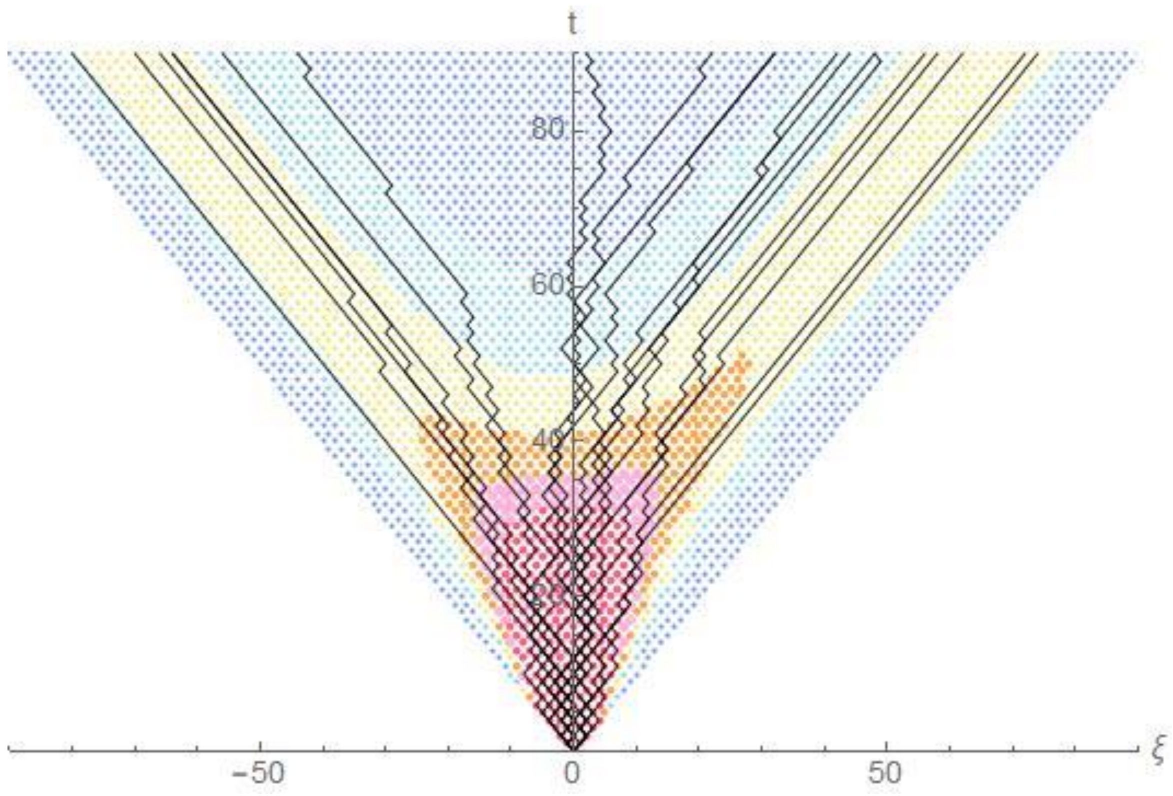


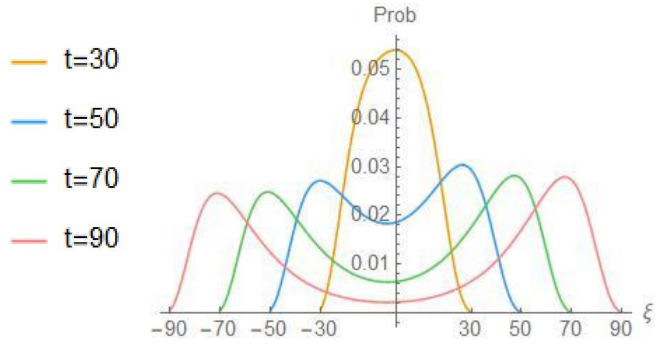


q=0.8, x=0.5, (a,b)=(0.4,-0.6)

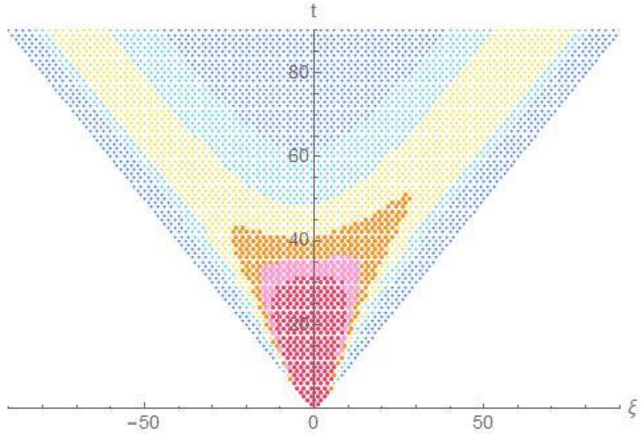
経路を20通りFortranで作成し,
Mathematicaでプロット







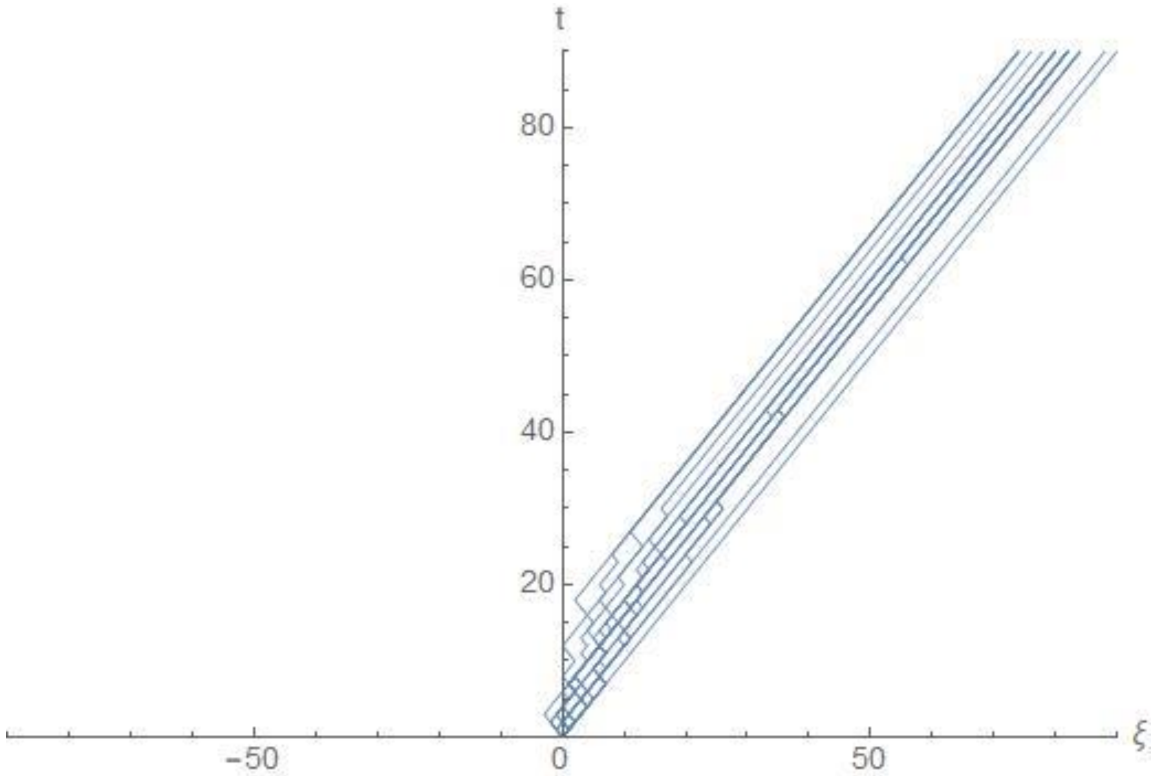
q=0.9, x=0.5, (a,b)=(0.5,-0.7)

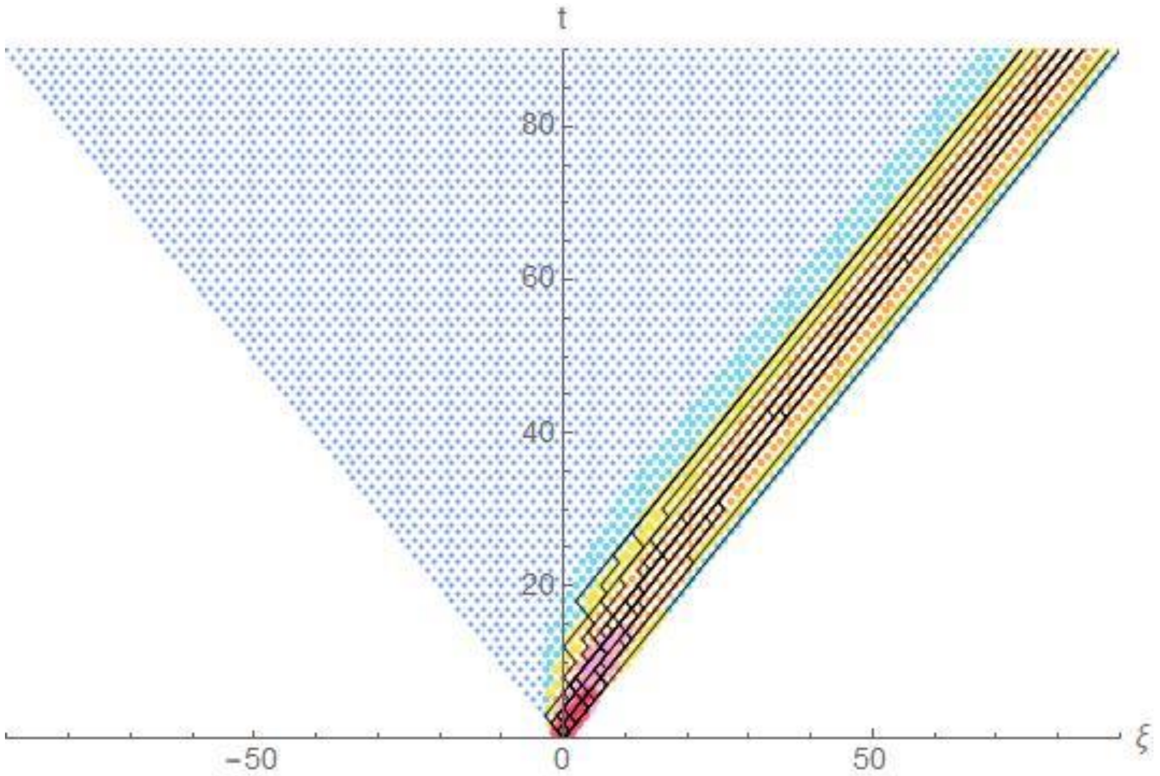


$0 \leq (\text{青}) < 0.01,\ 0.01 \leq (\text{水色}) < 0.02,$

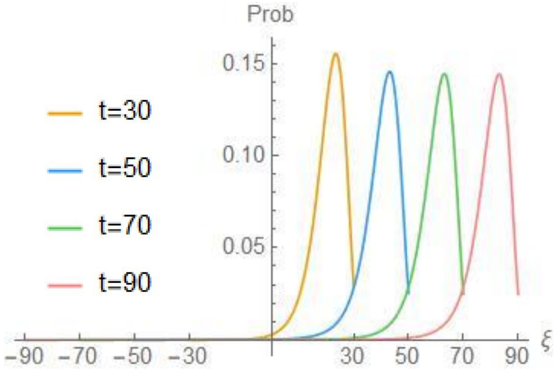
$0.02 \leq (\text{黄色}) < 0.03,\ 0.03 \leq (\text{橙色}) < 0.04,$

$0.04 \leq (\text{ピンク}) < 0.05,\ 0.05 \leq (\text{赤}) \leq 1.$

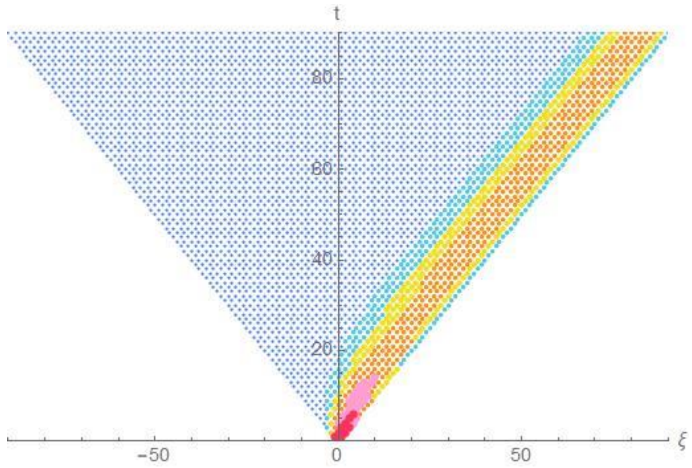


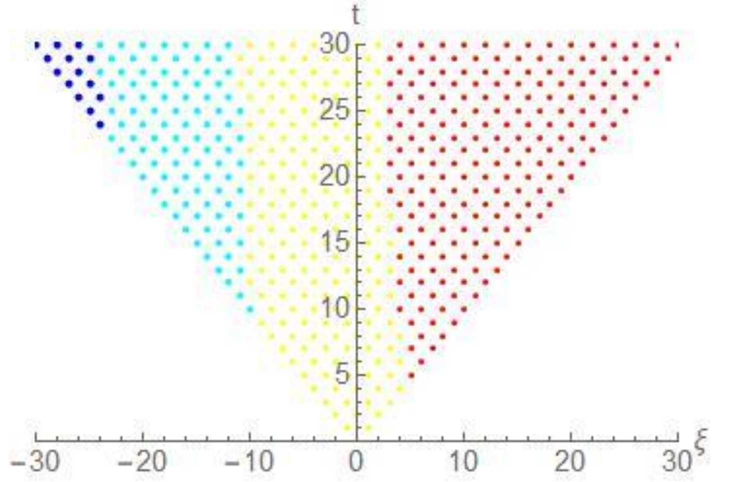


q=0.9, x=0.5, (a,b)=(0.2,-0.7)



$0 \leq$ (青) < 0.015, $0.015 \leq$ (水色) < 0.05,
$0.05 \leq$ (黄色) < 0.1, $0.1 \leq$ (橙色) < 0.2,
$0.2 \leq$ (ピンク) < 0.3, $0.3 \leq$ (赤) ≤ 1.





$0 \leq$ (青) < 0.2, $0.2 \leq$ (水色) < 0.5,
$0.5 \leq$ (黄色) < 0.8, $0.8 \leq$ (赤) ≤ 1.

4. 参考文献

[1] Chaundy, T. W., Bullard, J. E.: John Smith's problem. Math. Gazette **44**, 253--260 (1960)

[2] Koornwinder, T. H., Schlosser, M. J.: On an identity by Chaundy and Bullard. I. Indag. Mathem., N.S. **19** (2), 239--261 (2008)

[3] Koornwinder, T. H., Schlosser, M. J.: On an identity by Chaundy and Bullard. II. More history. Indag. Mathem., **24**, 174--180 (2013)

[4] Schlosser, M. J.: Elliptic enumeration of nonintersecting lattice paths. J. Combin. Theory Ser. A **114**, 505--521 (2007)