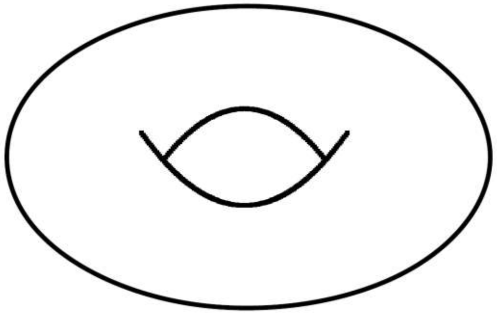
Orthogonal Theta Functions Associated with Affine Root Systems and Determinantal Point Processes

Makoto KATORI

(Chuo Univ., Tokyo)



Conference

Modern Analysis Related to Root Systems with Applications

18th – 22nd October , 2021

CIRM - Luminy

Centre International de Rencontres Mathématiques

Marseille, FRANCE

Plan

1. R_n Theta Functions of Rosengren and Schlosser

2. Orthogonality of R_n Theta Functions
  2.1 Orthonormal A_{n-1} Theta Function
  2.2 Other Orthonormal R_n Theta Functions
  2.3 Doubly-Quasi-Periodicity

3. Determinantal Point Processes (DPPs) on a 2-dim Torus and their Infinite Particle Limits
  3.1 A Brief Review of DPPs
  3.2 DPPs on a 2-dim Torus
  3.3 Infinite Particle Limits

1. R_n Theta Functions of Rosengren and Schlosser

- **We write the complex plane which is punctured at the origin as**

$$\mathbb{C}^\times := \mathbb{C} \setminus \{0\} = \{\zeta \in \mathbb{C} : 0 < |\zeta| < \infty\}.$$

- **Let $p \in \mathbb{C}$ be a fixed number so that $0 < |p| < 1$. The theta function with argument $\zeta \in \mathbb{C}^\times$ and nome p is defined by**

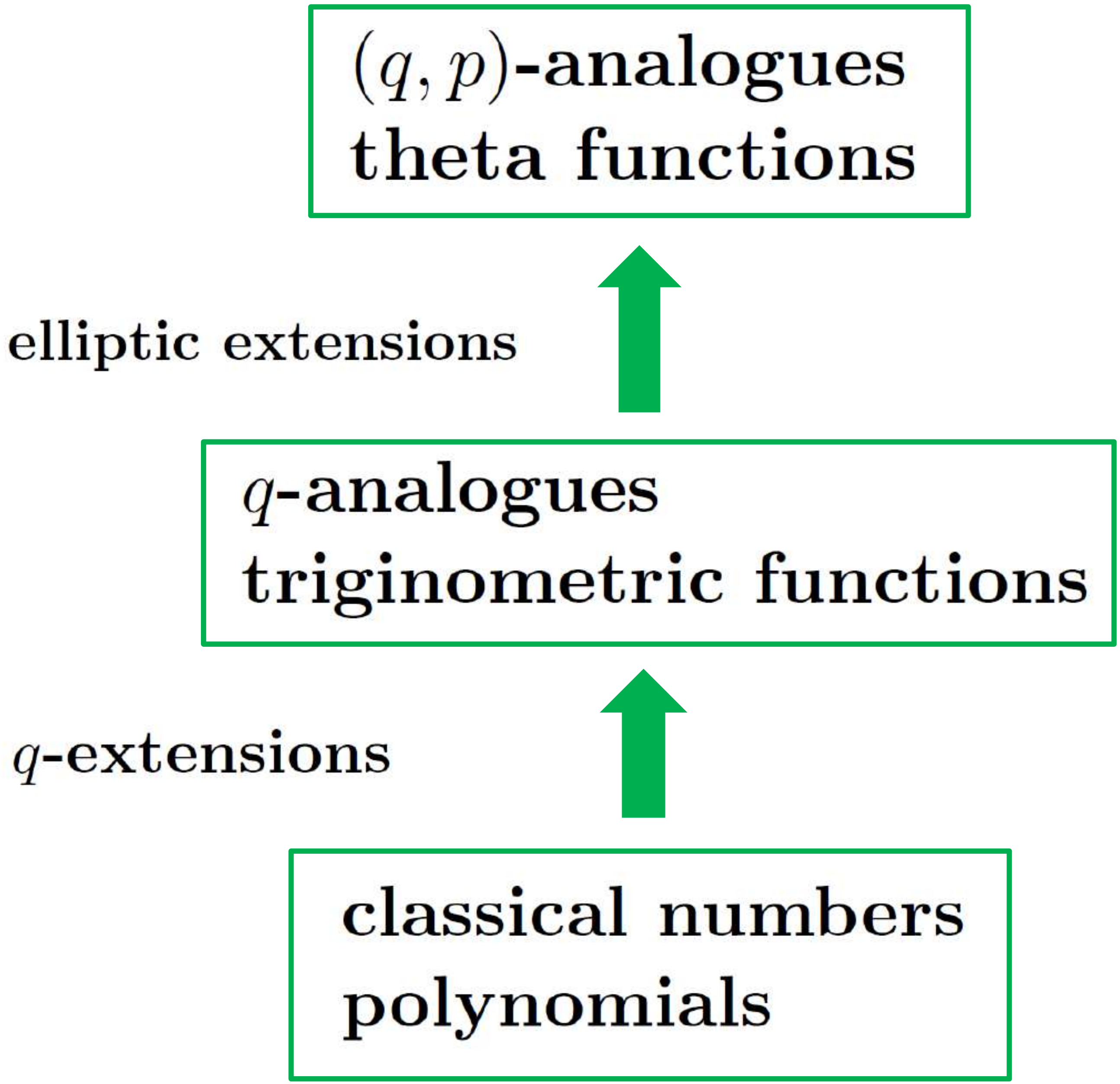
$$\theta(\zeta; p) = \prod_{j=0}^{\infty} \left(1 - \zeta p^j\right)\left(1 - \frac{p^{j+1}}{\zeta}\right).$$

- **By this definition, we can readily see that** $\lim_{p \to 0} \theta(\zeta; p) = 1 - \zeta.$

This implies that $\lim_{p \to 0} \frac{\theta(q^n; p)}{1-q} = \frac{1-q^n}{1-q} =: [n]_q$ **(*q*-analogue of $n \in \mathbb{N}$).**

If we consider $\theta(aq^n; p)$ with $a = e^{-2i\alpha}$ and $q = e^{-2i\phi}$, $\alpha, \phi \in [0, 2\pi)$, $i := \sqrt{-1}$,

$$\lim_{p \to 0} \frac{\theta(aq^n; p)}{2i\sqrt{aq^n}} = \sin(\alpha + n\phi) \quad (\textbf{triginometric function}).$$


(q, p)-analogues
theta functions
elliptic extensions
q-analogues
triginometric functions
q-extensions
classical numbers
polynomials

- **The fact $\lim_{p\to 0}\theta(\zeta;p) = 1-\zeta$ suggests that the theta function $\theta(\zeta;p)$ is an elliptic analogue of a linear function of ζ.**

- **What is the elliptic analogue of a polynomial of ζ?**

- **It might be given by a product of θ's. But should notice the equalities**

$$\theta(\zeta^k;p^k) = \prod_{j=0}^{k-1}\theta(\zeta\omega_k^j;p), \quad \theta(\zeta;p) = \prod_{j=0}^{k-1}\theta(\zeta p^j;p^k), \quad k\in\mathbb{N},$$

  Here ω_k denotes a primitive k-th root of unity.
  The degree of product of θ's depends on a choice of nome.

- **In order to define a degree of products of θ's with respect to a specified nome, Rosengren and Schlosser (2006) generalized the notion of the quasi-periodicity of the theta function,** $\theta(p\zeta;p) = -\dfrac{1}{\zeta}\theta(\zeta;p)$.

- **We notice that we have also the inversion formula $\theta(1/\zeta;p) = -\dfrac{1}{\zeta}\theta(\zeta;p)$, and the combination of these two proves the periodicity of the theta function,** $\theta(p/\zeta;p) = \theta(\zeta,p)$.

Definition 1.1 (Rosengren and Schlosser (2006)) Assume that $f(\zeta)$ is holomorphic in $\mathbb{C}^{\times}$. Then if there is a parameter $r \in \mathbb{C}^{\times}$ and f satisfies the equality,

$$f(p\zeta) = \frac{(-1)^n}{r\zeta^n} f(\zeta),$$

then f is said to be an $\boldsymbol{A}_{n-1}$ theta function of norm r. The space of all $\boldsymbol{A}_{n-1}$ theta functions with nome p and norm r is denoted by $\mathcal{E}_{p,r}^{\boldsymbol{A}_{n-1}}$.

[RS06] Rosengren, H., Schlosser, M.: Elliptic determinant evaluations and the Macdonald identities for affine root systems. Compositio Math. 142, 937–961 (2006)

- By this definition, we can say that $\theta(\zeta; p)$, satisfying the quasi-periodicity, $\theta(p\zeta; p) = -\frac{1}{\zeta}\theta(\zeta; p)$, is an A_0 theta function of norm $r = 1$.

- The following is proved.

Lemma 1.2 (Tarasov and Varchenko (1997)) The space $\mathcal{E}_{p,r}^{\boldsymbol{A}_{n-1}}$ is n dimensional and $\{\psi_j^{\boldsymbol{A}_{n-1}}(\zeta;p,r)\}_{j=1}^n$ defined below form a basis. For $j=1,\dots,n$,

$$\begin{aligned}\psi_j^{\boldsymbol{A}_{n-1}}(\zeta;p,r) &:= \zeta^{j-1}\theta(p^{j-1}(-1)^{n-1}r\zeta^n;p^n)\\ &= \zeta^{j-1}\prod_{k=0}^{n-1}\theta(\alpha^{j-1}\beta\omega_n^k\zeta;p),\end{aligned}$$

where α and β are complex numbers such that $\alpha^n=p$, $\beta^n=(-1)^{n-1}r$, respectively, and ω_n is a primitive n-th root of unity.

[TV97] Tarasov, V., Varchenko, A.: Geometry of q-hypergeometric functions, quantum affine algebras and elliptic quantum groups. Astérisque 246, 135 pages (1997)

- **By** $\lim_{p\to 0}\theta(\zeta;p)=1-\zeta$**, we see that**

$$\psi_j^{\boldsymbol{A}_{n-1}}(\zeta;0,r):=\lim_{p\to 0}\psi_j^{\boldsymbol{A}_{n-1}}(\zeta;p,r)=\begin{cases}1-(-1)^{n-1}r\zeta^n, & j=1,\\ \zeta^{j-1}, & j=2,\dots,n.\end{cases}$$

- **Hence** $\mathcal{E}_{0,r}^{\boldsymbol{A}_{n-1}}$ **spanned by them is a space of polynomials of degree** n **in the form**

$$c_0+c_1\zeta+\cdots+c_n\zeta^n \quad \textbf{with} \quad \frac{c_n}{c_0}\equiv(-1)^{n-1}r.$$

It implies that $\dim\mathcal{E}_{0,r}^{\boldsymbol{A}_{n-1}}=n$**.**

- **It is easy to verify that** $\det_{1\le j,k\le n}\left[\psi_j^{\boldsymbol{A}_{n-1}}(\zeta_k;0,r)\right]=\left(1-r\prod_{\ell=1}^{n}\zeta_\ell\right)W^{\boldsymbol{A}_{n-1}}(\boldsymbol{\zeta})$,

where $\quad W^{\boldsymbol{A}_{n-1}}(\boldsymbol{\zeta}):=\prod_{1\le j<k\le n}(\zeta_k-\zeta_j), \quad \boldsymbol{\zeta}=(\zeta_1,\dots,\zeta_n)\in\mathbb{C}^n$

(Vandermonde determinant =Weyl denominator of type $\boldsymbol{A}_{n-1}$**).**

- **If we set** $r=0$ **as well as** $p=0$, $\mathcal{E}_{0,0}^{\boldsymbol{A}_{n-1}}$ **is the space of all polynomials of degree** $n-1$ **without any restriction on coefficients. In this case, the above is reduced to** $\det_{1\le j,k\le n}\left[\zeta_k^{j-1}\right]=\prod_{1\le j<k\le n}(\zeta_k-\zeta_j)$ **which is known as the Weyl denominator formula of type** $\boldsymbol{A}_{n-1}$**.**

- The elliptic extension of $W^{\boldsymbol{A}_{n-1}}$ is defined as follows.

Definition 1.3 (Macdonald (1972)) For $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n) \in (\mathbb{C}^\times)^n$, the **Macdonald denominator of type A_{n-1}** is defined as

$$M^{\boldsymbol{A}_{n-1}}(\boldsymbol{\zeta}; p) := \prod_{1 \le j < k \le n} \zeta_k \theta(\zeta_j/\zeta_k; p).$$

- It is easy to confirm that $\lim_{p \to 0} M^{\boldsymbol{A}_{n-1}}(\boldsymbol{\zeta}; p) = W^{\boldsymbol{A}_{n-1}}(\boldsymbol{\zeta})$.

- The elliptic extension of the Weyl denominator formula of type $\boldsymbol{A}_{n-1}$ was proved by Rosengre and Schlosser [RS06]. See also Proposition 5.6.3 on page 216 of the textbook of Forrester [For10].

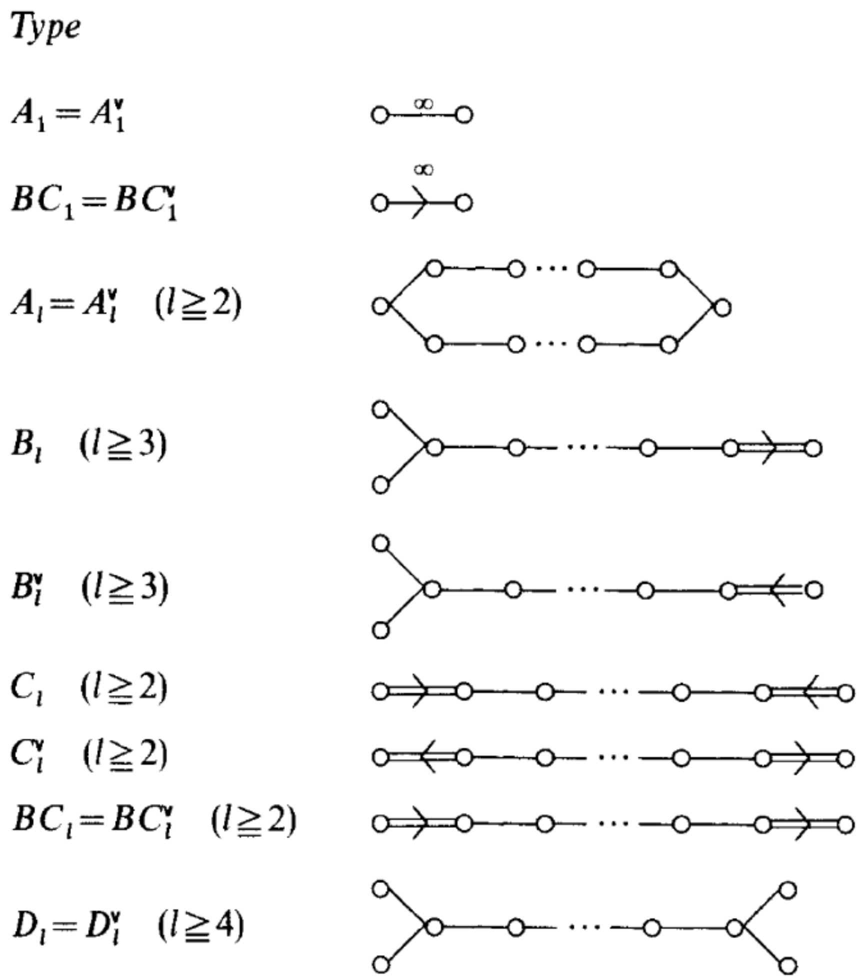
Proposition 1.4 (Rosengren and Schlosser (2006))

$$\det_{1 \le j,k \le n} \left[\psi_j^{A_{n-1}}(\zeta_k; p, r) \right] = \frac{(p;p)_\infty^n}{(p^n;p^n)_\infty^n} \theta\Big(r \prod_{\ell=1}^n \zeta_\ell; p \Big) M^{A_{n-1}}(\boldsymbol{\zeta}; p).$$

[Mac72] Macdonald, I. G.: Affine root systems and Dedekind's η-function. Invent. Math. 15, 91–143 (1972)

[For10] Forrester, P. J.: Log-Gases and Random Matrices. Princeton University Press, Princeton, NJ, (2010)

- There are seven infinite families of irreducible reduced affine root systems, which are denoted by A, B, $B^{\vee}$, C, $C^{\vee}$, BC and D [Mac72].



[Mac72] Macdonald, I. G.: Affine root systems and Dedekind's η-function. Invent. Math. 15, 91–143 (1972)

- There are seven infinite families of irreducible reduced affine root systems, which are denoted by A, B, $B^\vee$, C, $C^\vee$, BC and D [Mac72].

- Associated with them, Rosengren and Schlosser [RS06] defined seven types of theta functions. (The A_{n-1} theta function was already explained.)

Definition 1.5 (Rosengren and Schlosser (2006)) Assume that $f(\zeta)$ is holomorphic in $\mathbb{C}^\times$. For $R_n = B_n$, $B_n^\vee$, C_n, $C_n^\vee$, BC_n, D_n, we call f an R_n theta function if the following are satisfied,

$$f(p\zeta) = -\frac{1}{p^{n-1}\zeta^{2n-1}} f(\zeta), \quad f(1/\zeta) = -\frac{1}{\zeta} f(\zeta), \quad \text{if } R_n = B_n,$$
$$f(p\zeta) = -\frac{1}{p^{n}\zeta^{2n}} f(\zeta), \quad f(1/\zeta) = -f(\zeta), \quad \text{if } R_n = B_n^\vee,$$
$$f(p\zeta) = \frac{1}{p^{n+1}\zeta^{2n+2}} f(\zeta), \quad f(1/\zeta) = -f(\zeta), \quad \text{if } R_n = C_n,$$
$$f(p\zeta) = \frac{1}{p^{n-1/2}\zeta^{2n}} f(\zeta), \quad f(1/\zeta) = -\frac{1}{\zeta} f(\zeta), \quad \text{if } R_n = C_n^\vee,$$
$$f(p\zeta) = \frac{1}{p^{n}\zeta^{2n+1}} f(\zeta), \quad f(1/\zeta) = -\frac{1}{\zeta} f(\zeta), \quad \text{if } R_n = BC_n,$$
$$f(p\zeta) = \frac{1}{p^{n-1}\zeta^{2n-2}} f(\zeta), \quad f(1/\zeta) = f(\zeta), \quad \text{if } R_n = D_n.$$

The space of all R_n theta functions with nome p is denoted by $\mathcal{E}_p^{R_n}$.

- **In order to clarify the common structure, we introduce the notations,**

$$\mathcal{N} = \mathcal{N}^{\boldsymbol{R}_n} := \begin{cases} 2n-1, & \boldsymbol{R}_n = \boldsymbol{B}_n, \\ 2n, & \boldsymbol{R}_n = \boldsymbol{B}_n^{\vee}, \boldsymbol{C}_n^{\vee}, \\ 2n+2, & \boldsymbol{R}_n = \boldsymbol{C}_n, \\ 2n+1, & \boldsymbol{R}_n = \boldsymbol{BC}_n, \\ 2n-2, & \boldsymbol{R}_n = \boldsymbol{D}_n, \end{cases}$$

$$a = a^{\boldsymbol{R}_n} := \begin{cases} 1, & \boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{C}_n^{\vee}, \boldsymbol{BC}_n, \\ 0, & \boldsymbol{R}_n = \boldsymbol{B}_n^{\vee}, \boldsymbol{C}_n, \boldsymbol{D}_n, \end{cases}$$

$$\sigma_1 = \sigma_1^{\boldsymbol{R}_n} := \begin{cases} -1, & \boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{B}_n^{\vee}, \\ 1, & \boldsymbol{R}_n = \boldsymbol{C}_n, \boldsymbol{C}_n^{\vee}, \boldsymbol{BC}_n, \boldsymbol{D}_n. \end{cases}$$

$$\sigma_2 = \sigma_2^{\boldsymbol{R}_n} := \begin{cases} -1, & \boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{B}_n^{\vee}, \boldsymbol{C}_n, \boldsymbol{C}_n^{\vee}, \boldsymbol{BC}_n, \\ 1, & \boldsymbol{D}_n. \end{cases}$$

- **Then the above equations are simply expressed as**

$$f(p\zeta) = \sigma_1 \frac{f(\zeta)}{p^{(\mathcal{N}-a)/2}\zeta^{\mathcal{N}}}, \qquad f(1/\zeta) = \sigma_2 \frac{1}{\zeta^a} f(\zeta).$$

- **In addition to the above, we put**

$$\alpha_j = \alpha_j^{\boldsymbol{R}_n} := \begin{cases} j-n, & \boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{C}_n^\vee, \boldsymbol{BC}_n, \boldsymbol{D}_n, \\ j-n-1, & \boldsymbol{R}_n = \boldsymbol{B}_n^\vee, \boldsymbol{C}_n, \end{cases}$$

$$\beta_j(p) = \beta_j^{\boldsymbol{R}_n}(p) := -\sigma_1 p^{\alpha_j + (\mathcal{N}-a)/2}, \quad \boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{B}_n^\vee, \boldsymbol{C}_n, \boldsymbol{C}_n^\vee, \boldsymbol{BC}_n, \boldsymbol{D}_n,$$

$j = 1, \ldots, n$.

Lemma 1.6 (Rosengren and Schlosser (2006)) For $\boldsymbol{R}_n = \boldsymbol{B}_n$, $\boldsymbol{B}_n^\vee$, $\boldsymbol{C}_n$, $\boldsymbol{C}_n^\vee$, $\boldsymbol{BC}_n$, $\boldsymbol{D}_n$, the space $\mathcal{E}_p^{\boldsymbol{R}_n}$ is n dimensional and a basis is formed by

$$\psi_j^{\boldsymbol{R}_n}(\zeta;p) = \zeta^{\alpha_j}\theta(\beta_j(p)\zeta^{\mathcal{N}};p^{\mathcal{N}}) + \sigma_2\zeta^{-(\alpha_j-a)}\theta(\beta_j(p)\zeta^{-\mathcal{N}};p^{\mathcal{N}}), \quad j = 1, \ldots, n.$$

- **The explicit expressions are given below;**

$$\psi_j^{\boldsymbol{B}_n}(\zeta;p) := \zeta^{j-n}\theta(p^{j-1}\zeta^{2n-1};p^{2n-1}) - \zeta^{n+1-j}\theta(p^{j-1}\zeta^{1-2n};p^{2n-1}),$$
$$\psi_j^{\boldsymbol{B}_n^\vee}(\zeta;p) := \zeta^{j-n-1}\theta(p^{j-1}\zeta^{2n};p^{2n}) - \zeta^{n+1-j}\theta(p^{j-1}\zeta^{-2n};p^{2n}),$$
$$\psi_j^{\boldsymbol{C}_n}(\zeta;p) := \zeta^{j-n-1}\theta(-p^{j}\zeta^{2n+2};p^{2n+2}) - \zeta^{n+1-j}\theta(-p^{j}\zeta^{-2n-2};p^{2n+2}),$$
$$\psi_j^{\boldsymbol{C}_n^\vee}(\zeta;p) := \zeta^{j-n}\theta(-p^{j-1/2}\zeta^{2n};p^{2n}) - \zeta^{n+1-j}\theta(-p^{j-1/2}\zeta^{-2n};p^{2n}),$$
$$\psi_j^{\boldsymbol{BC}_n}(\zeta;p) := \zeta^{j-n}\theta(-p^{j}\zeta^{2n+1};p^{2n+1}) - \zeta^{n+1-j}\theta(-p^{j}\zeta^{-2n-1};p^{2n+1}),$$
$$\psi_j^{\boldsymbol{D}_n}(\zeta;p) := \zeta^{j-n}\theta(-p^{j-1}\zeta^{2n-2};p^{2n-2}) + \zeta^{n-j}\theta(-p^{j-1}\zeta^{-2n+2};p^{2n-2}).$$

- **In addition to the Macdonald denominator of type A_{n-1} given by Definition 1.3, the following other six kinds of Macdonald denominators are defined.**

Definition 1.7 (Macdonald (1972)) **For** $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n) \in (\mathbb{C}^\times)^n$, **let**

$$M^{B_n}(\boldsymbol{\zeta}; p) := \prod_{\ell=1}^{n} \theta(\zeta_\ell; p) \prod_{1 \le j < k \le n} \zeta_j^{-1} \theta(\zeta_j \zeta_k; p) \theta(\zeta_j / \zeta_k; p),$$

$$M^{B_n^\vee}(\boldsymbol{\zeta}; p) := \prod_{\ell=1}^{n} \zeta_\ell^{-1} \theta(\zeta_\ell^2; p^2) \prod_{1 \le j < k \le n} \zeta_j^{-1} \theta(\zeta_j \zeta_k; p) \theta(\zeta_j / \zeta_k; p),$$

$$M^{C_n}(\boldsymbol{\zeta}; p) := \prod_{\ell=1}^{n} \zeta_\ell^{-1} \theta(\zeta_\ell^2; p) \prod_{1 \le j < k \le n} \zeta_j^{-1} \theta(\zeta_j \zeta_k; p) \theta(\zeta_j / \zeta_k; p),$$

$$M^{C_n^\vee}(\boldsymbol{\zeta}; p) := \prod_{\ell=1}^{n} \theta(\zeta_\ell; p^{1/2}) \prod_{1 \le j < k \le n} \zeta_j^{-1} \theta(\zeta_j \zeta_k; p) \theta(\zeta_j / \zeta_k; p),$$

$$M^{BC_n}(\boldsymbol{\zeta}; p) := \prod_{\ell=1}^{n} \theta(\zeta_\ell; p) \theta(p \zeta_\ell^2; p^2) \prod_{1 \le j < k \le n} \zeta_j^{-1} \theta(\zeta_j \zeta_k; p) \theta(\zeta_j / \zeta_k; p),$$

$$M^{D_n}(\boldsymbol{\zeta}; p) := \prod_{1 \le j < k \le n} \zeta_j^{-1} \theta(\zeta_j \zeta_k; p) \theta(\zeta_j / \zeta_k; p).$$

They are called the Macdonald denominators of type R_n for $R_n = B_n$, $B_n^\vee$, C_n, $C_n^\vee$, BC_n, D_n, respectively.

- **The above are regarded as the elliptic extensions of the Weyl denominators of types B_n, C_n and D_n,**

$$W^{B_n}(\zeta) = \det_{1\le j,k\le n}\left[\zeta_k^{j-n} - \zeta_k^{n+1-j}\right] = \prod_{\ell=1}^{n}\zeta_\ell^{1-n}(1-\zeta_\ell)\prod_{1\le j<k\le n}(\zeta_k-\zeta_j)(1-\zeta_j\zeta_k)$$

$$= \lim_{p\to 0} M^{B_n}(\zeta;p) = \lim_{p\to 0} M^{C_n^\vee}(\zeta;p) = \lim_{p\to 0} M^{BC_n}(\zeta;p),$$

$$W^{C_n}(\zeta) = \det_{1\le j,k\le n}\left[\zeta_k^{j-n-1} - \zeta_k^{n+1-j}\right] = \prod_{\ell=1}^{n}\zeta_\ell^{-n}(1-\zeta_\ell^2)\prod_{1\le j<k\le n}(\zeta_k-\zeta_j)(1-\zeta_j\zeta_k)$$

$$= \lim_{p\to 0} M^{B_n^\vee}(\zeta;p) = \lim_{p\to 0} M^{C_n}(\zeta;p),$$

$$W^{D_n}(\zeta) = \det_{1\le j,k\le n}\left[\zeta_k^{j-n} + \zeta_k^{n-j}\right] = 2\prod_{\ell=1}^{n}\zeta_\ell^{1-n}\prod_{1\le j<k\le n}(\zeta_k-\zeta_j)(1-\zeta_j\zeta_k)$$

$$= 2\lim_{p\to 0} M^{D_n}(\zeta;p).$$

Notice the degeneracy in the limit $p \to 0$.

- **Rosengren and Schlosser proved the following.**

Proposition 1.8 (Rosengren and Schlosser (2006)) **The following equalities hold for $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in (\mathbb{C}^\times)^n$,**

$$\det_{1\le j,k\le n}\left[\psi_j^{\boldsymbol{B}_n}(\zeta_k;p)\right] = \frac{2(p;p)_\infty^n}{(p^{2n-1};p^{2n-1})_\infty^n} M^{\boldsymbol{B}_n}(\boldsymbol{\zeta};p),$$

$$\det_{1\le j,k\le n}\left[\psi_j^{\boldsymbol{B}_n^\vee}(\zeta_k;p)\right] = \frac{2(p^2;p^2)_\infty(p;p)_\infty^{n-1}}{(p^{2n};p^{2n})_\infty^n} M^{\boldsymbol{B}_n^\vee}(\boldsymbol{\zeta};p),$$

$$\det_{1\le j,k\le n}\left[\psi_j^{\boldsymbol{C}_n}(\zeta_k;p)\right] = \frac{(p;p)_\infty^n}{(p^{2n+2};p^{2n+2})_\infty^n} M^{\boldsymbol{C}_n}(\boldsymbol{\zeta};p),$$

$$\det_{1\le j,k\le n}\left[\psi_j^{\boldsymbol{C}_n^\vee}(\zeta_k;p)\right] = \frac{(p^{1/2};p^{1/2})_\infty(p;p)_\infty^{n-1}}{(p^{2n},p^{2n})_\infty^n} M^{\boldsymbol{C}_n^\vee}(\boldsymbol{\zeta};p),$$

$$\det_{1\le j,k\le n}\left[\psi_j^{\boldsymbol{BC}_n}(\zeta_k;p)\right] = \frac{(p;p)_\infty^n}{(p^{2n+1};p^{2n+1})_\infty^n} M^{\boldsymbol{BC}_n}(\boldsymbol{\zeta};p),$$

$$\det_{1\le j,k\le n}\left[\psi_j^{\boldsymbol{D}_n}(\zeta_k;p)\right] = \frac{4(p;p)_\infty^n}{(p^{2n-2};p^{2n-2})_\infty^n} M^{\boldsymbol{D}_n}(\boldsymbol{\zeta};p).$$

[RS06] Rosengren, H., Schlosser, M.: Elliptic determinant evaluations and the Macdonald identities for affine root systems. Compositio Math. 142, 937–961 (2006)

2. Orthogonality of *Rn* Theta Functions

- **Since the theta function $\theta(\zeta;p)$ is holomorphic for $\zeta \in \mathbb{C}^{\times}$, it allows the Laurent expansion,**

$$\theta(\zeta;p) = \frac{1}{(p;p)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n p^{\binom{n}{2}} \zeta^n,$$

where

$$(p;p)_\infty := \prod_{j=1}^{\infty} (1-p^j).$$

- **From now on we will assume $p \in \mathbb{R}$ and $r \in \mathbb{R}$. That is,**

$$p \in (0,1), \qquad r \in \mathbb{R} \setminus \{0\}.$$

- **In this case, $\overline{\theta(\zeta;p)} = \theta(\overline{\zeta};p)$.**

2.1 Orthonormal A_{n-1} Theta Function

- **We write the unit circle on the complex plane as**

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\} = \mathbb{R}/2\pi\mathbb{Z} \text{ (one-dimensional torus)}.$$

Each point in $\mathbb{T}$ is expressed by $e^{ix}, x \in [0, 2\pi)$.

- **For the space $\mathcal{E}_{p,r}^{\boldsymbol{A}_{n-1}}$ of the $\boldsymbol{A}_{n-1}$ theta functions, we introduce the following inner product,**

$$\begin{aligned}\langle f, g\rangle_{\mathbb{T}} &:= \frac{1}{2\pi}\int_{|\zeta|=1} f(\zeta)\overline{g(\zeta)}\ell(d\zeta) \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})\overline{g(e^{ix})}dx, \quad f, g \in \mathcal{E}_{p,r}^{\boldsymbol{A}_{n-1}},\end{aligned}$$

where ℓ denotes the arc length measure on $\mathbb{T}$ normalized as $\ell(\mathbb{T}) = 2\pi$.

Proposition 2.1 **Let** $p, \widehat{p} \in (0,1)$ **and** $r, \widehat{r} \in \mathbb{R} \setminus \{0\}$**. Then**

$$\langle \psi_j^{\boldsymbol{A}_{n-1}}(\cdot; p, r), \psi_k^{\boldsymbol{A}_{n-1}}(\cdot; \widehat{p}, \widehat{r}) \rangle_{\mathbb{T}} = h_j^{\boldsymbol{A}_{n-1}}(p, \widehat{p}, r\widehat{r}) \delta_{jk},$$

for $j, k = 1, \dots, n$**, where**

$$h_j^{\boldsymbol{A}_{n-1}}(p, \widehat{p}, r\widehat{r}) = \frac{((p\widehat{p})^n; (p\widehat{p})^n)_\infty}{(p^n; p^n)_\infty (\widehat{p}^n, \widehat{p}^n)_\infty} \theta(-(r\widehat{r})(p\widehat{p})^{j-1}; (p\widehat{p})^n).$$

Proof. **We apply the Laurent expansion of the theta function and obtain**

$$\psi_j^{\boldsymbol{A}_{n-1}}(e^{ix}; p, r) = \frac{e^{i(j-1)x}}{(p^n; p^n)_\infty} \sum_{\ell \in \mathbb{Z}} (-1)^\ell p^{n\binom{\ell}{2}} (-1)^{(n-1)\ell} r^\ell p^{(j-1)\ell} e^{in\ell x},$$

$$\overline{\psi_k^{\boldsymbol{A}_{n-1}}(e^{ix}; \widehat{p}, \widehat{r})} = \frac{e^{-i(k-1)x}}{(\widehat{p}^n; \widehat{p}^n)_\infty} \sum_{m \in \mathbb{Z}} (-1)^m \widehat{p}^{n\binom{m}{2}} (-1)^{(n-1)m} \widehat{r}^m \widehat{p}^{(k-1)m} e^{-inmx}.$$

The inner product of them includes the integrals

$$I_{jk,\ell m}^{\boldsymbol{A}_{n-1}} := \frac{1}{2\pi} \int_0^{2\pi} e^{i\{(j-k)+n(\ell-m)\}x} dx$$

as $\langle \psi_j^{\boldsymbol{A}_{n-1}}(\cdot; p, r), \psi_k^{\boldsymbol{A}_{n-1}}(\cdot; \widehat{p}, \widehat{r}) \rangle_{\mathbb{T}}$

$$= \frac{1}{(p^n; p^n)_\infty (\widehat{p}^n; \widehat{p}^n)_\infty} \sum_{\ell \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} (-1)^{n(\ell+m)} p^{n\binom{\ell}{2}} \widehat{p}^{n\binom{m}{2}} r^\ell \widehat{r}^m p^{(j-1)\ell} \widehat{p}^{(k-1)m} I_{jk,\ell m}^{\boldsymbol{A}_{n-1}}.$$

It is easy to verify that

$$\psi_j^{\boldsymbol{A}_{n-1}}(\zeta; p, r) := \zeta^{j-1}\theta(p^{j-1}(-1)^{n-1}r\zeta^n; p^n)$$

$$I_{jk,\ell m}^{\boldsymbol{A}_{n-1}} := \frac{1}{2\pi}\int_0^{2\pi} e^{i\{(j-k)+n(\ell-m)\}x}dx = \mathbf{1}((j-k)+n(\ell-m)=0).$$

Since $\ell, m \in \mathbb{Z}$, while $j, k \in \{1, \dots, n\}$, $|j-k| \le n-1 < n$ and thus the nonzero condition of $I_{jk,\ell m}^{\boldsymbol{A}_{n-1}}$ is satisfied if and only if $j-k=0$ and $\ell - m = 0$. That is, we have the equalities,

$$I_{jk,\ell m}^{\boldsymbol{A}_{n-1}} = \delta_{jk}\delta_{\ell m} \quad \text{for } j,k \in \{1,\dots,n\},\ \ell, m \in \mathbb{Z}.$$

Therefore, we obtain

$$\begin{aligned}
&\langle \psi_j^{\boldsymbol{A}_{n-1}}(\cdot; p, r), \psi_k^{\boldsymbol{A}_{n-1}}(\cdot; \widehat{p}, \widehat{r})\rangle_{\mathbb{T}} \\
&\quad = \frac{\delta_{jk}}{(p^n; p^n)_\infty(\widehat{p}^n; \widehat{p}^n)_\infty} \sum_{\ell \in \mathbb{Z}} (p\widehat{p})^{n\binom{\ell}{2}} (r\widehat{r})^\ell (p\widehat{p})^{(j-1)\ell}.
\end{aligned}$$

Again we use the Laurent-series expression of the theta function and the assertion is proved. ∎

- **When $p \neq \widehat{p}$, $r \neq \widehat{r}$, we have two distinct sets of functions $\{\psi_j^{\boldsymbol{A}_{n-1}}(\cdot; p, r)\}_{j=1}^n$ and $\{\psi_j^{\boldsymbol{A}_{n-1}}(\cdot; \widehat{p}, \widehat{r})\}_{j=1}^n$. In such a general case, the property**

$$\langle \psi_j^{\boldsymbol{A}_{n-1}}(\cdot; p, r), \psi_k^{\boldsymbol{A}_{n-1}}(\cdot; \widehat{p}, \widehat{r}) \rangle_{\mathbb{T}} = h_j^{\boldsymbol{A}_{n-1}}(p, \widehat{p}, r\widehat{r}) \delta_{jk}, \ j, k = 1, \dots, n,$$

shall be called biorthononality.

- **As a special case with $p = \widehat{p}$ and $r = \widehat{r}$, $\{\psi_j^{\boldsymbol{A}_{n-1}}(\cdot; p, r)\}_{j=1}^n$ makes an orthogonal basis of $\mathcal{E}_{p,r}^{\boldsymbol{A}_{n-1}}$. Here we consider this simplified situation.**

- **But now we replace $x \in \mathbb{R}$ by a complex argument $z = x + iy$, $x, y \in \mathbb{R}$.**

Lemma 2.2 **Let $p \in (0, 1)$ and $r \in \mathbb{R} \setminus \{0\}$. For $j, k = 1, \dots, n$, the orthogonal relations hold;**

$$\frac{1}{2\pi} \int_0^{2\pi} \psi_j^{\boldsymbol{A}_{n-1}}(e^{i(x+iy)}; p, r) \overline{\psi_k^{\boldsymbol{A}_{n-1}}(e^{i(x+iy)}; p, r)} dx = \widetilde{h}_j^{\boldsymbol{A}_{n-1}}(y; p, r) \delta_{jk},$$

with $\widetilde{h}_j^{\boldsymbol{A}_{n-1}}(y; p, r) = e^{-2(j-1)y} \dfrac{(p^{2n}; p^{2n})_\infty}{(p^n; p^n)_\infty^2} \theta(-r^2 e^{-2ny} p^{2(j-1)}; p^{2n}).$

<u>Proof.</u> **Just note** $\psi_j^{\boldsymbol{A}_{n-1}}(e^{i(x+iy)}; p, r) = e^{-(j-1)y} \psi_j^{\boldsymbol{A}_{n-1}}(e^{ix}; p, re^{-ny})$. ∎

- **For the nome $p \in \mathbb{C}$, $|p| < 1$, we define the nome modular parameter τ by**

$$p = e^{2\pi i \tau} =: p(\tau).$$

Lemma 2.3 (Jacobi's imaginary transformation) We define

$$\widetilde{p} := p(-1/\tau) = e^{-2\pi i/\tau}.$$

Then the following equality holds for $\zeta \in \mathbb{C}$,

$$\theta(e^{i\zeta}; p) = e^{\pi i/4} \frac{\widetilde{p}^{1/8}(\widetilde{p}; \widetilde{p})_\infty}{p^{1/8}(p; p)_\infty} \tau^{-1/2} e^{-i\zeta^2/4\pi\tau} e^{i\zeta/2} e^{-i\zeta/2\tau} \theta(e^{-i\zeta/\tau}; \widetilde{p}).$$

- **Since we have assumed $p \in (0,1)$, τ is pure imaginary and written as $\tau = i|\tau|$; that is, $p = e^{-2\pi|\tau|}$.**
- **Moreover, we fix the norm r as**

$$r = (-1)^n e^{-n\pi|\tau|} = (-1)^n p^{n/2}.$$

- **Then we find the following;**

$$\begin{aligned}\widetilde{h}_j^{\boldsymbol{A}_{n-1}}(y;p,r) &= C_1(n|\tau|)(e^{-\pi/n|\tau|}; e^{-\pi/n|\tau|})_\infty e^{2\pi|\tau|(j-1)^2/n + ny^2/2\pi|\tau|} \\ &\times \theta(C_2(n|\tau|, (j-1+n/2)|\tau|)e^{-iy/|\tau|}; e^{-\pi/n|\tau|}), \quad j = 1, \dots, n,\end{aligned}$$

where

$$C_1(t) := \frac{1}{\sqrt{2t}} \frac{e^{-\pi/t}}{(e^{-2\pi t}; e^{-2\pi t})_\infty^2}, \quad C_2(t,s) := e^{-(4is-1)\pi/2t}.$$

- There are three important points;

$$\widetilde{h}_j^{\boldsymbol{A}_{n-1}}(y;p,r) \propto \exp\left(\frac{2\pi|\tau|}{n}(j-1)^2 + \frac{n}{2\pi|\tau|}y^2\right)\theta\left(C_2 e^{-iy/|\tau|}; e^{-\pi/n|\tau|}\right).$$

Lemma 2.4 **Assume** $p = e^{-2\pi|\tau|}$ **and** $r = (-1)^n p^{n/2}$**. Then**

$$\frac{1}{2\pi|\tau|}\int_0^{2\pi|\tau|} e^{-ny^2/2\pi|\tau|}\widetilde{h}_j^{\boldsymbol{A}_{n-1}}(y;p,r)dy = C_1(n|\tau|)e^{2\pi|\tau|(j-1)^2/n}.$$

Proof. **When** C **does not depend on** y**,**

$$\frac{1}{2\pi|\tau|}\int_0^{2\pi|\tau|}\theta(Ce^{\pm iy/|\tau|};p)dy = \frac{1}{(p;p)_\infty}\sum_{k\in\mathbb{Z}}(-1)^k p^{\binom{k}{2}}C^k \times \frac{1}{2\pi|\tau|}\int_0^{2\pi|\tau|}e^{\pm iyk/|\tau|}dy$$
$$= \frac{1}{(p;p)_\infty}\sum_{k\in\mathbb{Z}}(-1)^k p^{\binom{k}{2}}C^k\delta_{k0} = \frac{1}{(p;p)_\infty}.$$

Then the assertion is immediately obtained. ∎

- **With** $p = e^{-2\pi|\tau|}$ **and** $C_1(t) = \frac{1}{\sqrt{2t}} \frac{e^{-\pi/t}}{(e^{-2\pi t}; e^{-2\pi t})_\infty^2}$**, define**

$$\begin{aligned} \Psi_j^{\boldsymbol{A}_{n-1}}(z) &= \Psi_j^{\boldsymbol{A}_{n-1}}(z; |\tau|) \\ &:= \frac{e^{-\pi|\tau|(j-1)^2/n}}{\sqrt{C_1(n|\tau|)}} e^{-ny^2/4\pi|\tau|} \psi_j^{\boldsymbol{A}_{n-1}}(e^{iz}; p, (-1)^n p^{n/2}), \quad z = x + iy \in \mathbb{C}. \end{aligned}$$

- **Consider a rectangular domain in** $\mathbb{C}$**,**

$$D_{(2\pi, 2\pi|\tau|)} := \Big\{ z = x + iy \in \mathbb{C} : 0 \le x < 2\pi, 0 \le y < 2\pi|\tau| \Big\}.$$

- **Introduce the inner product for holomorphic functions** f, g **on** $D_{(2\pi,2\pi|\tau|)}$**;**

$$\begin{aligned} \langle f, g \rangle_{D_{(2\pi,2\pi|\tau|)}} &:= \frac{1}{|D_{(2\pi,2\pi|\tau|)}|} \int_{D_{(2\pi,2\pi|\tau|)}} f(z) \overline{g(z)} dz \\ &= \frac{1}{(2\pi)^2 |\tau|} \int_0^{2\pi} dx \int_0^{2\pi|\tau|} dy\, f(x+iy) \overline{g(x+iy)}. \end{aligned}$$

The above results are summarized as follows.

Proposition 2.5 $\langle \Psi_j^{A_{n-1}}, \Psi_k^{A_{n-1}} \rangle_{D_{(2\pi,2\pi|\tau|)}} = \delta_{jk}, \quad j, k = 1, \dots, n.$

2.2 Other Orthonormal R_n Theta Functions

- **Let**

$$J(j) = J^{\boldsymbol{R}_n}(j) := \alpha_j - a/2 = \begin{cases} j-n-1/2, & \boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{C}_n^\vee, \boldsymbol{BC}_n, \\ j-n-1, & \boldsymbol{R}_n = \boldsymbol{B}_n^\vee, \boldsymbol{C}_n, \\ j-n, & \boldsymbol{R}_n = \boldsymbol{D}_n, \end{cases}$$

where $a = a^{\boldsymbol{R}_n}$ **and** $\alpha_j = \alpha_j^{\boldsymbol{R}_n}$ **are defined above, and**

$$c_j = c_j^{\boldsymbol{R}_n} := \begin{cases} 1, \quad j=1,\dots,n, & \boldsymbol{R}_n = \boldsymbol{C}_n, \boldsymbol{C}_n^\vee, \boldsymbol{BC}_n, \\ \begin{cases} 2, & j=1, \\ 1, & j=2,\dots,n, \end{cases} & \boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{B}_n^\vee, \\ \begin{cases} 2, & j=1,n, \\ 1, & j=2,\dots,n-1, \end{cases} & \boldsymbol{R}_n = \boldsymbol{D}_n. \end{cases}$$

Proposition 2.6 For $R_n = B_n,\ B_n^\vee,\ C_n,\ C_n^\vee,\ BC_n,\ D_n$, define

$$\Psi_j^{R_n}(z) = \Psi_j^{R_n}(z;|\tau|) := \frac{e^{-\pi|\tau|J(j)^2/\mathcal{N}}}{\sqrt{2c_j C_1(\mathcal{N}|\tau|)}} e^{-\mathcal{N}y^2/4\pi|\tau|+ay/2}\psi_j^{R_n}(e^{iz};p),$$

$j = 1, \ldots, n$, where $a = a^{R_n}$, $\mathcal{N} = \mathcal{N}^{R_n}$, $c_j = c_j^{R_n}$), and $J(j) = J^{R_n}(j)$ are given above. Then

$$\langle \Psi_j^{R_n}, \Psi_k^{R_n} \rangle_{D_{(2\pi,2\pi|\tau|)}} = \delta_{jk}, \quad j,k = 1, \ldots, n.$$

2.3 Doubly-Quasi-Periodicity

- The orthonormal functions $\{\Psi_j^{\boldsymbol{R}_n}\}_{j=1}^n$ have the following doubly-quasi-periodicity and can be extended for $z \in \mathbb{C}$.

Lemma 2.7 Assume $p = e^{-2\pi|\tau|}$. The following hold,

$$\Psi_j^{\boldsymbol{R}_n}(z+2\pi) = \Psi_j^{\boldsymbol{R}_n}(z),$$

$$\Psi_j^{\boldsymbol{A}_{n-1}}(z+2\pi i|\tau|) = e^{-inx}\Psi_j^{\boldsymbol{R}_n}(z), \quad \Psi_j^{\boldsymbol{R}_n}(z+2\pi i|\tau|) = \sigma_1 e^{-i\mathcal{N}x}\Psi_j^{\boldsymbol{R}_n}(z),$$

$j = 1, \ldots, n$, where $\mathcal{N} = \mathcal{N}^{\boldsymbol{R}_n}$ and $\sigma_1 = \sigma_1^{\boldsymbol{R}_n}$ as above.

Proof. The periodicity with period 2π is obvious, since $\{\Psi_j^{\boldsymbol{R}_n}(z)\}_{j=1}^n$ are functions of e^{iz}. For $\psi_j^{\boldsymbol{A}_{n-1}}(e^{iz}; p, (-1)^n p^{n/2}) \in \mathcal{E}_{p,(-1)^n p^{n/2}}^{\boldsymbol{A}_{n-1}}$, $\psi_j^{\boldsymbol{R}_n}(e^{iz}; p) \in \mathcal{E}_p^{\boldsymbol{R}_n}$,

$$\begin{aligned}
\psi_j^{\boldsymbol{A}_{n-1}}(e^{i(z+2\pi|\tau|i)}; p, (-1)^n p^{n/2}) &= \psi_j^{\boldsymbol{A}_{n-1}}(pe^{iz}; p, (-1)^n p^{n/2}) \\
&= \frac{(-1)^n}{(-1)^n p^{n/2}(e^{iz})^n}\psi_j^{\boldsymbol{A}_{n-1}}(e^{iz}; p, (-1)^n p^{n/2}) = e^{n\pi|\tau|}e^{ny}e^{-inx}\psi_j^{\boldsymbol{A}_{n-1}}(e^{iz}; p, r), \\
\psi_j^{\boldsymbol{R}_n}(e^{i(z+2\pi|\tau|i)}; p) &= \psi_j^{\boldsymbol{R}_n}(pe^{iz}; p) \\
&= \sigma_1 \frac{1}{p^{(\mathcal{N}-a)/2}(e^{iz})^{\mathcal{N}}}\psi_j^{\boldsymbol{R}_n}(e^{iz}; p) = \sigma_1 e^{(\mathcal{N}-a)\pi|\tau|}e^{\mathcal{N}y}e^{-i\mathcal{N}x}\psi_j^{\boldsymbol{R}_n}(e^{iz}; p),
\end{aligned}$$

for others. Irrelevant factors are cancelled by

$$e^{-n(y+2\pi|\tau|)^2/4\pi|\tau|} = e^{-n\pi|\tau|}e^{-ny}e^{-ny^2/4\pi|\tau|} \text{ and}$$

$$e^{-\mathcal{N}(y+2\pi|\tau|)^2/4\pi|\tau|+a(y+2\pi|\tau|)/2} = e^{-(\mathcal{N}-a)\pi|\tau|}e^{-\mathcal{N}y}e^{-\mathcal{N}y^2/4\pi|\tau|+ay/2}. \blacksquare$$

3. Determinantal Point Processes (DPPs) on a 2-Dim Torus and their Infinite Particle Limits

3.1 A Brief Review of DPPs

- Let a space S be a subset of $\mathbb{R}^d$ with $d \in \mathbb{N}$ equipped with a reference measure λ.

- A random point process with n points, $n \in \mathbb{N}$, on a space S is a statistical ensemble of nonnegative integer-valued Radon measures

$$\Xi(\cdot) = \sum_{j=1}^{n} \delta_{X_j}(\cdot).$$

- Here $\delta_y(\cdot), y \in S$ denotes the delta measure such that $\delta_y(\{x\}) = 1$ if $x = y$ and $\delta_y(\{x\}) = 0$ otherwise.

- In general, the configuration space of point process is given by

$$\mathrm{Conf}(S) := \left\{ \xi = \sum_{i} \delta_{x_i} : x_i \in S,\, \xi(\Lambda) < \infty \text{ for all bounded set } \Lambda \subset S \right\}.$$

- **The distribution of points $\{X_j\}_{j=1}^n$ on S is governed by a probability measure P. We assume P has probability density p with respect to $\lambda(dx)$.**

- **That is, for the set of points $\mathbf{X} := (X_1, \dots, X_n)$,**
$$\mathbf{P}(\mathbf{X} \in d\mathbf{x}) = \mathbf{p}(\mathbf{x})\lambda(d\mathbf{x}), \quad d\mathbf{x} \subset S^n.$$
Since the labeling order of n points $\{x_j\}_{j=1}^n$ is irrelevant for point configuration $\xi = \sum_{j=1}^n \delta_{x_j}$, the probability density should be normalized as
$$\frac{1}{n!} \int_{S^n} \mathbf{p}(\mathbf{x})\lambda(d\mathbf{x}) = 1.$$

- **The point process is denoted by a triplet $(\Xi, \mathbf{p}, \lambda(dx))$.**

- **For $(\Xi, \mathbf{p}, \lambda(dx))$, the m-point correlation function, $1 \le m \le n$, is defined by**

$$\rho_m(x_1, \dots, x_m) = \frac{1}{(n-m)!} \int_{S^{n-m}} \mathbf{p}(x_1, \dots, x_m, x_{m+1}, \dots, x_n) \prod_{j=m+1}^{n} \lambda(dx_j),$$

$(x_1, \dots, x_m) \in S^m$. **By definition, for any $m \in \{1, \dots, n\}$, the correlation function ρ_m is a symmetric function on S^m;**
$$\rho_m(x_{\sigma(1)}, \dots, x_{\sigma(m)}) = \rho_m(x_1, \dots, x_m) \quad \textbf{for all } \sigma \in \mathfrak{S}_m.$$

- **Let $\mathcal{B}_c(S)$ be the set of all bounded measurable complex functions on S of compact support. For $\xi \in \mathbf{Conf}(S)$ and $\phi \in \mathcal{B}_c(S)$ we set**

$$\langle \xi, \phi \rangle := \int_S \phi(x)\, \xi(dx) = \sum_{j=1}^{n} \phi(x_j).$$

- **Then the expectation of the random variable $\langle \Xi, \phi \rangle$ with respect to P is given by**

$$\mathbf{E}[\langle \Xi, \phi \rangle] = \int_S \phi(x) \rho_1(x) \lambda(dx).$$

  In other words, the first correlation function $\rho_1(x)$ gives the density of point at $x \in S$ with respect to the reference measure $\lambda(dx)$.

- **For $2 \leq m \leq n$, from $\xi \in \mathbf{Conf}(S)$ we define $\xi_m := \sum_{i_1, \dots, i_m : i_j \neq i_k, j \neq k} \delta_{x_{i_1}} \cdots \delta_{x_{i_m}}$. Then for all $\phi \in \mathcal{B}_c(S^m)$,**

$$\mathbf{E}[\langle \Xi_m, \phi \rangle] = \int_{S^m} \phi(x_1, \dots, x_m) \rho_m(x_1, \dots, x_m) \prod_{j=1}^{m} \lambda(dx_j).$$

- **With $\phi \in \mathcal{B}_c(S)$, $k \in \mathbb{R}$, the characteristic function of $(\Xi, \mathbf{p}, \lambda(dx))$ is defined by**

$$\Psi[\phi;\kappa] := \mathbf{E}\left[e^{\kappa\langle\Xi,\phi\rangle}\right] = \frac{1}{n!}\int_{S^n} e^{\kappa\langle\xi,\phi\rangle}\mathbf{p}(\mathbf{x})\lambda(d\mathbf{x}),$$

which can be regarded as the Laplace transform of the probability density function p.

- **Put**

$$\chi(x) = \chi(x;\kappa) := 1 - e^{\kappa\phi(x)}.$$

Then we can show that

$$\Psi[\phi;\kappa] = 1 + \sum_{m=1}^{n}(-1)^m \frac{1}{m!}\int_{S^m} \rho_m(x_1,\ldots,x_m)\prod_{k=1}^{m}\left\{\chi(x_k)\lambda(dx_k)\right\}.$$

This expression means that the characteristic function is regarded as the generating function of correlation functions.

- **If every correlation function is expressed by a determinant in the form**

$$\rho_m(x_1, \dots, x_m) = \det_{1 \le j,k \le m}[K(x_j, x_k)], \quad m = 1, \dots, n,$$

with a two-point continuous function $K(x, y)$, $x, y \in S$, then the point process is said to be a determinantal point process (DPP) and K is called the correlation kernel.

- **In particular, the density of point with respect to λ on S is given by**

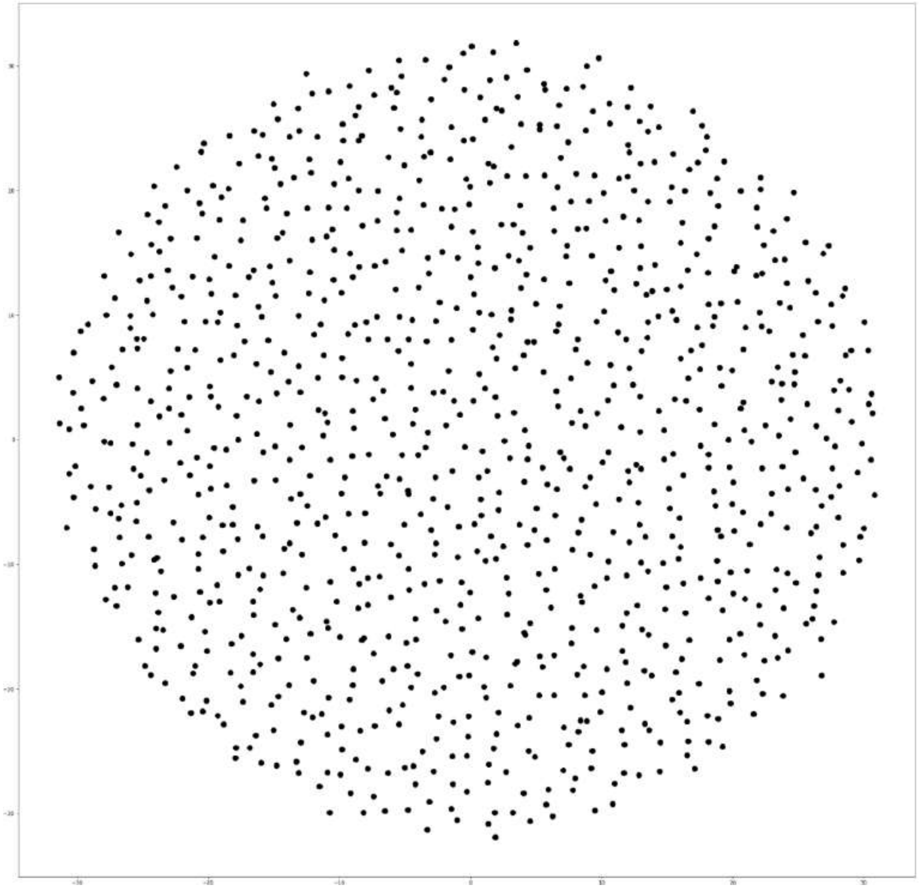
$$\rho_1(x) = K(x, x), \quad x \in S.$$

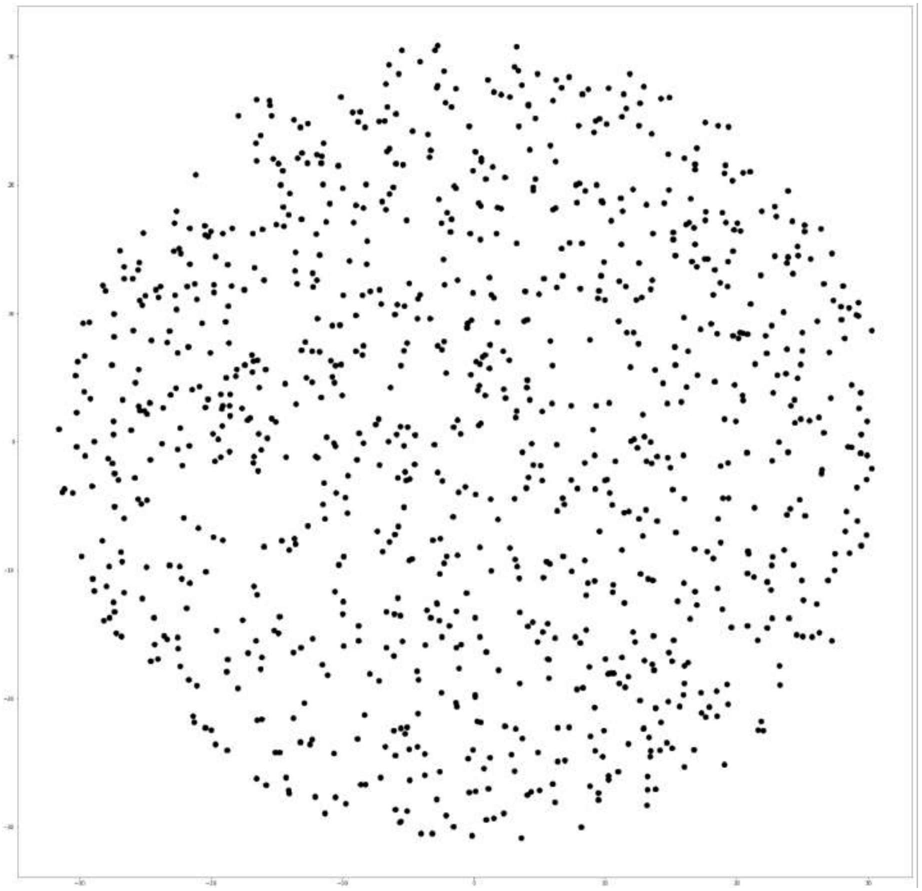
- **The characteristic function is given by**

$$\Psi[\phi; \kappa] = 1 + \sum_{m=1}^{n} (-1)^m \frac{1}{m!} \int_{S^m} \det_{1 \le j,k \le m}[K(x_j, x_k)\chi(x_k)] \prod_{\ell=1}^{m} \lambda(dx_\ell)$$

$$= \operatorname*{Det}_{(S,\lambda), x,y \in S} \Big[\delta(x - y) - K(x, y)\chi(y)\Big] \text{ (**Fredholm determinant**)}.$$

- **We denote the DPP by a triplet $(\Xi, K, \lambda(dx))$.**





an example of DPP (Ginibre DPP) **Poisson point process**

(Computer simulation by T. Matsui (Chuo U.))

another example of DPP

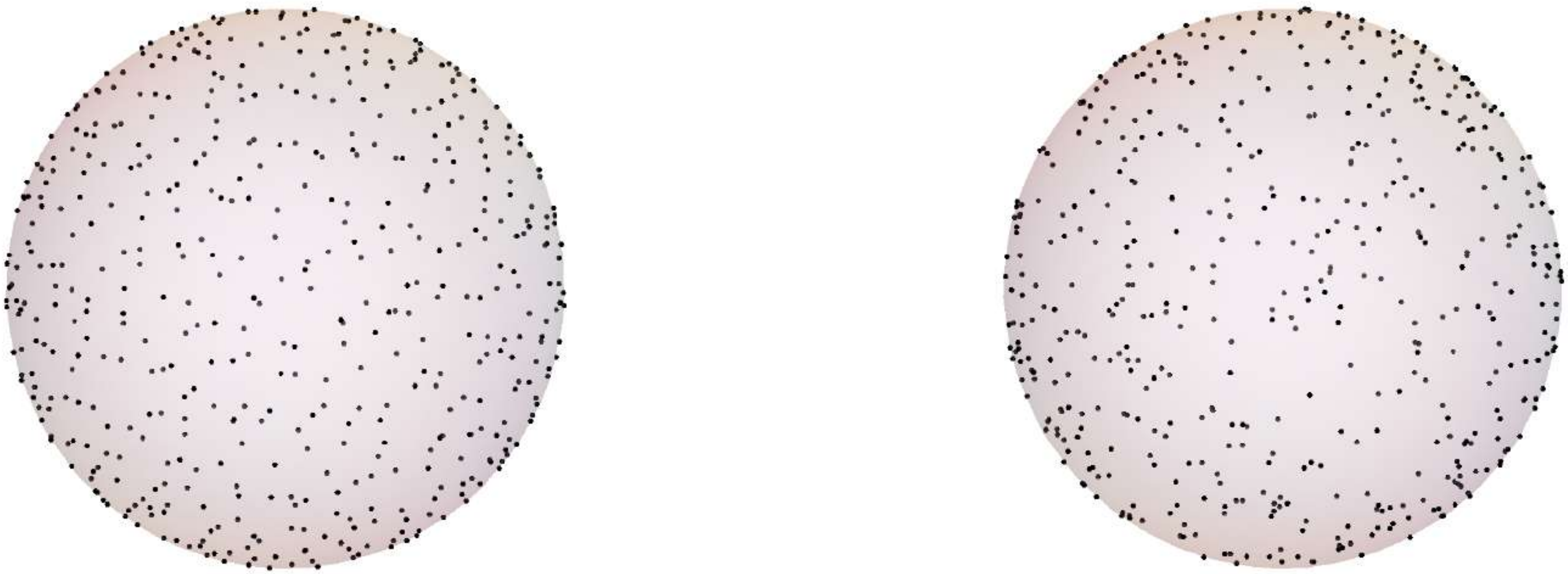


Figure: Spherical ensemble (left) and Poisson (right) ($N = 500$)

(Computer simulation by T. Shirai (Kyushu U.))

- **The following fact of DPPs is proved using a basic property of determinant.**

Lemma 3.1 Consider a non-vanishing function $f : S \to \mathbb{C}$. Even if the correlation kernel $K(x, y)$ is transformed as

$$K(x,y) \to K_f(x,y) := f(x)K(x,y)\frac{1}{f(y)}, \quad x, y \in S,$$

all correlation functions are the same and hence

$$(\Xi, K, \lambda(dx)) \stackrel{\text{(law)}}{=} (\Xi, K_f, \lambda(dx)).$$

- **The above transformation is called the Gauge transformation and the above property of DPP is referred to Gauge invariance.**

Theorem 3.2 Fix $n \in \mathbb{N}$ and assume that a set of functions $\{f_j\}_{j=1}^n$ on S with a reference measure $\lambda(dx)$ satisfies the orthogonality relation,

$$\int_S f_j(x)\overline{f_k(x)}\lambda(dx) = h_j \delta_{jk} \quad j, k \in \{1, \dots, n\}.$$

Then we can define a point process with n particles on S such that the probability density function with respect to $\lambda(dx)$ is given by

$$\mathbf{p}(\mathbf{x}) = \frac{1}{Z} \left| \det_{1 \leq j,k \leq n} [f_j(x_k)] \right|^2, \quad \mathbf{x} \in S^n,$$

where $1/Z$ is a normalization factor so that $(1/n!) \int_{S^n} \mathbf{p}(\mathbf{x}) \lambda(d\mathbf{x}) = 1$. Then this is a DPP $(\Xi, K, \lambda(dx))$ such that the correlation kernel is given by

$$K(x, y) = \sum_{\ell=1}^{n} \frac{1}{h_\ell} f_\ell(x) \overline{f_\ell(y)}, \quad x, y \in S.$$

- This theorem is well known in random matrix theory. For example, see Appendix C in [K19].

[K19] Katori, M.: Macdonald denominators for affine root systems, orthogonal theta functions, and elliptic determinantal point processes. J. Math. Phys. 60, 013301/1-27 (2019)

Theorem 3.2 Fix $n \in \mathbb{N}$ and assume that a set of functions $\{f_j\}_{j=1}^n$ on S with a reference measure $\lambda(dx)$ satisfies the orthogonality relation,

$$\int_S f_j(x)\overline{f_k(x)}\lambda(dx) = h_j \delta_{jk} \quad j,k \in \{1,\dots,n\}.$$

Then we can define a point process with n particles on S such that the probability density function with respect to $\lambda(dx)$ is given by

$$\mathbf{p}(\mathbf{x}) = \frac{1}{Z}\left|\det_{1\le j,k\le n}[f_j(x_k)]\right|^2, \quad \mathbf{x} \in S^n,$$

where $1/Z$ is a normalization factor so that $(1/n!)\int_{S^n}\mathbf{p}(\mathbf{x})\lambda(d\mathbf{x}) = 1$. Then this is a DPP $(\Xi, K, \lambda(dx))$ such that the correlation kernel is given by

$$K(x,y) = \sum_{\ell=1}^n \frac{1}{h_\ell} f_\ell(x)\overline{f_\ell(y)}, \quad x,y \in S.$$

- A general framework to construct DPPs based on the notion of partial isometry is given in [K-Shirai21] not only for DPPs with finite number of particles $n \in \mathbb{N}$, but also for DPPs with an infinite number of particles.

[K-Shirai21] Katori, M., Shirai, T.: Partial isometries, duality, and determinantal point processes. Random Matrices: Theory and Applications. 2250025, 70 pages (2021)

- **So far we have fixed $n \in \mathbb{N}$ for each system. We can consider a series of systems with increasing n; increasing the number of particles for DPPs.**

- **According to the change of n, we change the scale of coordinates, which is called dilatation of DPPs.**

Definition 3.3 **For a DPP $(\Xi, K, \lambda(dx))$ with $\Xi = \sum_j \delta_{X_j}$ on a space S, given a factor $c > 0$,**

$$c \circ \Xi := \sum_j \delta_{cX_j},$$
$$c \circ K(x, y) := K\left(\frac{x}{c}, \frac{y}{c}\right), \quad x, y \in cS := \{cx : x \in S\},$$
$$c \circ \lambda(dx) := \lambda(dx/c).$$

Then the dilatation by factor c of the DPP is defined by

$$c \circ (\Xi, K, \lambda(dx)) := (c \circ \Xi, c \circ K, c \circ \lambda(dx)).$$

- **Notice the following equivalence. For $g \in \mathcal{B}_c(S)$ such that $g : S \to (0, \infty)$,**

$$(\Xi, K(x, y), g(x)\lambda(dx)) \stackrel{\text{(law)}}{=} (\Xi, \sqrt{g(x)}K(x, y)\sqrt{g(y)}, \lambda(dx)).$$

- **I will give limit theorems for DPPs at the end of this talk.**

- **Consider a DPP which depends on a continuous parameter, or a series of DPPs labeled by a discrete parameter (e.g., the number of points $n \in \mathbb{N}$), and describe the system by $(\Xi, K_p, \lambda_p(dx))$ with the continuous or discrete parameter p.**

- **If $(\Xi, K_p, \lambda_p(dx))$ converges to a DPP, $(\Xi, K, \lambda(dx))$, as $p \to \infty$, weakly in the vague topology, we write this limit theorem as**

$$(\Xi, K_p, \lambda_p(dx)) \overset{p\to\infty}{\Longrightarrow} (\Xi, K, \lambda(dx)).$$

- **The weak convergence of DPPs is verified by the uniform convergence of the kernel $K_p \to K$ on each compact set $C \subset S \times S$ [ST03].**

[ST03] T. Shirai and Y. Takahashi, Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point process, J. Funct. Anal. 205 (2003) 414–463 (2003)

3.2 DPPs on a 2-dim Torus

$$\Psi_j^{\boldsymbol{R}_n}(z+2\pi i|\tau|) = \sigma_1 e^{-i\mathcal{N}x}\Psi_j^{\boldsymbol{R}_n}(z)$$

- **For $\boldsymbol{R}_n = \boldsymbol{A}_{n-1}, \boldsymbol{B}_n, \boldsymbol{B}_n^{\vee}, \boldsymbol{C}_n, \boldsymbol{C}_n^{\vee}, \boldsymbol{BC}_n, \boldsymbol{D}_n$, we put**

$$k^{\boldsymbol{R}_n}(z,z') := \sum_{j=1}^{n} \Psi_j^{\boldsymbol{R}_n}(z)\overline{\Psi_j^{\boldsymbol{R}_n}(z')}, \quad z, z' \in D_{(2\pi, 2\pi|\tau|)}.$$

By Lemma 2.7, the following double periodicity is proved ($\mathcal{N}^{\boldsymbol{A}_{n-1}} := n$),

$$\begin{aligned} k^{\boldsymbol{R}_n}(z+2\pi, z'+2\pi) &= k^{\boldsymbol{R}_n}(z,z'), \\ k^{\boldsymbol{R}_n}(z+2\pi|\tau|i, z'+2\pi|\tau|i) &= \frac{e^{-i\mathcal{N}x}}{e^{-i\mathcal{N}x'}} k^{\boldsymbol{R}_n}(z,z') \\ &\simeq k^{\boldsymbol{R}_n}(z,z') \quad \text{(by Gauge invariance)}, \quad z, z' \in \mathbb{C}. \end{aligned}$$

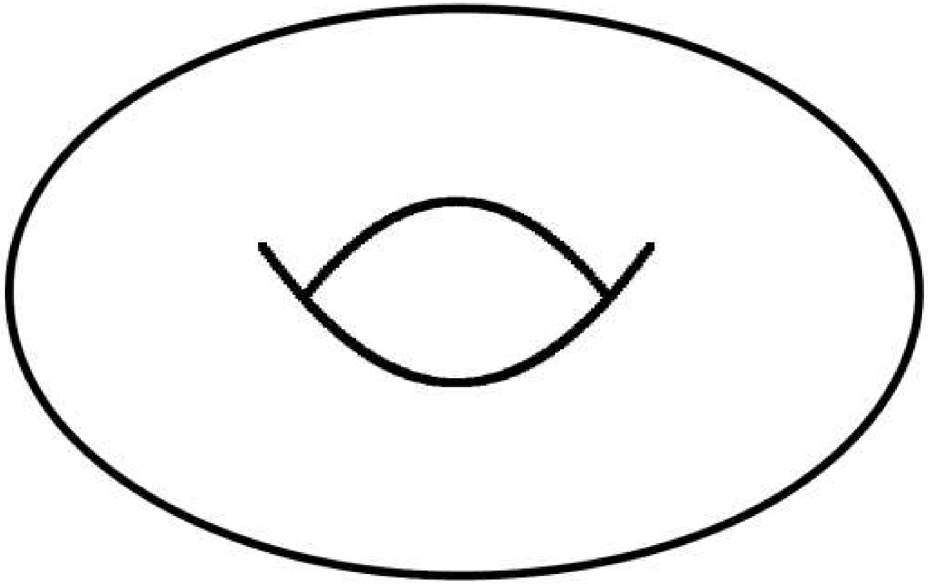
- **Now we apply Theorem 3.2 to our seven types of orthonormal theta functions. Then we obtain seven types of DPPs on $\mathbb{C}$ such that their correlation kernels are given by $k^{\boldsymbol{R}_n}$ satisfying the above double periodicity.**

- **In other words, if we define the two-dimensional torus denoted as**

$$\mathbb{T}^2 := \{z \in \mathbb{C} : z+2\pi = z, z+2\pi|\tau|i = z\} \simeq (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi|\tau|\mathbb{Z}),$$

then we have the seven types of DPPs on $\mathbb{T}^2$.

two-dimensional torus



$$\mathbb{T}^2 = \mathbb{T}^2_{|\tau|} := \{z \in \mathbb{C} : z + 2\pi = z, z + 2\pi|\tau|i = z\}$$
$$\simeq (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi|\tau|\mathbb{Z}),$$

Theorem 3.4 The seven types of point processes

$$\Xi_{\mathbb{T}^2}^{\boldsymbol{R}_n}(\cdot) = \sum_{j=1}^{n} \delta_{X_j^{\boldsymbol{R}_n}}(\cdot),$$

for $\boldsymbol{R}_n = \boldsymbol{A}_{n-1}, \boldsymbol{B}_n, \boldsymbol{B}_n^{\vee}, \boldsymbol{C}_n, \boldsymbol{C}_n^{\vee}, \boldsymbol{BC}_n, \boldsymbol{D}_n$ are well defined on $\mathbb{T}^2$ so that they have probability densities $\mathbf{p}_{\mathbb{T}^2}^{\boldsymbol{R}_n}(\mathbf{z})$

$$\mathbf{p}_{\mathbb{T}^2}^{\boldsymbol{R}_n}(\mathbf{z}) = \frac{1}{Z^{\boldsymbol{R}_n}} \left| \det_{1 \le j,k \le n} [\Psi_j^{\boldsymbol{R}_n}(z_k)] \right|^2, \quad \mathbf{z} = (z_1, \dots z_n) \in (\mathbb{T}^2)^n,$$

with respect to the Lebesgue measure $d\mathbf{z} = \prod_{j=1}^n d\Re z_j d\Im z_j$ on $(\mathbb{T}^2)^n$. Here $1/Z^{\boldsymbol{R}_n}$ are the normalization factors such that $(1/n!) \int_{(\mathbb{T}^2)^n} \mathbf{p}_{\mathbb{T}^2}^{\boldsymbol{R}_n}(\mathbf{z}) d\mathbf{z} = 1$. They are DPPs with the correlation kernels

$$K_{\mathbb{T}^2}^{\boldsymbol{R}_n}(z, z') := \frac{1}{(2\pi)^2 |\tau|} \sum_{j=1}^{n} \Psi_j^{\boldsymbol{R}_n}(z) \overline{\Psi_j^{\boldsymbol{R}_n}(z')}, \quad z, z' \in \mathbb{T}^2,$$

with respect to the Lebesgue measure $dz = d\Re z d\Im z$.

- **We define the reflection and shift of DPP on $\mathbb{T}^2$ as follows.**

Definition 3.5 **Consider a DPP $(\Xi, K, \lambda(dz))$ on $\mathbb{T}^2$, where we write $\Xi(\cdot) = \sum_j \delta_{Z_j}(\cdot)$.**

(i) **The inversion operator $\mathcal{R}$ is defined by**

$$\mathcal{R}\Xi := \sum_j \delta_{-Z_j}, \quad \mathcal{R}K(z, z') := K(-z, -z'), \quad \mathcal{R}\lambda(dz) := \lambda(-dz).$$

We write $(\mathcal{R}\Xi, \mathcal{R}K, \mathcal{R}\lambda(dz))$ simply as $\mathcal{R}(\Xi, K, \lambda(dz))$.

(ii) **For $u \in \mathbb{C}$, the shift operator $\mathcal{S}_u$ is defined by**

$$\mathcal{S}_u\Xi := \sum_j \delta_{Z_j - u}, \quad \mathcal{S}_u K(z, z') := K(z + u, z' + u), \quad \mathcal{S}_u\lambda(dz) := \lambda(u + dz).$$

We write $(\mathcal{S}_u\Xi, \mathcal{S}_u K, \mathcal{S}_u\lambda(dz))$ simply as $\mathcal{S}(\Xi, K, \lambda(dz))$.

We can prove the following symmetry which characterizes the seven types of DPPs on $\mathbb{T}^2$.

Proposition 3.6 (i) For $\boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{B}_n^\vee, \boldsymbol{C}_n, \boldsymbol{C}_n^\vee, \boldsymbol{BC}_n, \boldsymbol{D}_n$, the reflection invariance is established: $\mathcal{R}(\Xi, K, \lambda(dz)) \stackrel{\text{(law)}}{=} (\Xi, K, \lambda(dz))$.

(ii) The following shift invariance are satisfied:

$$\mathcal{S}_{2\pi/n}(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}, dz) \stackrel{\text{(law)}}{=} (\Xi, K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}, dz),$$

$$\mathcal{S}_{2\pi|\tau|i/n}(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}, dz) \stackrel{\text{(law)}}{=} (\Xi, K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}, dz),$$

$$\mathcal{S}_{\pi}(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{R}_n}, dz) \stackrel{\text{(law)}}{=} (\Xi, K_{\mathbb{T}^2}^{\boldsymbol{R}_n}, dz), \quad \boldsymbol{R}_n = \boldsymbol{B}_n^\vee, \boldsymbol{C}_n, \boldsymbol{D}_n,$$

$$\mathcal{S}_{\pi|\tau|i}(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{R}_n}, dz) \stackrel{\text{(law)}}{=} (\Xi, K_{\mathbb{T}^2}^{\boldsymbol{R}_n}, dz), \quad \boldsymbol{R}_n = \boldsymbol{C}_n, \boldsymbol{C}_n^\vee, \boldsymbol{BC}_n, \boldsymbol{D}_n.$$

(iii) The densities of points $\rho_{\mathbb{T}^2}^{\boldsymbol{R}_n}(z) = K_{\mathbb{T}^2}^{\boldsymbol{R}_n}(z, z), z \in \mathbb{T}^2$ with respect to the Lebesgue measure dz have the following zeros:

$$\begin{aligned}
&\rho_{\mathbb{T}^2}^{\boldsymbol{B}_n}(0) = 0, \\
&\rho_{\mathbb{T}^2}^{\boldsymbol{B}_n^\vee}(0) = \rho_{\mathbb{T}^2}^{\boldsymbol{B}_n^\vee}(\pi) = 0, \\
&\rho_{\mathbb{T}^2}^{\boldsymbol{R}_n}(0) = \rho_{\mathbb{T}^2}^{\boldsymbol{R}_n}(\pi|\tau|i) = 0, \quad \boldsymbol{R}_n = \boldsymbol{C}_n^\vee, \boldsymbol{BC}_n, \\
&\rho_{\mathbb{T}^2}^{\boldsymbol{C}_n}(0) = \rho_{\mathbb{T}^2}^{\boldsymbol{C}_n}(\pi) = \rho_{\mathbb{T}^2}^{\boldsymbol{C}_n}(\pi|\tau|i) = 0.
\end{aligned}$$

3.3 Infinite Particle Limits

- **We note that the periods $2\pi/n \in (0,\infty)$ and $2\pi|\tau|i/n \in i(0,\infty)$ of $(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}, dz)$ shown by Proposition 3.6 (ii) become zeros as $n \to \infty$. Hence, as the $n \to \infty$ limit of $(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}, dz)$, it is expected to obtain a translation-invariant system of infinite number of points on $\mathbb{C}$.**

- **Here we introduce three kinds of infinite DPPs on $\mathbb{C}$.**

- **Let the reference measure be the complex normal distribution,**

$$\lambda_{\mathrm{N}}(dz) := \frac{1}{\pi} e^{-|z|^2} dz.$$

- **Put**

$$\mathcal{K}^{A}_{\text{Ginibre}}(z,z') = e^{z\overline{z'}},$$
$$\mathcal{K}^{C}_{\text{Ginibre}}(z,z') = \sinh(z\overline{z'}) = \frac{1}{2}(e^{z\overline{z'}} - e^{-z\overline{z'}}),$$
$$\mathcal{K}^{D}_{\text{Ginibre}}(z,z') = \cosh(z\overline{z'}) = \frac{1}{2}(e^{z\overline{z'}} + e^{-z\overline{z'}}), \quad z, z' \in \mathbb{C}.$$

- **Then the Ginibre DPPs of type R are defined by $(\Xi, \mathcal{K}^{R}_{\text{Ginibre}}, \lambda_{\text{N}}(dz))$ for $R = A, C$, and D, respectively.**

- **The Ginibre DPP of type A describes the eigenvalue distribution of the Gaussian random complex matrix in the bulk scaling limit [Gin65]. The density of points is uniform with the Lebesgue measure dz on $\mathbb{C}$ and translation-invariant as**

$$\rho^{A}_{\text{Ginibre}}(x)dz = \mathcal{K}^{A}_{\text{Ginibre}}(z,z)\lambda_{\text{N}}(dz) = \frac{1}{\pi}dz, \quad z \in \mathbb{C}.$$

[Gin65] Ginibre, J.: Statistical ensembles of complex, quaternion, and real matrices, J. Math. Phys. 6 440–449 (1965)

- **Put**

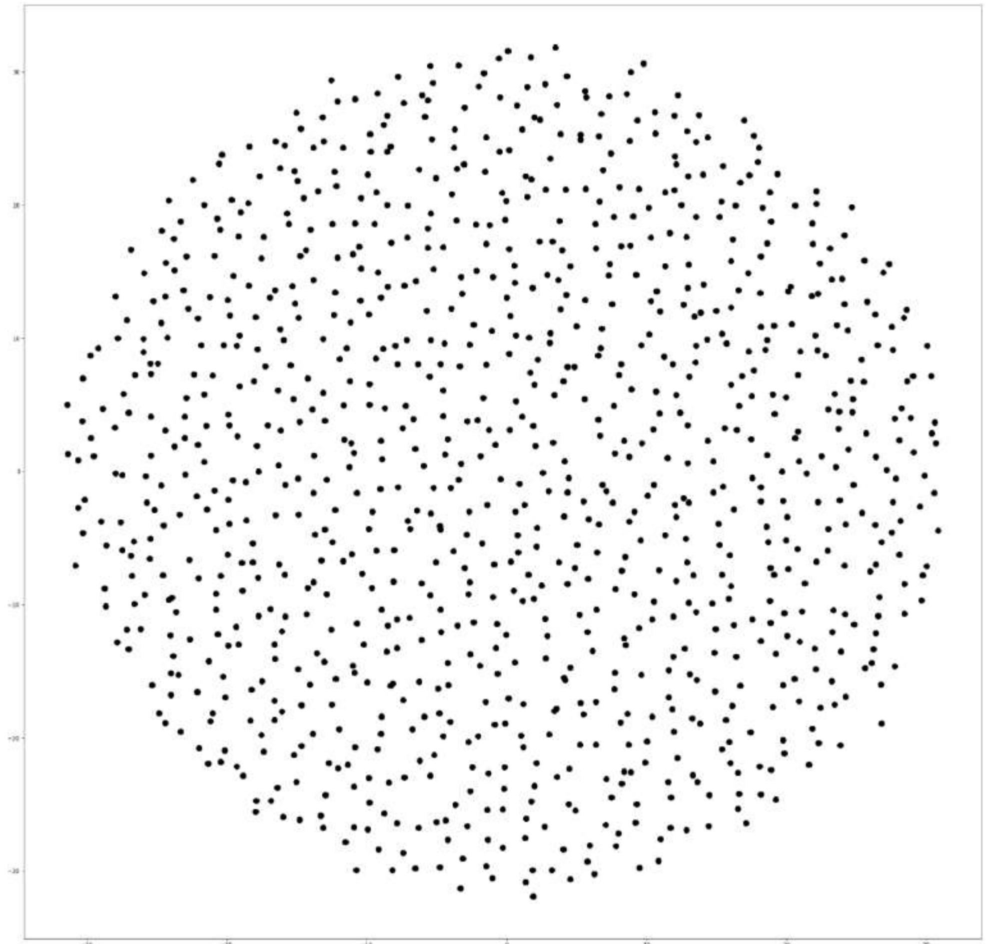
$$\mathcal{K}^{A}_{\text{Ginibre}}(z,z') = e^{z\overline{z'}},$$
$$\mathcal{K}^{C}_{\text{Ginibre}}(z,z') = \sinh(z\overline{z'}) = \frac{1}{2}(e^{z\overline{z'}} - e^{-z\overline{z'}}),$$
$$\mathcal{K}^{D}_{\text{Ginibre}}(z,z') = \cosh(z\overline{z'}) = \frac{1}{2}(e^{z\overline{z'}} + e^{-z\overline{z'}}), \quad z, z' \in \mathbb{C}.$$

- **Then the Ginibre DPPs of type R are defined by $(\Xi, \mathcal{K}^{R}_{\text{Ginibre}}, \lambda_{\text{N}}(dz))$ for $R = A, C$, and D, respectively.**

- **The Ginibre DPP of type A describes the eigenvalue distribution of the Gaussian random complex matrix in the bulk scaling limit [Gin65]. The density of points is uniform with the Lebesgue measure dz on $\mathbb{C}$ and translation-invariant as**

$$\rho^{A}_{\text{Ginibre}}(x)dz = \mathcal{K}^{A}_{\text{Ginibre}}(z,z)\lambda_{\text{N}}(dz) = \frac{1}{\pi}dz, \quad z \in \mathbb{C}.$$

[Gin65] Ginibre, J.: Statistical ensembles of complex, quaternion, and real matrices, J. Math. Phys. 6 440–449 (1965)



Finite approximation of Ginibre DPP of type A: eigenvalues of Gaussian random complex matrix

(Computer simulation by T. Matsui (Chuo U.))

- **On the other hands, the Ginibre DPPs of types C and D with the correlation kernels,**

$$\mathcal{K}^{C}_{\text{Ginibre}}(z, z') = \sinh(z\overline{z'}) = \frac{1}{2}(e^{z\overline{z'}} - e^{-z\overline{z'}}),$$
$$\mathcal{K}^{D}_{\text{Ginibre}}(z, z') = \cosh(z\overline{z'}) = \frac{1}{2}(e^{z\overline{z'}} + e^{-z\overline{z'}}), \quad z, z' \in \mathbb{C},$$

are rotationally symmetric around the origin, but non-uniform on $\mathbb{C}$ and the translation-symmetry is broken. The density of points with the Lebesgue measure dz on $\mathbb{C}$ are given by

$$\rho^{C}_{\text{Ginibre}}(x)dx = \mathcal{K}^{C}_{\text{Ginibre}}(x, x)\lambda_{\text{N}}(dx) = \frac{1}{2\pi}(1 - e^{-2|z|^2})dz, \quad z \in \mathbb{C},$$
$$\rho^{D}_{\text{Ginibre}}(z)dz = \mathcal{K}^{D}_{\text{Ginibre}}(z, z)\lambda_{\text{N}}(dz) = \frac{1}{2\pi}(1 + e^{-2|z|^2})dz, \quad z \in \mathbb{C}.$$

They were first obtained by the following limit theorems.

- **These three types of Ginibre DPPs on $\mathbb{C}$ are obtained by the infinite particle limits of our seven types of DPPs on $\mathbb{T}^2$.**

Proposition 3.7 The following weak convergence is established,

$$\frac{1}{2}\sqrt{\frac{n}{\pi|\tau|}} \circ \left(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}, dz\right) \stackrel{n\to\infty}{\Longrightarrow} \left(\Xi, \mathcal{K}_{\text{Ginibre}}^{A}, \lambda_{\text{N}}(dz)\right),$$

$$\sqrt{\frac{n}{2\pi|\tau|}} \circ \left(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{R}_n}, dz\right) \stackrel{n\to\infty}{\Longrightarrow} \left(\Xi, \mathcal{K}_{\text{Ginibre}}^{C}, \lambda_{\text{N}}(dz)\right), \quad \boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{B}_n^{\vee}, \boldsymbol{C}_n, \boldsymbol{C}_n^{\vee}, \boldsymbol{BC}_n,$$

$$\sqrt{\frac{n}{2\pi|\tau|}} \circ \left(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{D}_n}, dx\right) \stackrel{n\to\infty}{\Longrightarrow} \left(\Xi, \mathcal{K}_{\text{Ginibre}}^{D}, \lambda_{\text{N}}(dz)\right).$$

- **Remark that we have the degeneracy from the seven finite DPPs to three infinite DPPs.**

- **Here we show a sketch of proof for the first limit theorem by showing the convergence of the correlation kernels of type $\boldsymbol{A}_{n-1}$. Similar calculation proves the other two limit theorems.**

A Sketch of Proof.

- **By definition of** $\Psi_j^{\boldsymbol{A}_{n-1}}(z)$ **with the fact** $\lim_{n\to\infty}\theta(\zeta;p^n)=1-\zeta$ **for** $p=e^{-2\pi|\tau|}$**, we can evaluate**

$$\Psi_j^{\boldsymbol{A}_{n-1}}\left(2\sqrt{\frac{\pi|\tau|}{n}}z\right)\sim e^{-n\pi|\tau|/4}(2|\tau|)^{1/4}n^{1/4}e^{-y^2}$$
$$\times\exp\left[-\pi|\tau|\left(\frac{j-1-n/2}{\sqrt{n}}\right)^2+2i\sqrt{\pi|\tau|}z\frac{j-1-n/2}{\sqrt{n}}\right]$$
$$\times\left\{e^{-i\sqrt{n\pi|\tau|}z}\exp\left(\pi|\tau|\sqrt{n}\frac{j-1-n/2}{\sqrt{n}}\right)+e^{i\sqrt{n\pi|\tau|}z}\exp\left(-\pi|\tau|\sqrt{n}\frac{j-1-n/2}{\sqrt{n}}\right)\right\}$$

as $n\to\infty$**.**

- **Hence we have**

$$
\begin{aligned}
&K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}\left(2\sqrt{\frac{\pi|\tau|}{n}}z, 2\sqrt{\frac{\pi|\tau|}{n}}z'\right)\left(2\sqrt{\frac{\pi|\tau|}{n}}\right)^2 \\
&\quad \sim e^{-n\pi|\tau|/2}(2|\tau|)^{1/2}e^{-(y^2+y'^2)}4\pi|\tau|\frac{1}{(2\pi)^2|\tau|} \\
&\qquad \times \frac{1}{n^{1/2}}\sum_{j=0}^{n}\exp\left[-2\pi|\tau|\left(\frac{j-1-n/2}{\sqrt{n}}\right)^2 + 2i\sqrt{\pi|\tau|}\frac{j-1-n/2}{\sqrt{n}}(z-\overline{z'})\right] \\
&\qquad\qquad \times \left[e^{-i\sqrt{n\pi|\tau|}z}\exp\left(\pi|\tau|\sqrt{n}\frac{j-1-n/2}{\sqrt{n}}\right) + e^{i\sqrt{n\pi|\tau|}z}\exp\left(-\pi|\tau|\sqrt{n}\frac{j-1-n/2}{\sqrt{n}}\right)\right] \\
&\qquad\qquad \times \left[e^{i\sqrt{n\pi|\tau|}\overline{z'}}\exp\left(\pi|\tau|\sqrt{n}\frac{j-1-n/2}{\sqrt{n}}\right) + e^{-i\sqrt{n\pi|\tau|}\overline{z'}}\exp\left(-\pi|\tau|\sqrt{n}\frac{j-1-n/2}{\sqrt{n}}\right)\right] \\
&\quad \sim e^{-n\pi|\tau|/2}\frac{(2|\tau|)^{1/2}}{\pi}\left\{e^{-i\sqrt{n\pi|\tau|}(z-\overline{z'})}I_+ + 2\cos(\sqrt{n\pi|\tau|}(z+\overline{z'}))I_0 + e^{i\sqrt{n\pi|\tau|}(z-\overline{z'})}I_-\right\},
\end{aligned}
$$

as $n \to \infty$.

- **Here**

$$I_\pm := \int_{-\sqrt{n}/2}^{\sqrt{n}/2} e^{-2\pi|\tau|u^2+2i\sqrt{\pi|\tau|}(z-\overline{z'})u\pm 2\pi|\tau|\sqrt{n}u}du$$
$$= e^{n\pi|\tau|/2} e^{\pm i\sqrt{n\pi|\tau|}(z-\overline{z'})} e^{-(z-\overline{z'})^2/2} \int_{a_\pm - i(z-\overline{z'})/2\sqrt{\pi|\tau|}}^{b_\pm - i(z-\overline{z'})/2\sqrt{\pi|\tau|}} e^{-2\pi|\tau|v^2}dv$$

with $a_+ = -b_- = -\sqrt{n}$**,** $b_+ = a_- = 0$**, and**

$$I_0 := \int_{-\sqrt{n}/2}^{\sqrt{n}/2} e^{-2\pi|\tau|u^2+2i\sqrt{\pi|\tau|}(z-\overline{z'})u}du = e^{-(z-\overline{z'})^2/2} \int_{-\sqrt{n}/2-i(z-\overline{z'})/2\sqrt{\pi|\tau|}}^{\sqrt{n}/2-i(z-\overline{z'})/2\sqrt{\pi|\tau|}} e^{-2\pi|\tau|v^2}dv.$$

We see

$$I_\pm \sim e^{n\pi|\tau|/2}\frac{1}{2}\frac{1}{(2|\tau|)^{1/2}} e^{\pm i\sqrt{n\pi|\tau|}(z-\overline{z'})} e^{-(z-\overline{z'})^2/2},$$
$$I_0 \sim \frac{1}{(2|\tau|)^{1/2}} e^{-(z-\overline{z'})^2/2}, \quad n \to \infty.$$

- **Therefore, we have**

$$K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}\left(2\sqrt{\frac{\pi|\tau|}{n}}z, 2\sqrt{\frac{\pi|\tau|}{n}}z'\right)\left(2\sqrt{\frac{\pi|\tau|}{n}}\right)^2 \to \frac{1}{\pi}e^{-(y^2+y'^2)-(z-\overline{z'})^2/2}.$$

- **Since**

$$\begin{aligned}\frac{1}{\pi}e^{-(y^2+y'^2)-(z-\overline{z'})^2/2} &= \frac{e^{-ixy}}{e^{-ix'y'}}e^{z\overline{z'}}\frac{1}{\pi}e^{-(|z|^2+|z'|^2)/2} \\ &= \frac{e^{-ixy}}{e^{-ix'y'}} \times \sqrt{\frac{1}{\pi}e^{-|z|^2}} \times e^{z\overline{z'}} \times \sqrt{\frac{1}{\pi}e^{-|z'|^2}}\end{aligned}$$

with

$$\mathcal{K}_{\text{Ginibre}}^{A}(z,z') = e^{z\overline{z'}}, \quad \lambda_{\text{N}}(dz) = \frac{1}{\pi}e^{-|z|^2}dz,$$

the Gauge invariance of DPP (Lemma 3.1) and the equivalence

$$(\Xi, K(x,y), g(x)\lambda(dx)) \overset{\text{(law)}}{=} (\Xi, \sqrt{g(x)}K(x,y)\sqrt{g(y)}, \lambda(dx))$$

imply the assertion. ∎

Thank you very much for your attention.

[K19a] Macdonald denominators for affine root systems, orthogonal theta functions, and elliptic determinantal point processes. J. Math. Phys. 60, 013301/1-27 (2019)

[K19b] Two-dimensional elliptic determinantal point processes and related systems. Commun. Math. Phys. 371, 1283–1321 (2019)

[K-Shirai21] Katori, M., Shirai, T.: Partial isometries, duality, and determinantal point processes. Random Matrices: Theory and Applications. 2250025, 70 pages (2021)