Zeros of the i.i.d. Gaussian Laurent series on an annulus

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based on arXiv: 2008.04177

Conference

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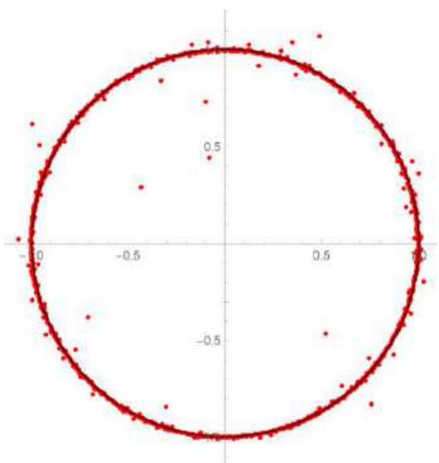
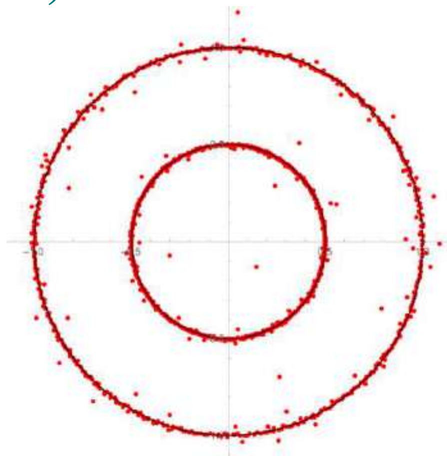




Plan

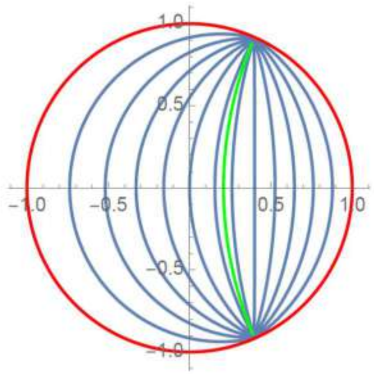
1. **Hilbert function spaces, reproducing kernels, conformal maps, and conditional Hilbert function spaces**

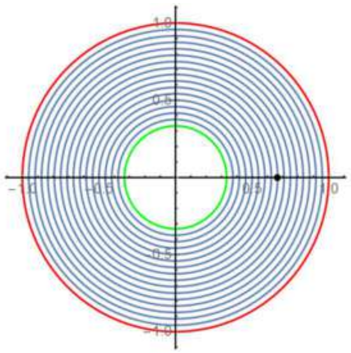
2. **Gaussian analytic functions (GAFs), Gaussian processes (GPs), and zero-point processes**

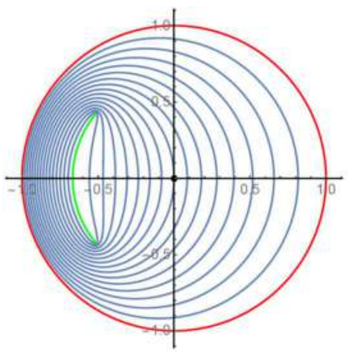
3. **Main results: zero-point processes of GAFs**

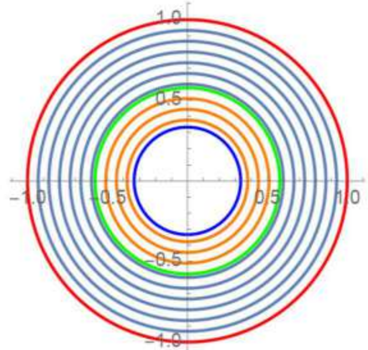
4. **A sketch of proof for the main theorem**

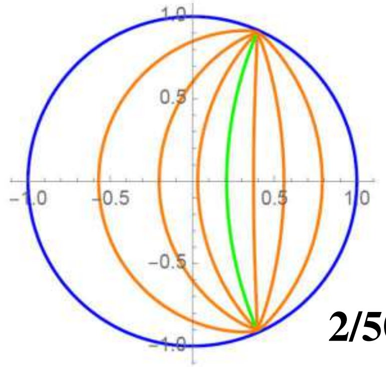
5. **And geometry?**











1. Hilbert Function Spaces, Reproducing Kernels, Conformal Maps, and Conditional Hilbert Function Spaces

$\mathcal{H}$: A **Hilbert space of holomorphic functions** on a domain D in $\mathbb{C}$

an inner product $\langle f, g\rangle_{\mathcal{H}}$

$$\begin{cases} \langle af + bg, h\rangle_{\mathcal{H}} = a\langle f, h\rangle_{\mathcal{H}} + b\langle g, h\rangle_{\mathcal{H}}, \\ \langle h, af + bg\rangle_{\mathcal{H}} = \overline{a}\langle h, f\rangle_{\mathcal{H}} + \overline{b}\langle h, g\rangle_{\mathcal{H}}, \quad f, g, h \in \mathcal{H},\ a, b \in \mathbb{C} \\ \langle g, f\rangle_{\mathcal{H}} = \overline{\langle f, g\rangle_{\mathcal{H}}} = \langle \overline{f}, \overline{g}\rangle_{\mathcal{H}}, \end{cases}$$

the norm $\|f\|_{\mathcal{H}} := \sqrt{\langle f, f\rangle_{\mathcal{H}}}$

For each point $w \in D$, there is **an element of $\mathcal{H}$**, $k_w \in \mathcal{H}$, with the property

$$\langle f, k_w\rangle_{\mathcal{H}} = f(w) \quad \forall f \in \mathcal{H}.$$

Since $k_z \in \mathcal{H}, z \in D$, if we put $f = k_z$ and write $k_{\mathcal{H}}(\cdot, w) := k_w(\cdot)$, then

$$\boxed{\langle k_{\mathcal{H}}(\cdot, z), k_{\mathcal{H}}(\cdot, w)\rangle_{\mathcal{H}} = k_{\mathcal{H}}(w, z), \quad z, w \in D.}$$

This two-point function $k_{\mathcal{H}}(z, w), z, w \in D$ is called the **reproducing kernel** of $\mathcal{H}$.

For a Hilbert space $\mathcal{H}$ of holomorphic functions on D,

there uniquely exists a kernel $k_{\mathcal{H}}(\cdot,\cdot)$ with the **reproducing property**

$$\boxed{\langle k_{\mathcal{H}}(\cdot,z), k_{\mathcal{H}}(\cdot,w)\rangle_{\mathcal{H}} = k_{\mathcal{H}}(w,z), \quad z,w \in D.}$$

By definition, the reproducing kernels is **Hermitian**: $\overline{k_{\mathcal{H}}(z,w)} = k_{\mathcal{H}}(w,z),\ z,w \in D.$

Semi-positivity of reproducing kernel is readily obtained by the reproducing property.

For an arbitrary $n \in \mathbb{N} := \{1,2,\dots\}$,

arbitrary n points $z_1,\dots,z_n \in D$ and n complex variables $\xi_1,\dots,\xi_n \in \mathbb{C}$,

$$\sum_{j=1}^{n}\sum_{k=1}^{n} k_{\mathcal{H}}(z_k,z_j)\xi_j\overline{\xi_k} = \sum_{j=1}^{n}\sum_{k=1}^{n}\langle k_{\mathcal{H}}(\cdot,z_j), k_{\mathcal{H}}(\cdot,z_k)\rangle_{\mathcal{H}}\xi_j\overline{\xi_k}$$

$$= \Big\langle \sum_{j=1}^{n} k_{\mathcal{H}}(\cdot,z_j)\xi_j, \sum_{k=1}^{n} k_{\mathcal{H}}(\cdot,z_k)\xi_k \Big\rangle_{\mathcal{H}} = \Big\| \sum_{j=1}^{n} k_{\mathcal{H}}(\cdot,z_j)\xi_j \Big\|_{\mathcal{H}}^2 \geq 0.$$

This is equivalent with the fact that $\boxed{\det_{1\leq j,k\leq n}\Big[k_{\mathcal{H}}(z_j,z_k)\Big] \geq 0\ \forall n \in \mathbb{N},\ \forall z_1,\dots,z_n \in D.}$

A **complete orthonormal system (CONS)** $\{e_n; n \in \mathcal{I}\} : \langle e_n, e_m \rangle_{\mathcal{H}} = \delta_{nm},\ n, m \in \mathcal{I}$

$$f \in \mathcal{H} \iff f = \sum_{n \in \mathcal{I}} c_n e_n \text{ with } (c_n)_{n \in \mathcal{I}} \in \ell^2(\mathcal{I})$$

The reproducing kernel has an expression, $\boxed{k_{\mathcal{H}}(\cdot, w) := \sum_{n \in \mathcal{I}} e_n(\cdot)\overline{e_n(w)},\ w \in D.}$

Actually, this gives

$$\langle f(\cdot), k_{\mathcal{H}}(\cdot, w) \rangle_{\mathcal{H}} = \left\langle \sum_{m \in \mathcal{I}} c_m e_m(\cdot), \sum_{n \in \mathcal{I}} e_n(\cdot)\overline{e_n(w)} \right\rangle_{\mathcal{H}} = \sum_{m \in \mathcal{I}} \sum_{n \in \mathcal{I}} c_m \langle e_m, e_n \rangle_{\mathcal{H}} e_n(w)$$

$$= \sum_{n \in \mathcal{I}} c_n e_n(w) = f(w) \ \forall f \in \mathcal{H},\ w \in D.$$

Example 1 Let $D = \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ (the unit disk)

$L^2_{\mathrm{B}}(\mathbb{D})$: the **Bergman space on $\mathbb{D}$**

:= the Hilbert space of holomorphic functions on $\mathbb{D}$,

which are square-integrable

with respect to the **Lebesgue measure $m(dz)$ on $\mathbb{C}$**.

inner product: $\langle f, g \rangle_{L^2_{\mathrm{B}}(\mathbb{D})} := \frac{1}{\pi} \int_{\mathbb{D}} f(z)\overline{g(z)} m(dz)$

CONS: $\widetilde{e_n}(z) := \sqrt{n+1} z^n, n \in \mathbb{N}_0 := \{0, 1, \dots\}$

reproducing kernel: the **Bergman kernel of $\mathbb{D}$**

$$K_{\mathbb{D}}(z, w) := k_{L^2_{\mathrm{B}}(\mathbb{D})}(z, w) = \sum_{n \in \mathbb{N}_0} (n+1)(z\overline{w})^n = \frac{1}{(1 - z\overline{w})^2}, \quad z, w \in \mathbb{D}$$

Example 2 Let $D = \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ (the unit disk)

$H^2(\mathbb{D})$: the **Hardy space on $\mathbb{D}$**

:= the Hilbert space of holomorphic functions on $\mathbb{D}$ such that

the Taylor coefficients form a square-summable series; $\sum_{n\in\mathbb{N}_0} |\widehat{f}(n)|^2 < \infty$

inner product:

$$\langle f, g\rangle_{H^2(\mathbb{D})} = \begin{cases} \sum_{n\in\mathbb{N}_0} \widehat{f}(n)\overline{\widehat{g}(n)}, \\ \frac{1}{2\pi}\int_0^{2\pi} f(e^{\sqrt{-1}\phi})\overline{g(e^{\sqrt{-1}\phi})}d\phi, \end{cases} \qquad f, g \in H^2(\mathbb{D})$$

(The latter is the integral **over boundary** with the arc length measure).

CONS: $e_n(z) = e_n^{(0,0)}(z) := z^n, n \in \mathbb{N}_0$

reproducing kernel: the **Szegő kernel of $\mathbb{D}$**

$$S_{\mathbb{D}}(z, w) := k_{H^2(\mathbb{D})}(z, w) = \sum_{n\in\mathbb{N}_0} (z\overline{w})^n = \frac{1}{1 - z\overline{w}}, \quad z, w \in \mathbb{D}$$

Let $f : D \to \widetilde{D}$ be a **conformal transformation**

between two bounded domains $D, \widetilde{D} \subsetneq \mathbb{C}$ with analytic boundaries.

The Szegő kernel is transformed by f as $(f'(z) := df(z)/dz)$

$$\boxed{S_D(z, w) = \sqrt{f'(z)}\overline{\sqrt{f'(w)}} S_{\widetilde{D}}(f(z), f(w)),\ z, w \in D.}$$

[Bel16] Bell S. R.: The Cauchy Transform, Potential Theory and Conformal Mapping. 2nd edn, CRC Press, Boca Raton, FL (2016)

Let $f : D \to \widetilde{D}$ be a **conformal transformation**

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The Szegő kernel is transformed by f as $(f'(z) := df(z)/dz)$

$$\boxed{S_D(z,w) = \sqrt{f'(z)}\overline{\sqrt{f'(w)}} S_{\widetilde{D}}(f(z), f(w)),\ z, w \in D.}$$

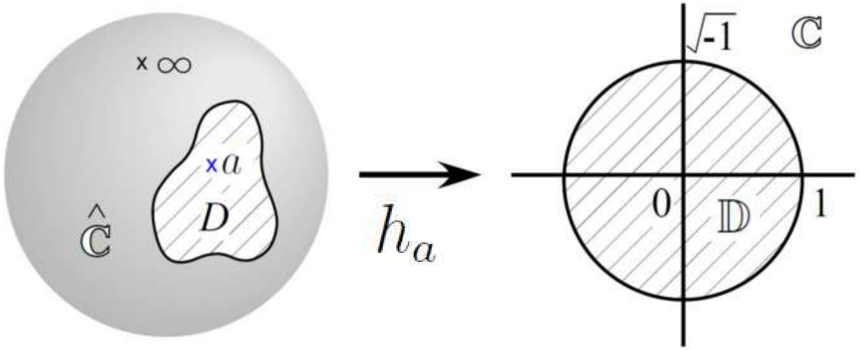
Consider the special case in which $D \subsetneq \mathbb{C}$ is a **simply connected domain** and $\widetilde{D} = \mathbb{D}$.

Remember that $S_{\mathbb{D}}(z,w) = \dfrac{1}{1 - z\overline{w}}$.

For each $a \in D$, **Riemann's mapping theorem** gives a unique conformal transformation:

$$\boxed{h_a : D \to \mathbb{D} \text{ (Riemann mapping function) such that } h_a(a) = 0,\ h_a'(a) > 0.}$$

$$\begin{aligned} S_D(z,w) &= \sqrt{h_a'(z)}\overline{\sqrt{h_a'(w)}} S_{\mathbb{D}}(h_a(z), h_a(w)) \\ &= \frac{S_D(z,a)\overline{S_D(w,a)}}{S_D(a,a)} \frac{1}{1 - h_a(z)\overline{h_a(w)}}, \quad z, w, a \in D. \end{aligned}$$



Example 3 Fix $q \in (0,1)$. Let $D = \mathbb{A}_q := \{z \in \mathbb{C} : q < |z| < 1\}$ (an annulus)

$H^2(\mathbb{A}_q)$: the **Hardy space on $\mathbb{A}_q$**

inner product: (given by an integral **over boundaries** with the arc length measure)

$$\langle f, g\rangle_{H^2(\mathbb{A}_q)} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\sqrt{-1}\phi})\overline{g(e^{\sqrt{-1}\phi})}d\phi + \frac{1}{2\pi}\int_0^{2\pi} f(qe^{\sqrt{-1}\phi})\overline{g(qe^{\sqrt{-1}\phi})}\, q d\phi,$$

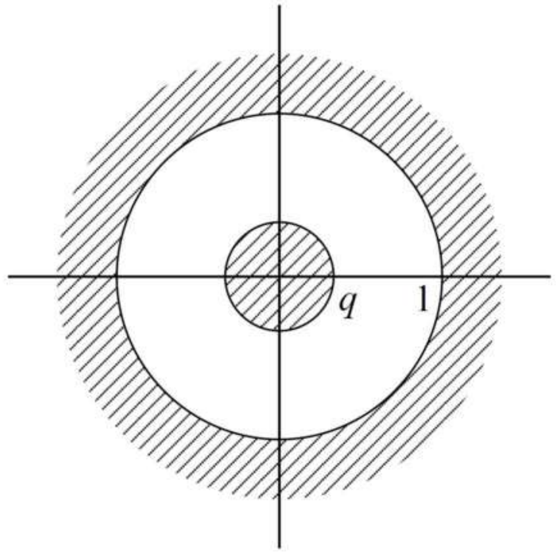
$$f, g \in H^2(\mathbb{A}_q)$$

CONS: $e_n(z) = e_n^{(q,q)}(z) := \dfrac{z^n}{\sqrt{1+q^{2n+1}}}, \quad n \in \mathbb{Z}$

reproducing kernel: the **Szegő kernel of $\mathbb{A}_q$**

$$S_{\mathbb{A}_q}(z,w) = \sum_{n\in\mathbb{Z}} e_n^{(q,q)}(z)\overline{e_n^{(q,q)}(w)} = \sum_{n=-\infty}^{\infty} \frac{(z\overline{w})^n}{1+q^{2n+1}}, \quad z, w \in \mathbb{A}_q$$

For each $a \in \mathbb{A}_q$, $S_{\mathbb{A}_q}(a, \widehat{a}) = S_{\mathbb{A}_q}(\widehat{a}, a) = 0$ with $\boxed{\widehat{a} = -\dfrac{q}{\overline{a}}.}$



Example 4 Fix $q \in (0,1)$. Let $D = \mathbb{A}_q := \{z \in \mathbb{C} : q < |z| < 1\}$ and assume $r > 0$.

$H^2_r(\mathbb{A}_q)$: the **Hardy space on $\mathbb{A}_q$ with parameter r**

inner product:

$$\langle f, g\rangle_{H^2_r(\mathbb{A}_q)} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\sqrt{-1}\phi})\overline{g(e^{\sqrt{-1}\phi})}d\phi + \frac{1}{2\pi}\int_0^{2\pi} f(qe^{\sqrt{-1}\phi})\overline{g(qe^{\sqrt{-1}\phi})}\boldsymbol{r}d\phi,$$

$$f, g \in H^2_r(\mathbb{A}_q)$$

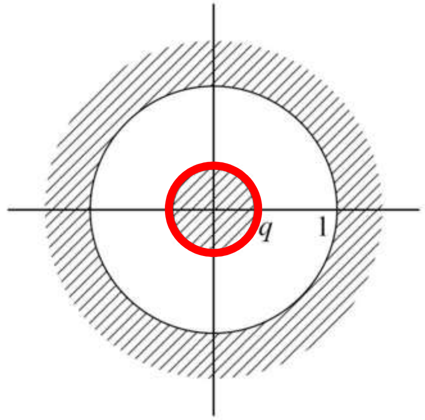
CONS: $e_n(z) = e_n^{(q,r)}(z) := \dfrac{z^n}{\sqrt{1 + rq^{2n}}}, \quad n \in \mathbb{Z}$

reproducing kernel: the **weighted Szegő kernel of $\mathbb{A}_q$ with weight parameter r**

$$S_{\mathbb{A}_q}(z, w; \boldsymbol{r}) = \sum_{n\in\mathbb{Z}} e_n^{(q,r)}(z)\overline{e_n^{(q,r)}(w)} = \sum_{n=-\infty}^{\infty} \frac{(z\overline{w})^n}{1 + \boldsymbol{r}q^{2n}}, \quad z, w \in \mathbb{A}_q$$

By definition $S_{\mathbb{A}_q}(z, w) = S_{\mathbb{A}_q}(z, w; q)$, $z, w \in \mathbb{A}_q$.

[MS94] **Mccullough, S, Shen, L.C.: On the Szegő kernel of an annulus. Proc. Amer. Math. Soc. 121, 1111–1121 (1994)**



Let $f : D \to \widetilde{D}$ be a **conformal transformation**

between two bounded domains $D, \widetilde{D} \subsetneq \mathbb{C}$ with analytic boundaries.

Here we consider the case in which D is a **2-connected domain** and $\widetilde{D} = \mathbb{D}$.

For each $a \in D$,

$f_a : D \to \mathbb{D}$ (**Ahlfors mapping function**)

a **branched 2 to 1** covering map of D to $\mathbb{D}$,

in which a unique point $\widehat{a} = \widehat{a}(a) \neq a$, $\widehat{a} \in D$ exists such that $f_a(a) = f_a(\widehat{a}) = 0$.

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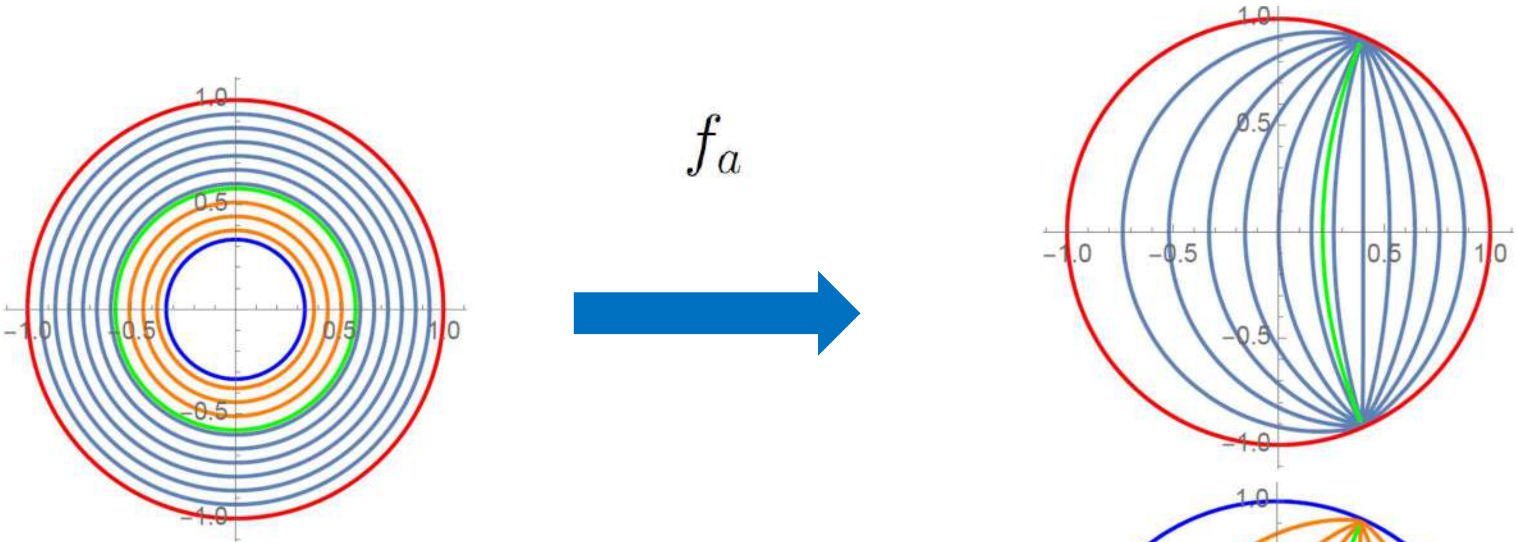
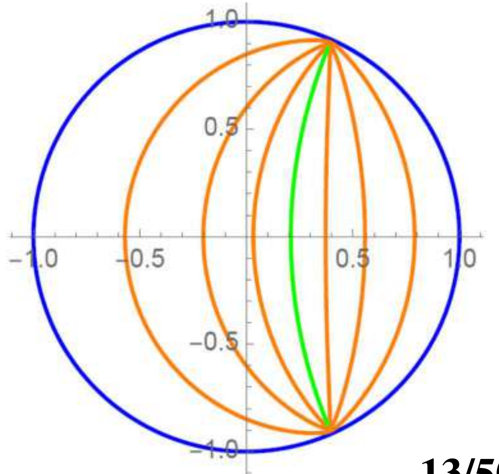


Ahlfors map f_a for $D = \mathbb{A}_q$ with $q = \dfrac{1}{3}$ and $a = \dfrac{1}{2}$.

In this case $\boxed{\widehat{a} = -\dfrac{q}{\overline{a}}} = -\dfrac{1}{3} \times 2 = -\dfrac{2}{3}$.



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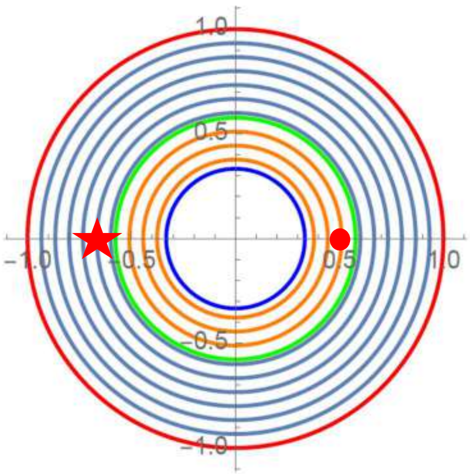
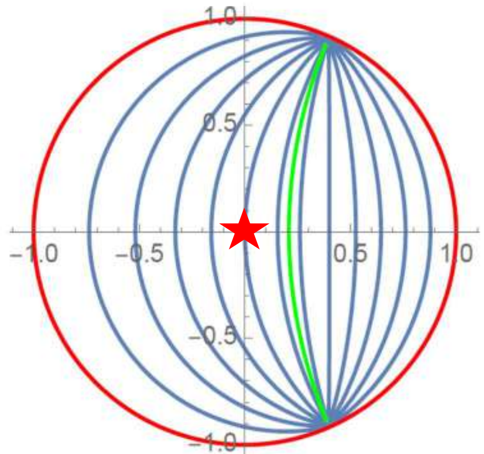
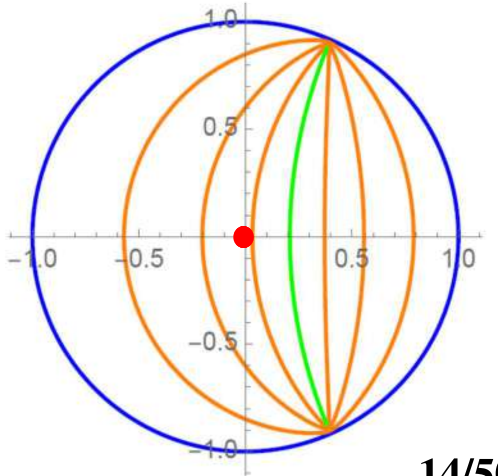


f_a →





Ahlfors map f_a for $D = \mathbb{A}_q$ with $q = \dfrac{1}{3}$ and $a = \dfrac{1}{2}$.

In this case $\boxed{\widehat{a} = -\dfrac{q}{a}} = -\dfrac{1}{3} \times 2 = -\dfrac{2}{3}$.

Consider again **Example 3**: the **Hardy space on $\mathbb{A}_q$**, $H^2(\mathbb{A}_q)$

inner product:

$$\langle f, g\rangle_{H^2(\mathbb{A}_q)} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\sqrt{-1}\phi})\overline{g(e^{\sqrt{-1}\phi})}d\phi + \frac{1}{2\pi}\int_0^{2\pi} f(qe^{\sqrt{-1}\phi})\overline{g(qe^{\sqrt{-1}\phi})}qd\phi,$$

$$f, g \in H^2(\mathbb{A}_q)$$

the previous CONS: $e_n(z) = e^{(q,q)}(z) := \dfrac{z^n}{\sqrt{1+q^{2n+1}}}, \quad n \in \mathbb{Z}$

Remember that for each $a \in \mathbb{A}_q$, $S_{\mathbb{A}_q}(a, \widehat{a}) = S_{\mathbb{A}_q}(\widehat{a}, a) = 0$ with $\boxed{\widehat{a} = \widehat{a}(a) = -\dfrac{q}{\overline{a}}.}$

For an arbitrary but fixed $a \in \mathbb{A}_q$, we write $a_0 := a$ and $a_1 := \widehat{a}$.

new CONS:

$$\widehat{e}_{jn}(z) := \frac{S_{\mathbb{A}_q}(z, a_j)}{\sqrt{S_{\mathbb{A}_q}(a_j, a_j)}} f_a(z)^n, \quad j = 0, 1, \quad n \in \mathbb{N}_0$$

Proof: Put $c_j := 1/\sqrt{S_{\mathbb{A}_q}(a_j, a_j)}, j = 0, 1$. Suppose first $n > m$. On $\partial\mathbb{A}_q$, $\overline{f_a} = 1/f_a$.

$$\langle \widehat{e}_{jn}, \widehat{e}_{km}\rangle_{H^2(\mathbb{A}_q)} = c_j c_k \Big\langle S_{\mathbb{A}_q}(\cdot, a_j) f_a(\cdot)^{n-m}, S_{\mathbb{A}_q}(\cdot, a_k)\Big\rangle_{H^2(\mathbb{A}_q)}$$

$= c_j c_k S_{\mathbb{A}_q}(a_k, a_j) f_a(a_k)^{n-m}$, by the **reproducing property** of $S_{\mathbb{A}_q}(\cdot, a_k)$.

Since $f_a(a) = f_a(\widehat{a}) = 0$, this is zero.

If $n = m$, the above equals to $c_j\overline{c_k} S_{\mathbb{A}_q}(a_k, a_j) = |c_j|^2 S_{\mathbb{A}_q}(a_j, a_j)\delta_{jk}$. ∎

Consider again **Example 3**: the **Hardy space on $\mathbb{A}_q$**, $H^2(\mathbb{A}_q)$

inner product:

$$\langle f, g\rangle_{H^2(\mathbb{A}_q)} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\sqrt{-1}\phi})\overline{g(e^{\sqrt{-1}\phi})}d\phi + \frac{1}{2\pi}\int_0^{2\pi} f(qe^{\sqrt{-1}\phi})\overline{g(qe^{\sqrt{-1}\phi})}qd\phi,$$
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For an arbitrary but fixed $a \in \mathbb{A}_q$, we write $a_0 := a$ and $a_1 := \widehat{a}$.

new CONS: $\widehat{e}_{jn}(z) := \dfrac{S_{\mathbb{A}_q}(z, a_j)}{\sqrt{S_{\mathbb{A}_q}(a_j, a_j)}} f_a(z)^n, \quad j = 0, 1, \quad n \in \mathbb{N}_0$

expression of $S_{\mathbb{A}_q}$ using the Ahlfors mapping function:

$$S_{\mathbb{A}_q}(z, w) = \sum_{j=0}^{1}\sum_{n=0}^{\infty} \widehat{e}_{jn}(z)\overline{\widehat{e}_{jn}(w)}$$
$$= \left(\frac{S_{\mathbb{A}_q}(z, a)\overline{S_{\mathbb{A}_q}(w, a)}}{S_{\mathbb{A}_q}(a, a)} + \frac{S_{\mathbb{A}_q}(z, \widehat{a})\overline{S_{\mathbb{A}_q}(w, \widehat{a})}}{S_{\mathbb{A}_q}(\widehat{a}, \widehat{a})}\right) \sum_{n=0}^{\infty} f_a(z)^n \overline{f_a(z)^n}$$

$$= \left(\frac{S_{\mathbb{A}_q}(z, a)\overline{S_{\mathbb{A}_q}(w, a)}}{S_{\mathbb{A}_q}(a, a)} + \frac{S_{\mathbb{A}_q}(z, \widehat{a})\overline{S_{\mathbb{A}_q}(w, \widehat{a})}}{S_{\mathbb{A}_q}(\widehat{a}, \widehat{a})}\right) \frac{1}{1 - f_a(z)\overline{f_a(w)}}, \; z, w \in \mathbb{A}_q.$$

[Bell16] Bell S. R..: The Cauchy Transform, Potential Theory and Conformal Mapping. 2nd edn, CRC Press, Boca Raton, FL (2016)

We have obtained the following expressions for the Szegő kernels:

For a simply connected domain $D \ni a$,

$$S_D(z,w) = \frac{S_D(z,a)\overline{S_D(w,a)}}{S_D(a,a)} \frac{1}{1 - h_a(z)\overline{h_a(w)}}, \quad z,w \in D.$$

For an annulus $\mathbb{A}_q \ni a$ with $\widehat{a} = -q/\overline{a}$,

$$S_{\mathbb{A}_q}(z,w) = \left(\frac{S_{\mathbb{A}_q}(z,a)\overline{S_{\mathbb{A}_q}(w,a)}}{S_{\mathbb{A}_q}(a,a)} + \frac{S_{\mathbb{A}_q}(z,\widehat{a})\overline{S_{\mathbb{A}_q}(w,\widehat{a})}}{S_{\mathbb{A}_q}(\widehat{a},\widehat{a})} \right) \frac{1}{1 - f_a(z)\overline{f_a(w)}}, \quad z,w \in \mathbb{A}_q.$$

They are written as follows.

For a simply connected domain $D \ni a$,

$$S_D^a(z,w) := S_D(z,w) - \frac{S_D(z,a)\overline{S_D(w,a)}}{S_D(a,a)} = S_D(z,w)h_a(z)\overline{h_a(w)}, \quad z,w \in D.$$

For an annulus $\mathbb{A}_q \ni a$ with $\widehat{a} = -q/\overline{a}$,

$$\begin{aligned} S_{\mathbb{A}_q}^{a,\widehat{a}}(z,w) &:= S_{\mathbb{A}_q}(z,w) - \left(\frac{S_{\mathbb{A}_q}(z,a)\overline{S_{\mathbb{A}_q}(w,a)}}{S_{\mathbb{A}_q}(a,a)} + \frac{S_{\mathbb{A}_q}(z,\widehat{a})\overline{S_{\mathbb{A}_q}(w,\widehat{a})}}{S_{\mathbb{A}_q}(\widehat{a},\widehat{a})} \right) \\ &= S_{\mathbb{A}_q}(z,w)f_a(z)\overline{f_a(w)}, \quad z,w \in \mathbb{A}_q. \end{aligned}$$

For a simply connected domain $D \ni a, z, w$,

$$\boxed{S_D^a(z,w) := S_D(z,w) - \frac{S_D(z,a)\overline{S_D(w,a)}}{S_D(a,a)} = S_D(z,w)h_a(z)\overline{h_a(w)}.}$$

This is a reproducing kernel for the **Hilbert subspace** $H^2_a(D) := \{f \in H^2(D) : f(a) = 0\}$.

For an annulus $\mathbb{A}_q \ni a, w, z$ with $\widehat{a} = -q/\overline{a}$,

$$\boxed{S_{\mathbb{A}_q}^{a,\widehat{a}}(z,w) := S_{\mathbb{A}_q}(z,w) - \left(\frac{S_{\mathbb{A}_q}(z,a)\overline{S_{\mathbb{A}_q}(w,a)}}{S_{\mathbb{A}_q}(a,a)} + \frac{S_{\mathbb{A}_q}(z,\widehat{a})\overline{S_{\mathbb{A}_q}(w,\widehat{a})}}{S_{\mathbb{A}_q}(\widehat{a},\widehat{a})}\right) = S_{\mathbb{A}_q}(z,w)f_a(z)\overline{f_a(w)}.}$$

This is a reproducing kernel for the **Hilbert subspace**

$$H^2_{a,\widehat{a}}(\mathbb{A}_q) := \{f \in H^2(\mathbb{A}_q) : f(a) = f(\widehat{a}) = 0\}.$$

For a simply connected domain $D \ni a, z, w$, **with the Riemann map h_a,**

$$S_D^a(z,w) := S_D(z,w) - \frac{S_D(z,a)\overline{S_D(w,a)}}{S_D(a,a)} = S_D(z,w)\boldsymbol{h_a(z)\overline{h_a(w)}}.$$

This is a reproducing kernel for the Hilbert subspace $H_a^2(D) := \{f \in H^2(D) : f(a) = 0\}$.

For an annulus $\mathbb{A}_q \ni a, w, z$ with $\widehat{a} = -q/\overline{a}$, **with the Ahlfors map f_a,**

$$S_{\mathbb{A}_q}^{a,\widehat{a}}(z,w) := S_{\mathbb{A}_q}(z,w) - \left(\frac{S_{\mathbb{A}_q}(z,a)\overline{S_{\mathbb{A}_q}(w,a)}}{S_{\mathbb{A}_q}(a,a)} + \frac{S_{\mathbb{A}_q}(z,\widehat{a})\overline{S_{\mathbb{A}_q}(w,\widehat{a})}}{S_{\mathbb{A}_q}(\widehat{a},\widehat{a})} \right) = S_{\mathbb{A}_q}(z,w)\boldsymbol{f_a(z)\overline{f_a(w)}}.$$

This is a reproducing kernel for the Hilbert subspace

$$H_{a,\widehat{a}}^2(\mathbb{A}_q) := \{f \in H^2(\mathbb{A}_q) : f(a) = f(\widehat{a}) = 0\}.$$

For a simply connected domain $D \ni a, z, w$, **with the Riemann map h_a,**

$$S_D^a(z,w) := S_D(z,w) - \frac{S_D(z,a)\overline{S_D(w,a)}}{S_D(a,a)} = S_D(z,w)\boldsymbol{h_a(z)\overline{h_a(w)}}.$$

This is a reproducing kernel for the Hilbert subspace $H_a^2(D) := \{f \in H^2(D) : f(a) = 0\}$.

For an annulus $\mathbb{A}_q \ni a, w, z$ with $\widehat{a} = -q/\overline{a}$, **with the Ahlfors map f_a,**

$$S_{\mathbb{A}_q}^{a,\widehat{a}}(z,w) := S_{\mathbb{A}_q}(z,w) - \left(\frac{S_{\mathbb{A}_q}(z,a)\overline{S_{\mathbb{A}_q}(w,a)}}{S_{\mathbb{A}_q}(a,a)} + \frac{S_{\mathbb{A}_q}(z,\widehat{a})\overline{S_{\mathbb{A}_q}(w,\widehat{a})}}{S_{\mathbb{A}_q}(\widehat{a},\widehat{a})} \right) = S_{\mathbb{A}_q}(z,w)\boldsymbol{f_a(z)\overline{f_a(w)}}.$$

This is a reproducing kernel for the Hilbert subspace

$$H_{a,\widehat{a}}^2(\mathbb{A}_q) := \{f \in H^2(\mathbb{A}_q) : f(a) = f(\widehat{a}) = 0\}.$$

For an annulus $\mathbb{A}_q \ni a, w, z$, **how about for $H_a^2(\mathbb{A}_q) := \{f \in H^2(\mathbb{A}_q) : f(a) = 0\}$?**

$$S_{\mathbb{A}_q}^a(z,w) := S_{\mathbb{A}_q}(z,w) - \frac{S_{\mathbb{A}_q}(z,a)\overline{S_{\mathbb{A}_q}(w,a)}}{S_{\mathbb{A}_q}(a,a)} = \boldsymbol{???}$$

For a simply connected domain $D \ni a, z, w$, **with the Riemann map h_a,**

$$S_D^a(z,w) := S_D(z,w) - \frac{S_D(z,a)\overline{S_D(w,a)}}{S_D(a,a)} = S_D(z,w)\boldsymbol{h_a(z)\overline{h_a(w)}}.$$

For an annulus $\mathbb{A}_q \ni a, w, z$ with $\widehat{a} = -q/\overline{a}$, **with the Ahlfors map f_a,**

$$S_{\mathbb{A}_q}^{a,\widehat{a}}(z,w) := S_{\mathbb{A}_q}(z,w) - \left(\frac{S_{\mathbb{A}_q}(z,a)\overline{S_{\mathbb{A}_q}(w,a)}}{S_{\mathbb{A}_q}(a,a)} + \frac{S_{\mathbb{A}_q}(z,\widehat{a})\overline{S_{\mathbb{A}_q}(w,\widehat{a})}}{S_{\mathbb{A}_q}(\widehat{a},\widehat{a})}\right) = S_{\mathbb{A}_q}(z,w)\boldsymbol{f_a(z)\overline{f_a(w)}}.$$

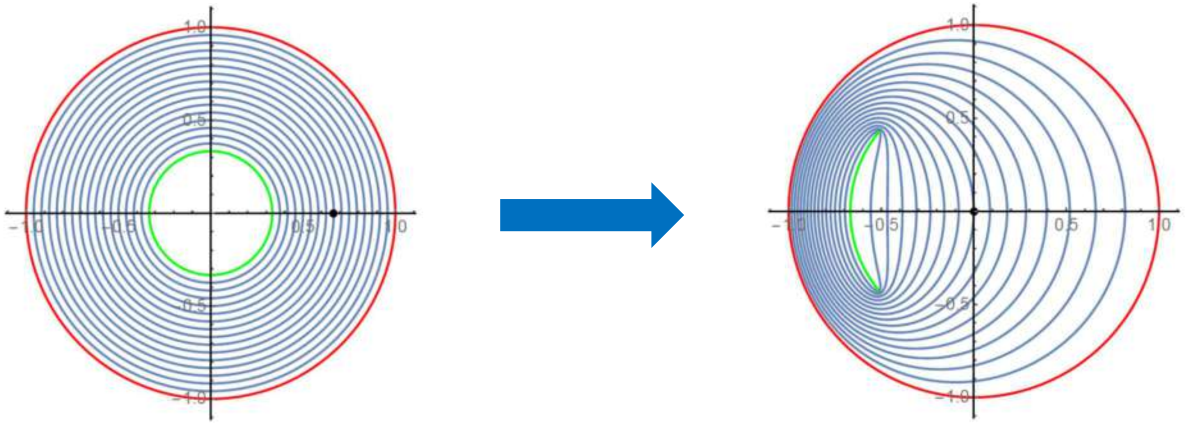
For an annulus $\mathbb{A}_q \ni a, w, z$, **how about for $H_a^2(\mathbb{A}_q) := \{f \in H^2(\mathbb{A}_q) : f(a) = 0\}$?**

The answer was found in [MS94].

[MS94] Mccullough, S, Shen, L.C.: On the Szegő kernel of an annulus. Proc. Amer. Math. Soc. 121, 1111–1121 (1994)

$$S_{\mathbb{A}_q}^a(z,w) := S_{\mathbb{A}_q}(z,w) - \frac{S_{\mathbb{A}_q}(z,a)\overline{S_{\mathbb{A}_q}(w,a)}}{S_{\mathbb{A}_q}(a,a)} = \boldsymbol{S_{\mathbb{A}_q}(z,w;q|a|^2)h_a^q(z)\overline{h_a^q(z)}},$$

where **$h_a^q(z)$ is a conformal map from $\mathbb{A}_q$ to $\mathbb{D} \setminus \{$a circular slit$\}$.**



For an annulus $\mathbb{A}_q \ni a, w, z$, and **with $r > 0$**, more generally,

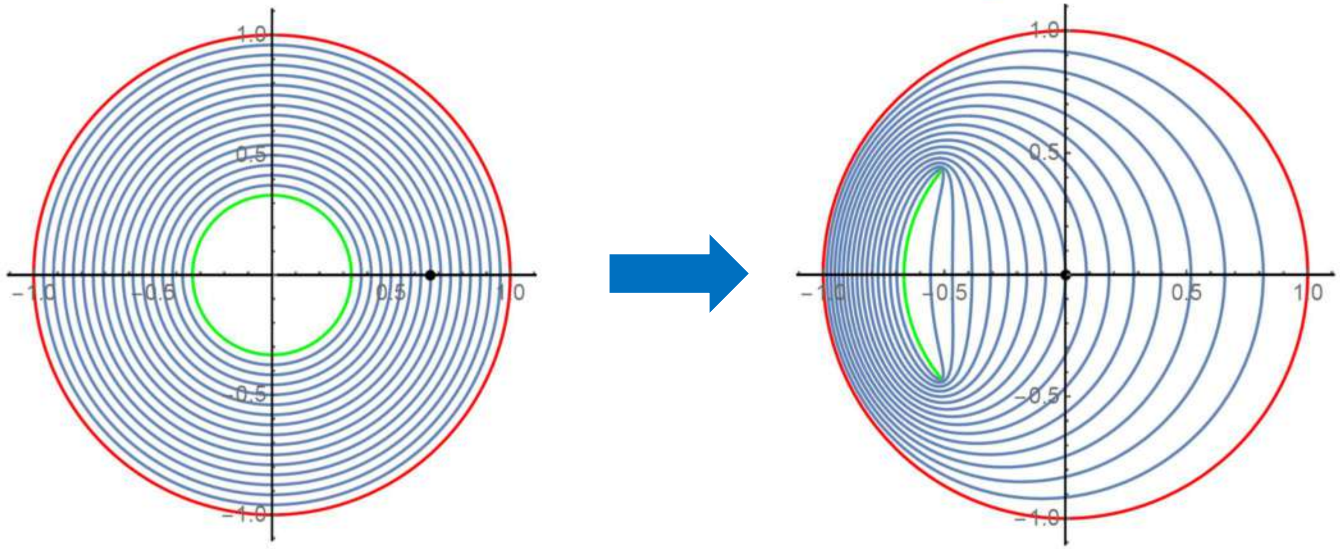
[MS94] Mccullough, S, Shen, L.C.: On the Szegő kernel of an annulus. Proc. Amer. Math. Soc. <u>121</u>, 1111–1121 (1994)

$$S^a_{\mathbb{A}_q}(z, w; \boldsymbol{r}) := S_{\mathbb{A}_q}(z, w; r) - \frac{S_{\mathbb{A}_q}(z, a; r)\overline{S_{\mathbb{A}_q}(w, a; r)}}{S_{\mathbb{A}_q}(a, a; r)} = S_{\mathbb{A}_q}(z, w; \boldsymbol{r|a|^2}) h^q_a(z) \overline{h^q_a(z)},$$

where $h^q_a(z)$ is a conformal map from $\mathbb{A}_q$ to $\mathbb{D} \setminus \{\text{a circular slit}\}$.

Remember that $S_{\mathbb{A}_q}(z, w; r)$ is the **weighted Szegő kernel with weight r**,

and the original one is given by $S_{\mathbb{A}_q}(z, w) = S_{\mathbb{A}_q}(z, w; \boldsymbol{q})$.



Conformal map $h^q_a : \mathbb{A}_q \to \mathbb{D} \setminus \{\text{a circular slit}\}$ is illustrated for $q = 1/3$ and $a = 2/3$. The point $a = 2/3$ in $\mathbb{A}_{1/3}$ is mapped to the origin. The outer boundary of $\mathbb{A}_{1/3}$ (denoted by a red circle) is mapped to a unit circle (a red circle) making the boundary of $\mathbb{D}$. The image of the inner boundary of $\mathbb{A}_{1/3}$ (a green circle) makes a circular slit inside of $\mathbb{D}$ (denoted by a green arc)

2. Gaussian Analytic Functions (GAFs), Gaussian Processes (GPs), and Zero-Point Processes

Let $i := \sqrt{-1}$.

The Lebesgue measure on $\mathbb{C}$ is denoted by $m(dz) = d\mathrm{Re}\, z d\mathrm{Im}\, z$, $z \in \mathbb{C}$.

Complex standard normal random variable (distribution)

$$\zeta = \mathrm{Re}\,\zeta + i\mathrm{Im}\,\zeta \sim \mathrm{N}_{\mathbb{C}}(0,1)$$

$$\iff \quad \text{the probability density function: } p(z) := \frac{1}{\sqrt{\pi}} e^{-(\mathrm{Re}\, z)^2} \times \frac{1}{\sqrt{\pi}} e^{-(\mathrm{Im}\, z)^2} = \frac{1}{\pi} e^{-|z|^2}$$

$$\iff \quad \mathbf{P}(\zeta \in D) = \int_D p(z) m(dz), \quad D \subset \mathbb{C}$$

$$\mathbf{E}[f(\zeta)] := \int f(z) p(z) m(dz)$$

In particular, $\mathbf{E}[\zeta] = \mathbf{E}[\overline{\zeta}\,] = 0$, $\quad \mathbf{E}[\zeta^2] = \mathbf{E}[(\mathrm{Re}\,\zeta)^2 + 2i\mathrm{Re}\,\zeta\mathrm{Im}\,\zeta - (\mathrm{Im}\,\zeta)^2] = \frac{1}{2} + 0 - \frac{1}{2} = 0,$

$$\mathbf{E}[|\zeta|^2] = \mathrm{E}[(\mathrm{Re}\,\zeta)^2 + (\mathrm{Im}\,\zeta)^2] = \frac{1}{2} + \frac{1}{2} = 1.$$

Consider $\{\zeta_n\}_{n\in\mathbb{Z}}$:

a series of **independently and identically distributed (i.i.d.)** complex standard normal random variables

$$\iff \quad \zeta_n \sim \mathcal{N}_{\mathbb{C}}(0,1), \ \forall n \in \mathbb{Z}$$

and $\quad \zeta_n \perp \zeta_m \ \forall n \neq m$

that is, $\quad \mathbf{E}[f(\zeta_n)g(\zeta_m)] = \mathbf{E}[f(\zeta_n)]\mathbf{E}[g(\zeta_m)] \ \forall n \neq m$

In particular, $\mathbf{E}[\zeta_n\overline{\zeta_m}] = \delta_{nm}\mathbf{E}[|\zeta_n|^2] = \delta_{nm}, \ n, m \in \mathbb{Z}$.

Examples of **Gaussian analytic functions (GAFs)**:

Example 2' : $H^2(\mathbb{D})$: the Hardy space on $\mathbb{D}$.

$$f \in H^2(\mathbb{D}) \iff f(z) = \sum_{n \in \mathbb{N}_0} c_n e_n(z) = \sum_{n=0}^{\infty} c_n z^n$$

$$\boxed{X_{\mathbb{D}}(z) := \sum_{n=0}^{\infty} \zeta_n z^n, \quad z \in \mathbb{D}}$$

This random Taylor series converges $\forall z \in \mathbb{D}$ with probability 1.

Covariance function: for $z, w \in \mathbb{D}$,

$$\mathbf{E}[X_{\mathbb{D}}(z)\overline{X_{\mathbb{D}}(w)}] = \mathbf{E}\left[\sum_{n=0}^{\infty} \zeta_n z^n \sum_{m=0}^{\infty} \overline{\zeta}^m \overline{w}^m\right] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathbf{E}[\zeta_n \overline{\zeta_m}] z^n \overline{w}^m$$

$$= \sum_{n=0}^{\infty} (z\overline{w})^n = \frac{1}{1 - z\overline{w}} = S_{\mathbb{D}}(z, w).$$

Examples 3' and 4' : $H_r^2(\mathbb{A}_q)$: the Hardy space on $\mathbb{A}_q$ with weight $r > 0$.

$$f \in H_r^2(\mathbb{A}_q) \quad \Longleftrightarrow \quad f(z) = \sum_{n \in \mathbb{Z}} c_n e_n^{(q,r)}(z) = \sum_{n=-\infty}^{\infty} c_n \frac{z^n}{\sqrt{1 + rq^{2n}}}$$

$$\boxed{X_{\mathbb{A}_q}^r(z) := \sum_{n=-\infty}^{\infty} \zeta_n \frac{z^n}{\sqrt{1 + rq^{2n}}}, \quad z \in \mathbb{A}_q}$$

This random Laurent series converges $\forall z \in \mathbb{A}_q$ with probability 1.

In particular, by setting $r = q$,

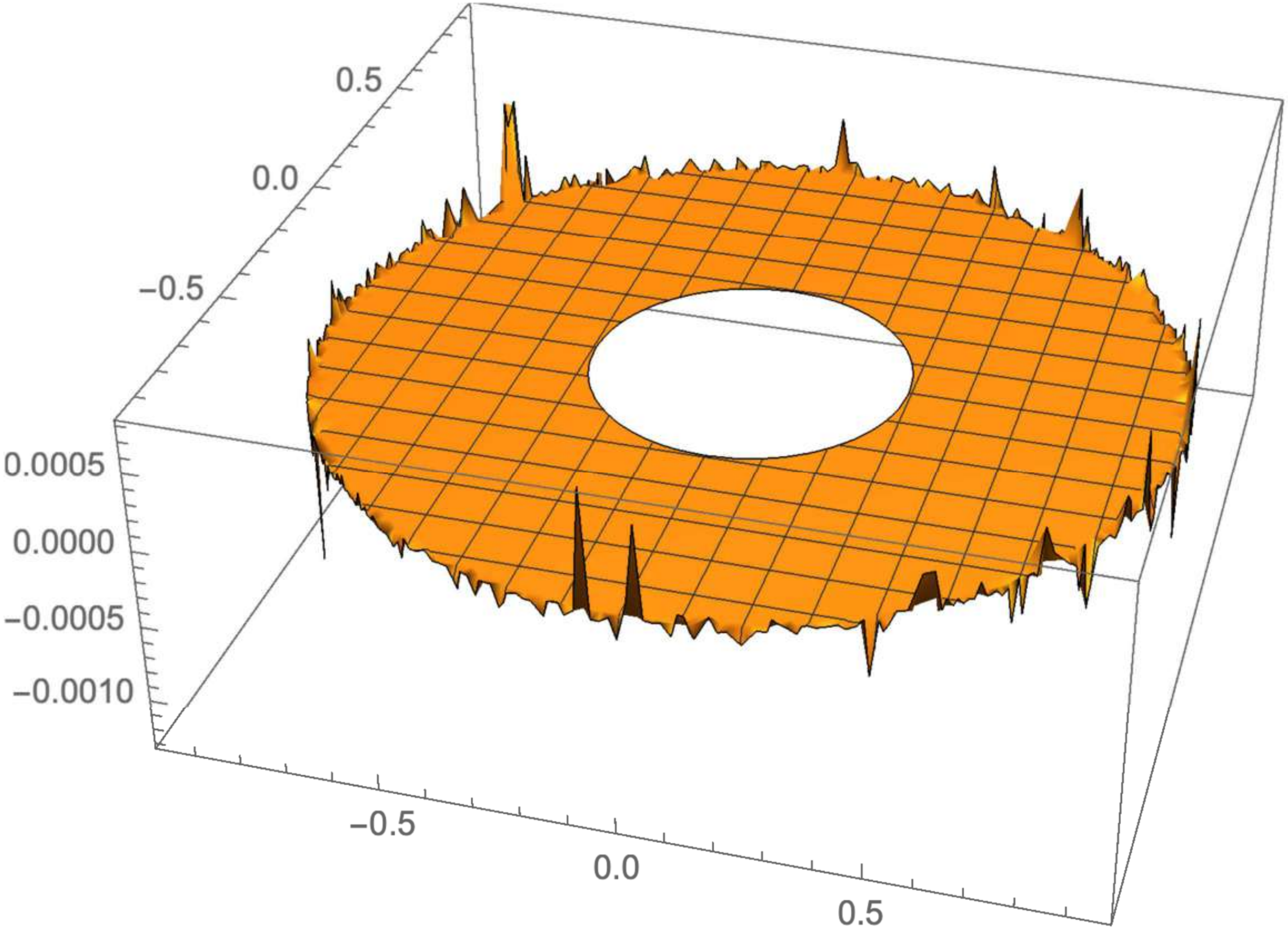
$$\boxed{X_{\mathbb{A}_q}(z) := X_{\mathbb{A}_q}^q(z) = \sum_{n=-\infty}^{\infty} \zeta_n \frac{z^n}{\sqrt{1 + q^{2n+1}}}, \quad z \in \mathbb{A}_q.}$$

Covariance function: for $z, w \in \mathbb{A}_q$,

$$\mathbf{E}[X_{\mathbb{A}_q}^r(z)\overline{X_{\mathbb{A}_q}^r(w)}] = \sum_{n=-\infty}^{\infty} \frac{1}{1 + rq^{2n}} (z\overline{w})^n = S_{\mathbb{A}_q}(z, w; r).$$

In particular, by setting $r = q$,

$$\mathbf{E}[X_{\mathbb{A}_q}(z)\overline{X_{\mathbb{A}_q}(w)}] = \sum_{n=-\infty}^{\infty} \frac{1}{1 + q^{2n+1}} (z\overline{w})^n = S_{\mathbb{A}_q}(z, w).$$



A sample of the real part of GAF $\{X_{\mathbb{A}_q}(z)\}_{z \in \mathbb{A}_q}$.
Due to bursts near the outer boundary,
all fine structures are smeared out in this picture.

GAFs $X_D(z) := \sum_{n \in \mathcal{I}} \zeta_n e_n(z), z \in D$ are **(centered) Gaussian processes (GPs)**.

$\iff$ For an arbitrary $n \in \mathbb{N} := \{1, 2, \dots\}$ and an arbitrary set of points $z_1, \dots, z_n \in D$,

$$\boxed{(X_D(z_1), \dots, X_D(z_n)) \sim \mathrm{N}^n_{\mathbb{C}}(\mathbf{0}, \Sigma_n),}$$

where $\Sigma_n := (\Sigma_n(z_j, z_k))_{1 \leq j,k \leq n} = (k_{\mathcal{H}}(z_j, z_k))_{1 \leq j,k \leq n} =: (k_{\mathcal{H}})_n$

with $k_{\mathcal{H}}(z, w) = \mathbf{E}[X_D(z)\overline{X_D(w)}], \quad z, w \in D.$

$$\iff \begin{bmatrix} X_D(z_1) \\ X_D(z_2) \\ \cdots \\ X_D(z_n) \end{bmatrix} \sim \mathrm{N}^n_{\mathbb{C}} \left(\begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \begin{bmatrix} k_{\mathcal{H}}(z_1, z_1) & k_{\mathcal{H}}(z_1, z_2) & \cdots & k_{\mathcal{H}}(z_1, z_n) \\ k_{\mathcal{H}}(z_2, z_1) & k_{\mathcal{H}}(z_2, z_2) & \cdots & k_{\mathcal{H}}(z_2, z_n) \\ & \cdots & \cdots & \\ k_{\mathcal{H}}(z_n, z_1) & k_{\mathcal{H}}(z_n, z_2) & \cdots & k_{\mathcal{H}}(z_n, z_n) \end{bmatrix} \right)$$

$$\iff \mathbf{P}\Big(X_D(z_1) \in d\xi_1, X_D(z_2) \in d\xi_2, \dots, X_D(z_n) \in d\xi_n\Big)$$

$$= \frac{1}{\pi^n \det[(k_{\mathcal{H}})_n]} \exp\Big(-\boldsymbol{\xi}^T (k_{\mathcal{H}})_n^{-1} \boldsymbol{\xi}\Big) d\boldsymbol{\xi},$$

$$\boldsymbol{\xi} := (\xi_1, \xi_2, \dots, \xi_n) \in D^n, \ d\boldsymbol{\xi} := \prod_{j=1}^n d\xi_j$$

covariance kernel of the GP = **reproducing kernel** of the Hilbert space

Choose one point and write it as $z_1 = a$,

$$\begin{bmatrix} X_D(a) \\ X_D(z_2) \\ \cdots \\ X_D(z_n) \end{bmatrix} \sim \mathrm{N}_{\mathbb{C}}^n \left(\begin{matrix} 0, \\ \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix} \end{matrix}, \begin{matrix} k_{\mathcal{H}}(a,a) \\ \begin{bmatrix} k_{\mathcal{H}}(z_2,a) \\ \cdots \\ k_{\mathcal{H}}(z_n,a) \end{bmatrix} \end{matrix} \begin{matrix} \begin{bmatrix} k_{\mathcal{H}}(a,z_2) & \cdots & k_{\mathcal{H}}(a,z_n) \end{bmatrix} \\ \begin{bmatrix} k_{\mathcal{H}}(z_2,z_2) & \cdots & k_{\mathcal{H}}(z_2,z_n) \\ & \ddots & \\ k_{\mathcal{H}}(z_n,z_2) & \cdots & k_{\mathcal{H}}(z_n,z_n) \end{bmatrix} \end{matrix} \right).$$

Assume that $k_{\mathcal{H}}(a,a) > 0$.

$\Longrightarrow$ **Under the condition $X_D(a) = 0$**, $(X_D(z_2), \dots, X_D(z_n))$ is **again a centered GP**

$$\begin{bmatrix} X_D(z_2) \\ \cdots \\ X_D(z_n) \end{bmatrix} \sim \mathrm{N}_{\mathbb{C}}^{n-1} \left(\begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}, \left[\left(\frac{\det \begin{bmatrix} k_{\mathcal{H}}(z_j,z_k) & k_{\mathcal{H}}(z_j,a) \\ k_{\mathcal{H}}(a,z_k) & k_{\mathcal{H}}(a,a) \end{bmatrix}}{k_{\mathcal{H}}(a,a)} \right)_{2 \le j,k \le n} \right] \right).$$

$$\text{with the correlation kernel} = \frac{\det \begin{bmatrix} k_{\mathcal{H}}(z,w) & k_{\mathcal{H}}(z,a) \\ k_{\mathcal{H}}(a,w) & k_{\mathcal{H}}(a,a) \end{bmatrix}}{k_{\mathcal{H}}(a,a)}$$

$$= k_{\mathcal{H}}(z,w) - \frac{k_{\mathcal{H}}(z,a)\overline{k_{\mathcal{H}}(w,a)}}{k_{\mathcal{H}}(a,a)} =: k_{\mathcal{H}}^a(z,w), \quad z,w \in D.$$

Choose two points and write them as $z_1 = a$ and $z_2 = b$,

$$\begin{bmatrix} X_D(a) \\ X_D(b) \\ X_D(z_3) \\ \cdots \\ X_D(z_n) \end{bmatrix} \sim \mathrm{N}_{\mathbb{C}}^n \left(\begin{matrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} k_{\mathcal{H}}(a,a) & k_{\mathcal{H}}(a,b) \\ k_{\mathcal{H}}(b,a) & k_{\mathcal{H}}(b,b) \end{bmatrix} & \begin{bmatrix} k_{\mathcal{H}}(a,z_3) & \cdots & k_{\mathcal{H}}(a,z_n) \\ k_{\mathcal{H}}(b,z_3) & \cdots & k_{\mathcal{H}}(b,z_n) \end{bmatrix} \\ \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}, & \begin{bmatrix} k_{\mathcal{H}}(z_3,a) & k_{\mathcal{H}}(z_3,b) \\ \cdots & \\ k_{\mathcal{H}}(z_n,a) & k_{\mathcal{H}}(z_n,b) \end{bmatrix} & \begin{bmatrix} k_{\mathcal{H}}(z_3,z_3) & \cdots & k_{\mathcal{H}}(z_3,z_n) \\ & \ddots & \\ k_{\mathcal{H}}(z_n,z_3) & \cdots & k_{\mathcal{H}}(z_n,z_n) \end{bmatrix} \end{matrix} \right).$$

Assume that $\det \begin{bmatrix} k_{\mathcal{H}}(a,a) & k_{\mathcal{H}}(a,b) \\ k_{\mathcal{H}}(b,a) & k_{\mathcal{H}}(b,b) \end{bmatrix} > 0.$

$\Longrightarrow$ **Under the condition $X_D(a) = X_D(b) = 0$**, $(X_D(z_3), \ldots, X_D(z_n))$ is **again a centered GP**.

$$\begin{bmatrix} X_D(z_3) \\ \cdots \\ X_D(z_n) \end{bmatrix} \sim \mathrm{N}_{\mathbb{C}}^{n-2} \left(\begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}, \left[\left(\frac{\det \begin{bmatrix} k_{\mathcal{H}}(z_j,z_k) & k_{\mathcal{H}}(z_j,a) & k_{\mathcal{H}}(z_j,b) \\ k_{\mathcal{H}}(a,z_k) & k_{\mathcal{H}}(a,a) & k_{\mathcal{H}}(a,b) \\ k_{\mathcal{H}}(b,z_k) & k_{\mathcal{H}}(b,a) & k_{\mathcal{H}}(b,b) \end{bmatrix}}{\det \begin{bmatrix} k_{\mathcal{H}}(a,a) & k_{\mathcal{H}}(a,b) \\ k_{\mathcal{H}}(b,a) & k_{\mathcal{H}}(b,b) \end{bmatrix}} \right)_{3 \le j,k \le n} \right] \right).$$

Choose two points and write them as $z_1 = a$ and $z_2 = b$,

Under the condition $X_D(a) = X_D(b) = 0$, $(X_D(z_3), \dots, X_D(z_n))$ is **again a centered GP**.

$$\begin{bmatrix} X_D(z_3) \\ \cdots \\ X_D(z_n) \end{bmatrix} \sim \mathrm{N}_{\mathbb{C}}^{n-2}\left(\begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}, \left[\left(\frac{\det \begin{bmatrix} k_{\mathcal{H}}(z_j, z_k) & k_{\mathcal{H}}(z_j, a) & k_{\mathcal{H}}(z_j, b) \\ k_{\mathcal{H}}(a, z_k) & k_{\mathcal{H}}(a, a) & k_{\mathcal{H}}(a, b) \\ k_{\mathcal{H}}(b, z_k) & k_{\mathcal{H}}(b, a) & k_{\mathcal{H}}(b, b) \end{bmatrix}}{\det \begin{bmatrix} k_{\mathcal{H}}(a, a) & k_{\mathcal{H}}(a, b) \\ k_{\mathcal{H}}(b, a) & k_{\mathcal{H}}(b, b) \end{bmatrix}} \right)_{3 \le j,k \le n} \right] \right).$$

Moreover, if we assume that **$b = \widehat{a}$ and hence $k_{\mathcal{H}}(a, b) = k_{\mathcal{H}}(b, a) = 0$**, then

$$\text{the correlation kernel} = \frac{\det \begin{bmatrix} k_{\mathcal{H}}(z, w) & k_{\mathcal{H}}(z, a) & k_{\mathcal{H}}(z, \widehat{a}) \\ k_{\mathcal{H}}(a, w) & k_{\mathcal{H}}(a, a) & 0 \\ k_{\mathcal{H}}(\widehat{a}, w) & 0 & k_{\mathcal{H}}(\widehat{a}, \widehat{a}) \end{bmatrix}}{\det \begin{bmatrix} k_{\mathcal{H}}(a, a) & 0 \\ 0 & k_{\mathcal{H}}(\widehat{a}, \widehat{a}) \end{bmatrix}}$$

$$= k_{\mathcal{H}}(z, w) - \left(\frac{k_{\mathcal{H}}(z, a)\overline{k_{\mathcal{H}}(w, a)}}{k_{\mathcal{H}}(a, a)} + \frac{k_{\mathcal{H}}(z, \widehat{a})\overline{k_{\mathcal{H}}(w, \widehat{a})}}{k_{\mathcal{H}}(\widehat{a}, \widehat{a})} \right) =: k_{\mathcal{H}}^{a,\widehat{a}}(z, w), \quad z, w \in D.$$

the **covariance kernel of GAF** under $\boldsymbol{X_D(a) = 0}$

$\iff$ the **reproducing kernel** of the conditional Hilbert space $\boldsymbol{\mathcal{H}^a}$

$$\frac{\det \begin{bmatrix} k_{\mathcal{H}}(z,w) & k_{\mathcal{H}}(z,a) \\ k_{\mathcal{H}}(a,w) & k_{\mathcal{H}}(a,a) \end{bmatrix}}{k_{\mathcal{H}}(a,a)} = k_{\mathcal{H}}(z,w) - \frac{k_{\mathcal{H}}(z,a)\overline{k_{\mathcal{H}}(w,a)}}{k_{\mathcal{H}}(a,a)} =: k_{\mathcal{H}}^{a}(z,w), \quad z,w \in D$$

the **covariance kernel of GAF** under $\boldsymbol{X_D(a) = X_D(\widehat{a}) = 0}$

$\iff$ the **reproducing kernel** of the conditional Hilbert space $\boldsymbol{\mathcal{H}^{a,\widehat{a}}}$

$$\frac{\det \begin{bmatrix} k_{\mathcal{H}}(z,w) & k_{\mathcal{H}}(z,a) & k_{\mathcal{H}}(z,\widehat{a}) \\ k_{\mathcal{H}}(a,w) & k_{\mathcal{H}}(a,a) & 0 \\ k_{\mathcal{H}}(\widehat{a},w) & 0 & k_{\mathcal{H}}(\widehat{a},\widehat{a}) \end{bmatrix}}{\det \begin{bmatrix} k_{\mathcal{H}}(a,a) & 0 \\ 0 & k_{\mathcal{H}}(\widehat{a},\widehat{a}) \end{bmatrix}} = k_{\mathcal{H}}(z,w) - \left(\frac{k_{\mathcal{H}}(z,a)\overline{k_{\mathcal{H}}(w,a)}}{k_{\mathcal{H}}(a,a)} + \frac{k_{\mathcal{H}}(z,\widehat{a})\overline{k_{\mathcal{H}}(w,\widehat{a})}}{k_{\mathcal{H}}(\widehat{a},\widehat{a})} \right)$$

$$=: k_{\mathcal{H}}^{a,\widehat{a}}(z,w), \quad z,w \in D$$

3. Main Results: Zero-Point Processes of GAFs

We study **a zero set** of GAF $\{X_D(z)\}_{z \in D}$, which is regarded as a **point process** on D. It is denoted by a nonnegative-integer-valued Radon measure,

$$\mathcal{Z}_{X_D}(\cdot) = \sum_{z \in D: X_D(z)=0} \delta_z(\cdot),$$

which we simply call a **zero-point process** of the GAF.

Zero-point processes of GAFs have been extensively studied in **quantum and statistical physics** as solvable models of quantum chaotic systems and interacting particle systems.

e.g., Bogomolny–Bohigas–Lebœuf (1992), Hannay (1996), Forrester (2010)

Many important characterizations of their probability laws have been reported in **probability theory**

e.g., Edelman–Kostlan (1995), Bleher–Shiffman–Zelditch (2000), Sodin–Tsirelson (2004), Peres–Virág (2005), Shirai (2012), Matsumoto–Shirai (2013)

The following monograph is very useful:

[HKPV09] Hough, J. B., Krishnapur, M., Peres, Y., Virág, B.:
Zeros of Gaussian Analytic Functions and Determinantal Point Processes.
University Lecture Series, Vol. 51, Amer. Math. Soc., Providence, RI (2009)

The configuration space of zero-point process $\mathcal{Z}_{X_D}(\cdot)$ is given by

$$\mathrm{Conf}(D) = \Big\{ \xi = \sum_j \delta_{z_j} : z_j \in D,\ \xi(\Lambda) < \infty \text{ for all bounded set } \Lambda \subset D \Big\}.$$

Let $\mathcal{B}_{\mathrm{c}}(D)$ be the set of all bounded measurable complex functions on D of compact support.

For $\xi \in \mathrm{Conf}(D)$ and $\phi \in \mathcal{B}_{\mathrm{c}}(D)$, we set $\boxed{\langle \xi, \phi \rangle := \int_D \phi(z)\, \xi(dz) = \sum_j \phi(z_j).}$

For a point process $\mathcal{Z}_{X_D}$, if there exists a non-negative measurable function $\rho^{(1)}_{\mathcal{Z}_{X_D}}$ such that

$$\mathbf{E}[\langle \mathcal{Z}_{X_D}, \phi \rangle] = \int_D \phi(z) \rho^{(1)}_{\mathcal{Z}_{X_D}}(z) m(dz)/\pi \quad \forall \phi \in \mathcal{B}_{\mathrm{c}}(D),$$

$\rho^{(1)}_{\mathcal{Z}_{X_D}}(z)$ is called the **first correlation function** of $\mathcal{Z}_{X_D}$ with respect to the measure m/π.

By definition, $\rho^{(1)}_{\mathcal{Z}_{X_D}}(z)$ gives the **density of point** at $z \in D$ with respect to $m(dz)/\pi$.

For $n = 2, 3, \dots$, from $\xi \in \mathrm{Conf}(D)$ we define $\xi_n := \displaystyle\sum_{j_1,\dots,j_n : j_k \neq j_\ell, k \neq \ell} \delta_{z_{j_1}} \cdots \delta_{z_{j_n}}$,

and denote the n-product measure of m by $m^{\otimes n}(dz_1 \cdots dz_n) := \displaystyle\prod_{j=1}^{n} m(dz_j)$.

For a point process $\mathcal{Z}_{X_D}$,

if there exists a symmetric, non-negative measurable function $\rho^{(n)}_{\mathcal{Z}_{X_D}}$ on D^n such that

$$\mathbf{E}[\langle(\mathcal{Z}_{X_D})_n, \phi\rangle] = \int_{D^n} \phi(z_1, \dots, z_n) \rho^{(n)}_{\mathcal{Z}_{X_D}}(z_1, \dots, z_n) m^{\otimes n}(dz_1 \cdots dz_n)/\pi^n \quad \forall \phi \in \mathcal{B}_{\mathrm{c}}(D^n),$$

we say $\rho^{(n)}_{\mathcal{Z}_{X_D}}(z_1, \dots, z_n)$ is the ***n*-th correlation function** of $\mathcal{Z}_{X_D}$ with respect to m/π.

In order to describe our main theorem, we introduce some notations and functions.

Determinant and **permanent** are defined for an $n \times n$ matrix $M = (m_{jk})_{1 \le j,k \le n}$ as

$$\det M = \det_{1 \le j,k \le n} [m_{jk}] := \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{\ell=1}^{n} m_{\ell \sigma(\ell)},$$

$$\operatorname{per} M = \operatorname{per}_{1 \le j,k \le n} [m_{jk}] := \sum_{\sigma \in \mathfrak{S}_n} \prod_{\ell=1}^{n} m_{\ell \sigma(\ell)},$$

where $\mathfrak{S}_n$ denotes the symmetric group of order n.

We introduce the following notation.

$$\boxed{\operatorname{perdet} M = \operatorname{perdet}_{1 \le j,k \le n} [m_{jk}] := \operatorname{per} M \det M,}$$

that is, **perdet M** denotes per M multiplied by det M.

Assume that $p \in \mathbb{C}$ is a fixed number such that $0 < |p| < 1$.

We use the following standard notations (the p-Pochhammer symbols),

$$(a;p)_n := \prod_{m=0}^{n-1}(1-ap^m), \ (a;p)_\infty := \prod_{m=0}^{\infty}(1-ap^m), \ (a_1,\dots,a_n;p)_\infty := \prod_{j=1}^{n}(a_j;p)_\infty.$$

The **theta function** with argument z and nome p is defined by

$$\boxed{\theta(z;p) := (z, p/z; p)_\infty.}$$

We often use the shorthand notation: $\theta(z_1,\dots,z_n;p) := \prod_{j=1}^{n}\theta(z_j;p)$.

In the following, we only consider the case in which $p = q^2$, and set $\boxed{\theta(z) := \theta(z;q^2).}$

We will use the following **theta-function representations** of the Szegő kernel;

$$\boxed{S_{\mathbb{A}_q}(z,w;r) = \frac{q_0^2\theta(-rz\overline{w})}{\theta(-r, z\overline{w})}, \quad z,w \in \mathbb{A}_q,\ r>0 \text{ with } q_0 := (q^2;q^2)_\infty = \prod_{n=1}^{\infty}(1-q^{2n}),}$$

and the conformal map of Mccullough–Shen;

$$\boxed{h_a^q(z) = z\frac{\theta(a/z)}{\theta(\overline{a}z)}, \quad {h_a^q}'(a) = \frac{q_0^2}{\theta(|a|^2)}, \quad z,a \in \mathbb{A}_q.}$$

As a main theorem, here we give a result for the zero-point process $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$ of the GAF associated with the **weighted Szegő kernel** $S_{\mathbb{A}_q}(\cdot,\cdot;r)$.
We will show the results for other processes $\mathcal{Z}_{X_{\mathbb{A}_q}}$ and $\mathcal{Z}_{X_{\mathbb{D}}}$ are reduced from this main result.

Theorem 3.1 Consider the zero-point process $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$ on $\mathbb{A}_q$ with $r > 0$. Then, it is a **permanental-determinantal point process (PDPP)** in the sense that it has correlation functions $\{\rho^{(n)}_{\mathcal{Z}_{X^r_{\mathbb{A}_q}}}\}_{n\in\mathbb{N}}$ given by

$$\rho^{(n)}_{\mathcal{Z}_{X^r_{\mathbb{A}_q}}}(z_1,\dots,z_n) = \frac{\theta(-r)}{\theta(-r\prod_{m=1}^n |z_m|^4)} \operatorname*{perdet}_{1\le j,k\le n}\left[S_{\mathbb{A}_q}\left(z_j, z_k; r\prod_{\ell=1}^n |z_\ell|^2\right)\right]$$

for every $n \in \mathbb{N}$ and $z_1,\dots,z_n \in \mathbb{A}_q$ with respect to m/π.

Due to the determinantal factor in perdet, PDPP is **simple**, *i.e.*, no multiple point.
It is verified that $\rho^{(n)}_{\mathcal{Z}_{X^r_{\mathbb{A}_q}}}(z_1,\dots,z_n) > 0, \forall n \in \mathbb{N}, z_1,\dots,z_n \in \mathbb{A}_q$ by this explicit expression, which implies that this PDPP has **an infinite number of points**; $\boxed{\mathcal{Z}_{X^r_{\mathbb{A}_q}}(\mathbb{A}_q) = \infty \text{ a.s.}}$
In our paper (arXiv:2008.04177), we proved that $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$ is **essentially PDPP**; that is, this cannot be reduced to any permanental point process nor determinantal point process.

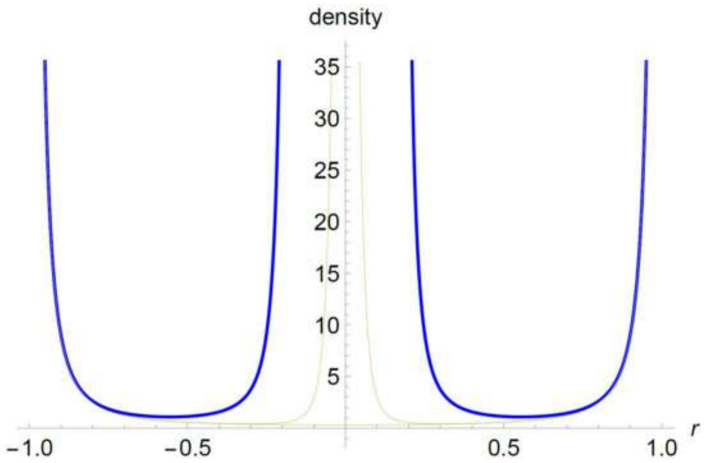
The **density of zeros** on $\mathbb{A}_q$ with respect to m/π is given by

$$\rho^{(1)}_{\mathcal{Z}_{X^r_{\mathbb{A}_q}}}(z) = \frac{\theta(-r)}{\theta(-r|z|^4)} S_{\mathbb{A}_q}(z, z; r|z|^2)^2 = \frac{q_0^4 \theta(-r, -r|z|^4)}{\theta(-r|z|^2, |z|^2)^2}, \quad z \in \mathbb{A}_q.$$

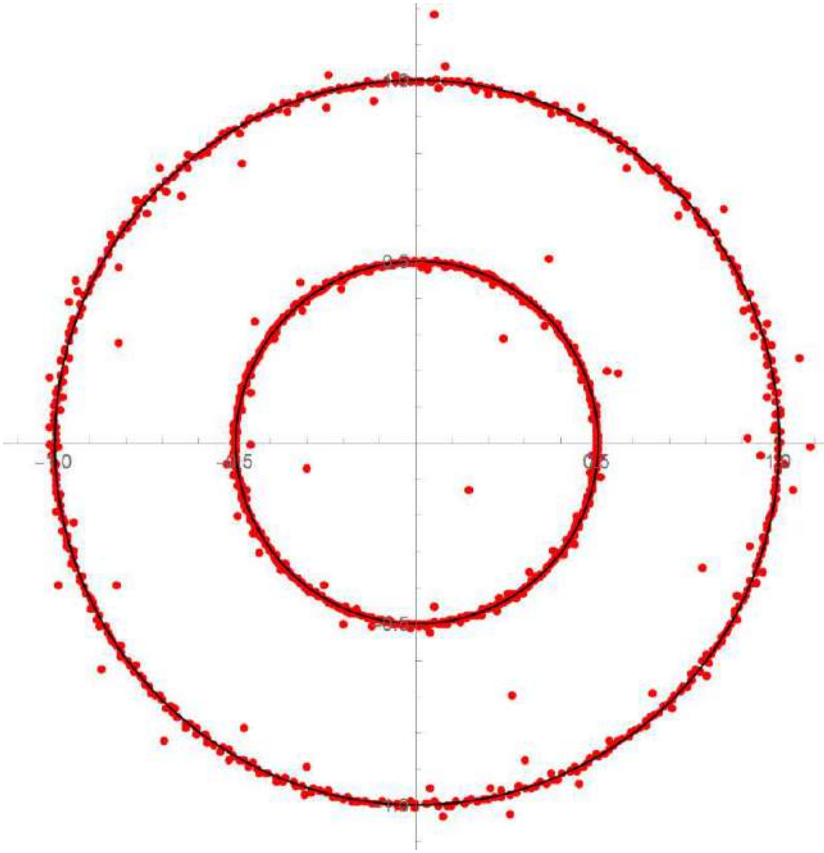
Since $\rho^{(1)}_{\mathcal{Z}_{X^r_{\mathbb{A}_q}}}$ depends only on $|z| \in (q, 1)$, this PDPP is **rotationally invariant**.

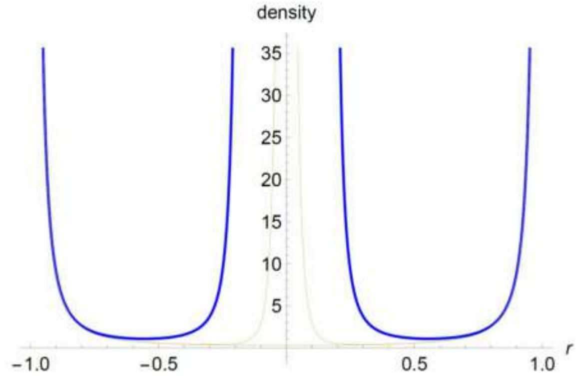
The density shows divergence both at the inner and outer boundaries as

$$\rho^{(1)}_{\mathcal{Z}_{X^r_{\mathbb{A}_q}}}(z) \sim \begin{cases} \dfrac{q^4}{(|z|^2 - q^2)^2}, & |z| \downarrow q, \\ \dfrac{1}{(1 - |z|^2)^2}, & |z| \uparrow 1. \end{cases} \qquad \text{(This is independent of } r.\text{)}$$



Density of zeros $\mathcal{Z}_{X^q_{\mathbb{A}_q}}$ when $q = 1/6$





We consider a **finite-series approximation** of the GAF $X_{\mathbb{A}_q}$:

$$X_{\mathbb{A}_q}^{(N)}(z) := \sum_{n=-N}^{N} \zeta_n \frac{z^n}{\sqrt{1+q^{2n+1}}}$$

A sample of zeros in the case $q = 1/2$ and $N = 500$ is shown. $2N = 1000$ zeros are plotted. We see a lot of zeros near the inner and the outer boundaries of $\mathbb{A}_q$. Due to the finiteness of N; $N < \infty$, we have **outliers** (the zeros inside of the inner circle and outside of the outer circle), which shall vanish in the limit $N \to \infty$.

In the **limit $q \to 0$**, Theorem 3.1 is much simplified by the formula $\boxed{\lim_{q\to 0}\theta(z;q^2) = 1 - z.}$

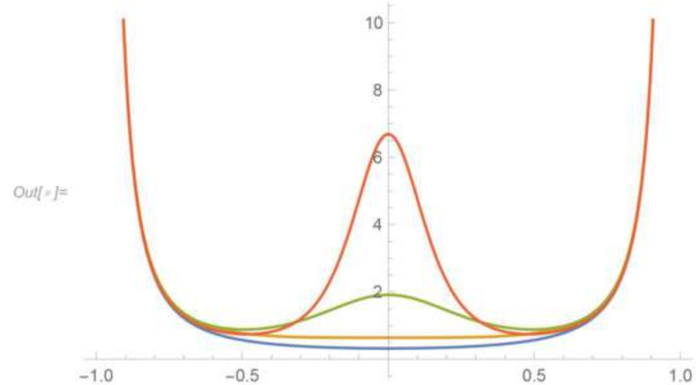
Corollary 3.2 Assume that $r > 0$. In the limit $q \to 0$, $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$ is reduced to $\mathcal{Z}_{X^r_{\mathbb{D}^\times}}$ on $\mathbb{D}^\times := \{z \in \mathbb{C} : 0 < |z| < 1\}$, which is a PDPP with the correlation functions

$$\rho^{(n)}_{\mathcal{Z}_{X^r_{\mathbb{D}^\times}}}(z_1, \dots, z_n) = \frac{1+r}{1 + r\prod_{m=1}^n |z_m|^4} \operatorname*{perdet}_{1\le j,k\le n}\left[S_{\mathbb{D}^\times}\left(z_j, z_k; r\prod_{\ell=1}^n |z_\ell|^2\right)\right]$$

for every $n \in \mathbb{N}$ and $z_1, \dots, z_n \in \mathbb{D}^\times$ with respect to m/π. Here

$$S_{\mathbb{D}^\times}(z, w) = \frac{1 + rz\overline{w}}{(1+r)(1 - z\overline{w})}, \quad z, w \in \mathbb{D}^\times.$$

In particular, the density of zeros on $\mathbb{D}^\times$ is given by $\rho^{(1)}_{\mathcal{Z}_{X^r_{\mathbb{D}^\times}}}(z) = \dfrac{(1+r)(1+r|z|^4)}{(1+r|z|^2)^2(1-|z|^2)^2}$, $z \in \mathbb{D}^\times$.



Density of zeros $\mathcal{Z}_{X^r_{\mathbb{D}^\times}}$ with $r = 0$ (blue), $r = 1$ (orange), $r = 5$ (green), and $r = 20$ (red)

If we take the **further limit $r \to 0$**, we obtain the Szegő kernel of $\mathbb{D}$.

$$S_{\mathbb{D}^\times}(z,w) = \frac{1 + rz\overline{w}}{(1+r)(1-z\overline{w})} \quad \Longrightarrow \quad S_{\mathbb{D}}(z,w) = \frac{1}{1 - z\overline{w}}$$

Since the matrix $(S_{\mathbb{D}}(z_j, z_k)^{-1})_{1 \leq j,k \leq n} = (1 - z_j \overline{z_k})_{1 \leq j,k \leq n}$ has **rank 2**, the following equality called **Borchardt's identity** holds,

$$\boxed{\operatorname*{perdet}_{1 \leq j,k \leq n} \left[\frac{1}{1 - z_j \overline{z_k}} \right] = \det_{1 \leq j,k \leq n} \left[\frac{1}{(1 - z_j \overline{z_k})^2} \right].}$$

This implies that the PDPP is reduced to a **determinantal point process (DPP)**.

Moreover, by the relation

$$\boxed{S_{\mathbb{D}}(z,w)^2 = \frac{1}{(1 - z\overline{w})^2} = K_{\mathbb{D}}(z,w), \quad z, w \in \mathbb{D},}$$

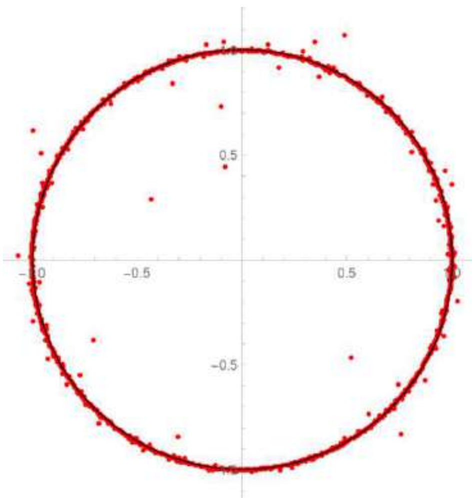
we see that the $r \to 0$ limit of $\mathcal{Z}_{X^r_{\mathbb{D}^\times}}$ is the DPP $\mathcal{Z}_{X_{\mathbb{D}}}$ on $\mathbb{D}$ whose correlation functions with respect to m/π are given by

$$\rho^{(n)}_{\mathcal{Z}_{X_{\mathbb{D}}}}(z_1, \dots, z_n) = \det_{1 \leq j,k \leq n} [K_{\mathbb{D}}(z_j, z_k)], \quad n \in \mathbb{N}, \quad z_1, \dots, z_n \in \mathbb{D}.$$

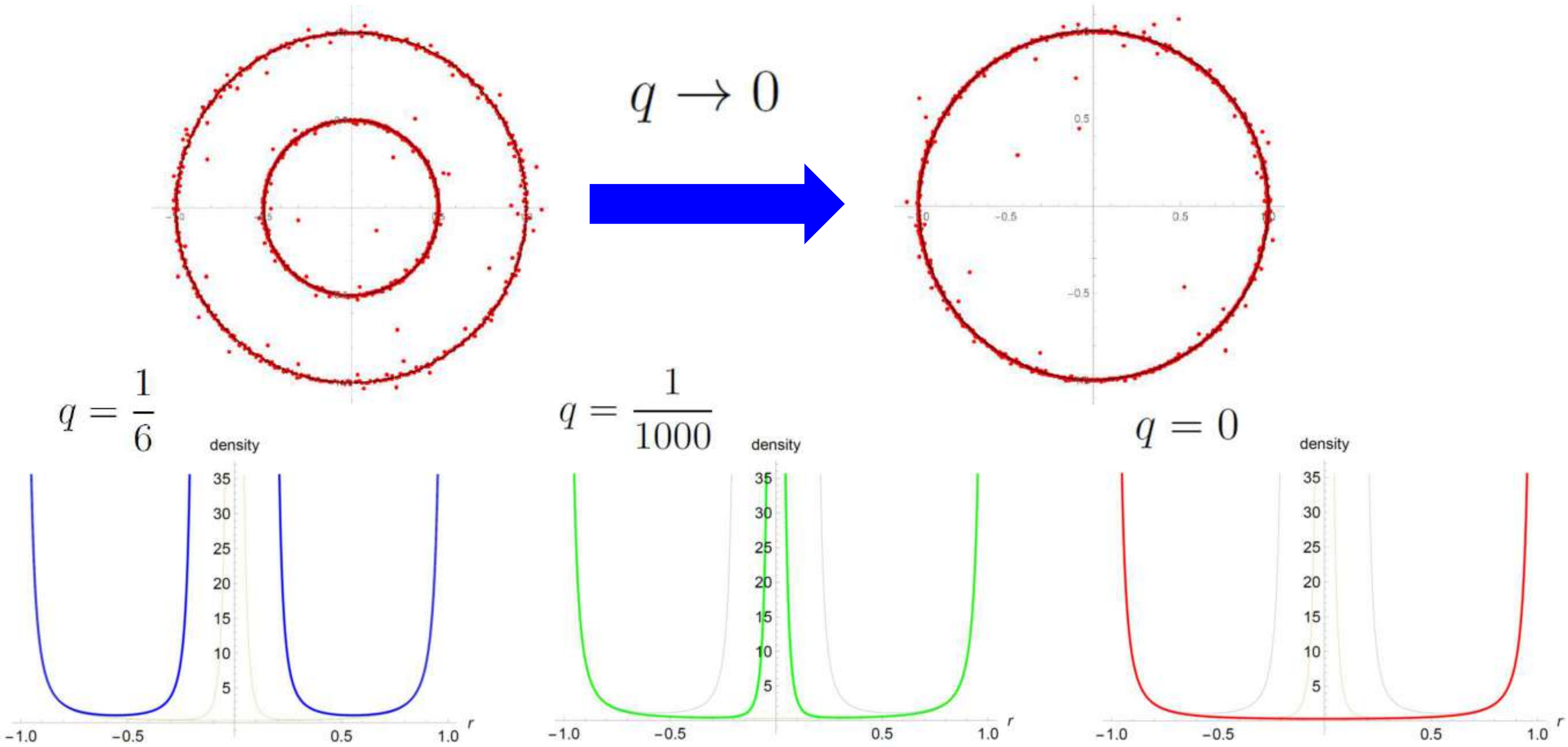
This is the **beautiful result by Peres and Virág (2005)**.

[PV05] Peres, Y., Virág, B.: Zeros of the i.i.d. Gaussian power series. A conformally invariant determinantal process. Acta Math. 194, 1–35 (2005)

Corollary 3.3 (Peres–Virág (2005)) Consider a GAF on $\mathbb{D}$ defined by $X_{\mathbb{D}}(z) := \sum_{n=0}^{\infty} \zeta_n z^n$. This is a **GP** with the covariance kernel $S_{\mathbb{D}}(z,w) = \dfrac{1}{1 - z\overline{w}}$, which is equal to the **Szegő kernel of $\mathbb{D}$** (that is, the reproducing kernel of the Hardy space $H^2(\mathbb{D})$.) The zero-point process $\mathcal{Z}_{X_{\mathbb{D}}}$ of $\{X_{\mathbb{D}}(z)\}_{z\in\mathbb{D}}$ is a **determinantal point process (DPP)** with the correlation kernel $K_{\mathbb{D}}(z,w) = \dfrac{1}{(1 - z\overline{w})^2}$, which is equal to the **Bergman kernel of $\mathbb{D}$** (that is, the reproducing kernel of the Bergman space $L^2_{\mathrm{B}}(\mathbb{D})$).



The asymptotics shows that the density of zeros of $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$ diverges at the inner boundary $\{z : |z| = q\}$ for each $q > 0$, while the density of $\mathcal{Z}_{X^r_{\mathbb{D}^\times}}$ is finite at the origin. Therefore **infinitely many zeros near the inner boundary seem to vanish in the limit $q \to 0$**. This is why we write $\mathbb{D}^\times$ instead of $\mathbb{D}$ for the limit domain of $\mathbb{A}_q$. Indeed, in the vague topology, with which we equip a configuration space, we cannot see configurations outside each compact set, hence infinitely many zeros are not observed on each compact set in $\mathbb{D}^\times$ (not $\mathbb{D}$) for any sufficiently small $q > 0$.



As another corollary of Theorem 3.1, we can also obtain the following.

Corollary 3.4 Consider the **pair-zero point process** of the GAF $X_{\mathbb{A}_q}$ on $\mathbb{A}_q$, which is denoted by $\mathcal{Z}_{X_{\mathbb{A}_q}}^{\text{pair}}$. This is a PDPP in the sense that the pair-point correlation functions are given as follows,

$$\rho_{\mathcal{Z}_{X_{\mathbb{A}_q}}^{\text{pair}}}^{(2n)}\Big((z_1, \widehat{z}_1), \dots, (z_n, \widehat{z}_n)\Big) = \operatorname*{perdet}_{1 \le j,k \le n} \begin{bmatrix} S_{\mathbb{A}_q}(z_j, z_k) & S_{\mathbb{A}_q}(z_j, \widehat{z}_k) \\ S_{\mathbb{A}_q}(\widehat{z}_j, z_k) & S_{\mathbb{A}_q}(\widehat{z}_j, \widehat{z}_k) \end{bmatrix}$$

for every $n \in \mathbb{N}$ and $z_1, \dots, z_n \in \mathbb{A}_q$ with respect to m, where $\widehat{z}_j = -\dfrac{q}{z_j}, j = 1, \dots, n$.

Note that $S_{\mathbb{A}_q}(z_j, \widehat{z}_j) = S_{\mathbb{A}_q}(\widehat{z}_j, z_j) = 0, j = 1, 2, \dots, n$. Then $X_{\mathbb{A}_q}(z_j)$ and $X_{\mathbb{A}_q}(\widehat{z}_j)$ are **uncorrelated**. In the above we put zeros both on these independent points.

4. A Sketch of Proof for Theorem 3.1

- We recall a general formula for correlation functions of zero-point process of a GAF, which is found in [PV05].
- But here we use a slightly different expression given by Proposition 6.1 of [Shi12].

Proposition 4.1 The correlation functions of $\mathcal{Z}_{X_D}$ of the GAF X_D on $D \subsetneq \mathbb{C}$ with covariance kernel $S_D(z, w)$ are given by

$$\rho^{(n)}_{\mathcal{Z}_{X_D}}(z_1, \dots, z_n) = \frac{\mathrm{per}_{1 \le j,k \le n} \left[(\partial_z \partial_{\overline{w}} S_D^{z_1, \dots, z_n})(z_j, z_k) \right]}{\det_{1 \le j,k \le n} \left[S_D(z_j, z_k) \right]}, \quad n \in \mathbb{N}, \quad z_1, \dots, z_n \in D,$$

with respect to m/π, whenever $\det_{1 \le j,k \le n} [S_D(z_j, z_k)] > 0$.

[Shi12] Shirai, T.: Limit theorems for random analytic functions and their zeros. In: Functions in Number Theory and Their Probabilistic Aspects – Kyoto 2010, RIMS Kôkyûroku Bessatsu 34, 335–359 (2012)

- Let $\gamma_n^q(z) = \gamma^q_{\{z_\ell\}_{\ell=1}^n} := \prod_{\ell=1}^n h^q_{z_\ell}(z), \quad \gamma_n^{q\prime}(z) := \dfrac{d\gamma_n^q(z)}{dz}, \quad z \in \mathbb{A}_q.$

  Then, for the conditional Szegő kernel with zeros at $z_1, \dots z_n$, we have

$$S_{\mathbb{A}_q}^{z_1,\dots,z_n}(z,w;r) = S_{\mathbb{A}_q}\Big(z,w; r\prod_{\ell=1}^n |z_\ell|^2\Big)\gamma_n^q(z)\overline{\gamma_n^q(w)}, \quad z,w,z_1,\dots,z_n \in \mathbb{A}_q.$$

  This formula gives $(\partial_z \partial_{\overline{w}} S_{\mathbb{A}_q}^{z_1,\dots,z_n})(z_j,z_k;r) = S_{\mathbb{A}_q}\Big(z_j,z_k; r\prod_{\ell=1}^n |z_\ell|^2\Big)\gamma_n^{q\prime}(z_j)\overline{\gamma_n^{q\prime}(z_k)}.$

- Therefore, Shirai's proposition gives now

$$\rho^{(n)}_{\mathcal{Z}_{X^r_{\mathbb{A}_q}}}(z_1,\dots,z_n) = \frac{\mathrm{per}_{1\le j,k\le n}\left[S_{\mathbb{A}_q}(z_j,z_k; r\prod_{\ell=1}^n |z_\ell|^2)\right]\prod_{m=1}^n |\gamma_n^{q\prime}(z_m)|^2}{\det_{1\le j,k\le n}\Big[S_{\mathbb{A}_q}(z_j,z_k;r)\Big]} \quad \text{with}$$

$$\prod_{j=1}^n |\gamma_n^{q\prime}(z_j)|^2 = \prod_{j=1}^n \Big|\Big(\prod_{1\le k\le n, k\ne j} h^q_{z_k}(z_j)\Big) h^q_{z_j}{}'(z_j)\Big|^2 = \prod_{j=1}^n \Big|\Big(\prod_{1\le k\le n, k\ne j} z_j \frac{\theta(z_k/z_j)}{\theta(\overline{z_k}z_j)}\Big)\frac{q_0^2}{\theta(|z_j|^2)}\Big|^2$$

$$= \left|\frac{q_0^{2n}\prod_{1\le j<k\le n} z_j\theta(z_k/z_j)\cdot \prod_{1\le j'<k'\le n} z_{k'}\theta(z_{j'}/z_{k'})}{\prod_{j=1}^n\prod_{k=1}^n \theta(z_j\overline{z_k})}\right|^2$$

$$= q_0^{4n}\Big(\frac{\prod_{1\le j<k\le n}|z_k|^2\theta(z_j/z_k, \overline{z_j}/\overline{z_k})}{\prod_{j=1}^n\prod_{k=1}^n \theta(z_j\overline{z_k})}\Big)^2.$$

- The following identity is known as an **elliptic extension of Cauchy's evaluation of determinant due to Frobenius**,

$$\boxed{\det_{1\le j,k\le n}\left[\frac{\theta(tx_ja_k)}{\theta(t,x_ja_k)}\right]=\frac{\theta(t\prod_{k=1}^n x_ka_k)}{\theta(t)}\frac{\prod_{1\le j<k\le n}x_ka_k\theta(x_j/x_k,a_j/a_k)}{\prod_{j=1}^n\prod_{k=1}^n\theta(x_ja_k)}.}$$

- Using the expression of $S_{\mathbb{A}_q}(\cdot,\cdot;r)$ by the theta functions, we have

$$q_0^{2n}\frac{\prod_{1\le j<k\le n}|z_k|^2\theta(z_j/z_k,\overline{z_j}/\overline{z_k})}{\prod_{j=1}^n\prod_{k=1}^n\theta(z_j\overline{z_k})}=\frac{\theta(-s)}{\theta(-s\prod_{\ell=1}^n|z_\ell|^2)}\det_{1\le j,k\le n}\left[S_{\mathbb{A}_q}(z_j,z_k;s)\right],\quad \forall s>0.$$

Then

$$\prod_{j=1}^n|\gamma_n^{q'}(z_i)|^2=\frac{\theta(-r)}{\theta(-r\prod_{\ell=1}^n|z_\ell|^4)}\det_{1\le j,k\le n}[S_{\mathbb{A}_q}(z_j,z_k;r)]\det_{1\le j,k\le n}\left[S_{\mathbb{A}_q}\Big(z_j,z_k;r\prod_{\ell=1}^n|z_\ell|^2\Big)\right].$$

Applying the above to

$$\rho^{(n)}_{\mathcal{Z}_{X^r_{\mathbb{A}_q}}}(z_1,\dots,z_n)=\frac{\operatorname{per}_{1\le j,k\le n}\left[S_{\mathbb{A}_q}(z_j,z_k;r\prod_{\ell=1}^n|z_\ell|^2)\right]\prod_{k=1}^n|\gamma_n^{q'}(z_k)|^2}{\det_{1\le j,k\le n}\Big[S_{\mathbb{A}_q}(z_j,z_k;r)\Big]},$$

the correlation functions in Theorem 3.1 are obtained. ∎

5. And Geometry?

- As shown by

$$\rho^{(1)}_{\mathcal{Z}_{X^r_{\mathbb{A}_q}}}(z) = \frac{q_0^4 \theta(-r, -r|z|^4)}{\theta(-r|z|^2, |z|^2)^2} \sim \begin{cases} \dfrac{q^4}{(|z|^2 - q^2)^2}, & |z| \downarrow q, \\ \dfrac{1}{(1 - |z|^2)^2}, & |z| \uparrow 1, \end{cases}$$

  the asymptotics of the density of zeros $\rho^{(1)}_{\mathcal{Z}_{X^r_{\mathbb{A}_q}}}(z) \sim (1-|z|^2)^{-2}$ with respect to $m(dz)/\pi$ in the vicinity of the outer boundary of $\mathbb{A}_q$ can be identified with the metric in the hyperbolic plane called the **Poincaré disk model**.

- The zero-point process $\mathcal{Z}_{X_{\mathbb{D}}}$ of Peres and Virág can be regarded as a **uniform DPP on the Poincaré disk model**.

- **Are there meaningful geometrical spaces in which the present zero-point processes $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$, $\mathcal{Z}^{\text{pair}}_{X_{\mathbb{A}_q}}$, and $\mathcal{Z}_{X^r_{\mathbb{D}}}$ seem to be uniform?**

**Thank you very much
for your attention.**

Appendix: On Physical Realization of Random Polynomials and Observation of their Zeros

Physicists have realized the **ideal Bose gas confined in a harmonic potential** in 3D.

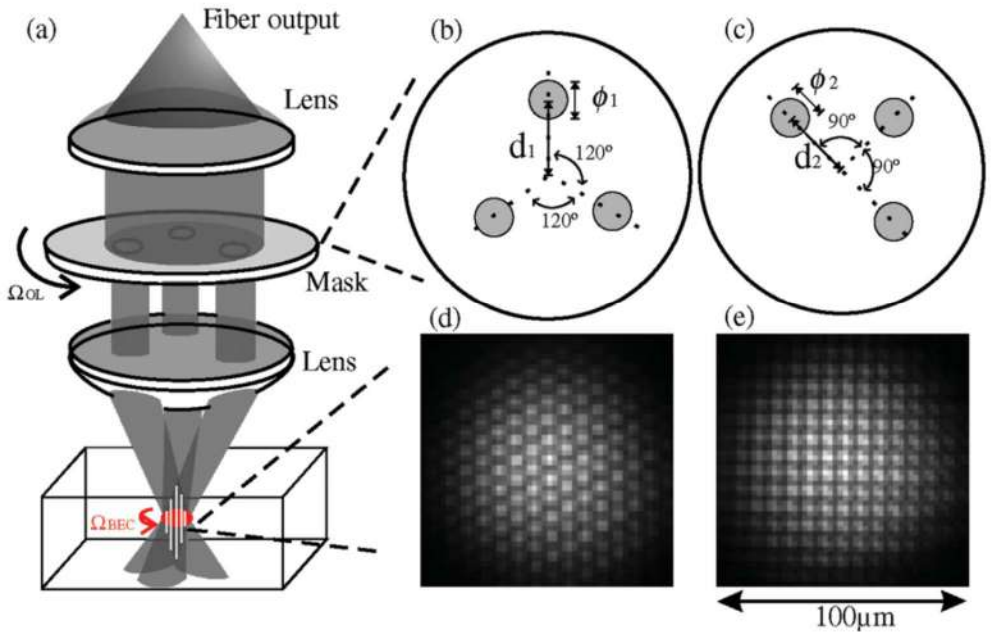
Let ω= the oscillation frequency in the xy plane, ω_z= that along the z-axis

for the harmonic potential $\frac{1}{2}m\omega^2(x^2+y^2)+\frac{1}{2}m\omega_z^2 z^2$.

Set in **low temperatures** ($k_{\rm B}T \ll \hbar\omega_z$) so that the z degree of freedom is **frozen**

$\Longrightarrow$ **the gas is kinetically two dimensional**

Consider the thermal equilibrium in the frame **rotating at frequency Ω along z-axis**.



Tung, S. et al.: Phys. Rev. Lett. 97, 240402 (2006) Fig. 1

Hamiltonian (energy operator) : $\boxed{\mathcal{H} = \frac{\boldsymbol{p}^2}{2m} + \frac{1}{2} m\omega^2(x^2+y^2) - \Omega L_z,}$

$\boldsymbol{p}$ = momentum operator, L_z= z-component of the angular momentum operator

A system of common-eigenfunctions of L_z and $\mathcal{H}$, $a := \sqrt{\frac{\hbar}{m\omega}}$

$$\phi_{j,k}(x,y) \propto e^{(x^2+y^2)/2a^2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)^j \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)^k e^{-(x^2+y^2)/2a^2},$$

$$j \in \mathbb{N}_0, \quad k = \textbf{Landau level} \in \mathbb{N}_0,$$

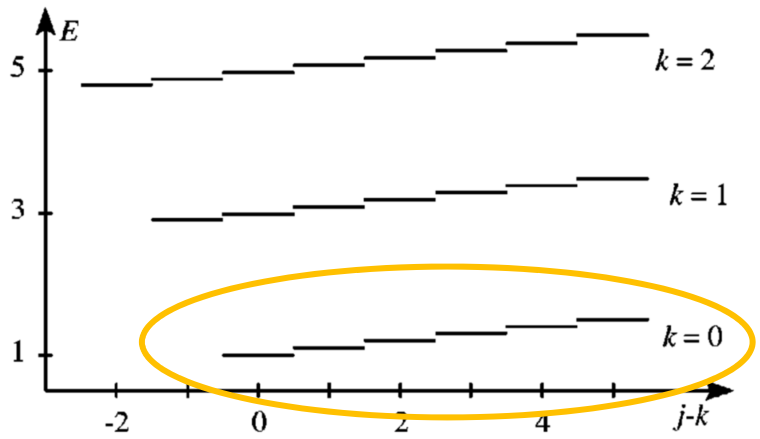
the eigenvalue of $\mathcal{H} = E_{j,k} := \hbar\omega + \hbar(\omega - \Omega)j + \hbar(\omega+\Omega)k$, the e.v. of $L_z = \hbar(j-k)$

fast rotation : $\Omega \lesssim \omega \iff \hbar(\omega - \Omega) \ll \hbar\omega < \hbar(\omega+\Omega)$

If $k_{\mathrm{B}}T$ and μ (chemical potential) $\ll \hbar\omega$,

then, the state can be actually described $\phi_{j,k}$ with **$k = 0$ (lowest-Landau-level) only**.

$$\boxed{\phi_{j,0}(x,y) = e^{(x^2+y^2)/2a^2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)^j e^{-(x^2+y^2)/2a^2} \propto (x+iy)^j e^{-(x^2+y^2)/2a^2}, \quad j \in \mathbb{N}_0.}$$



Afalion, A. et al. : Phys. Rev. A72, 023611 (2005) Fig. 2

LLL (Lowest-Landau-Level) state $(z := x + iy)$

$$\psi(x,y) = \text{a linear combination of the } \phi_{j,0}\text{'s} \quad (\phi_{j,0}(z) \propto z^j e^{-|z|^2/2a^2})$$

$$= e^{-|z|^2/2a^2} P(z) \quad (P(z):=\text{a polynomial of } z)$$

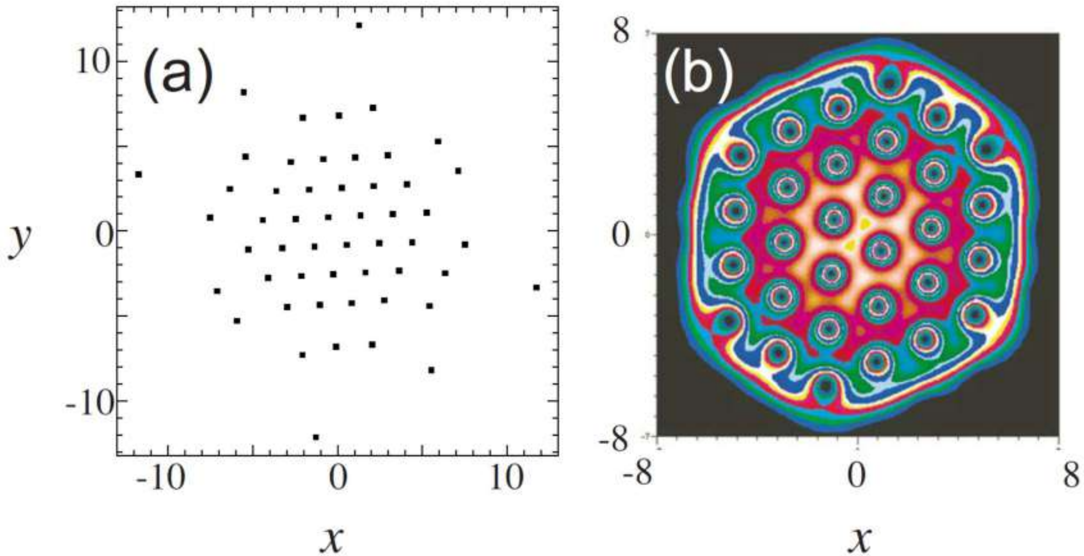
$$= e^{-|z|^2/2a^2} \sum_{n=0}^{N} c_n z^n \quad (\textbf{thermal noise} \Longrightarrow \mathbf{c_n} \textbf{ are i.i.d. Gaussian r.v.'s})$$

$$\propto e^{-|z|^2/2a^2} \prod_{\ell=1}^{N} (z - z_\ell)$$

The **phase** of $\psi(x,y)$ changes by 2π along a closed contour encircling z_ℓ.

a zero of $P(z)$ $\Longleftrightarrow$ a location of a single-charged, positive **vortex**

zero-point process of $P(z)$ $\Longleftrightarrow$ **vertex distribution in the 2D Bose gas in fast rotation**



Aftalion, A. et al. : Phys. Rev. A72, 023611 (2005) : Fig.1

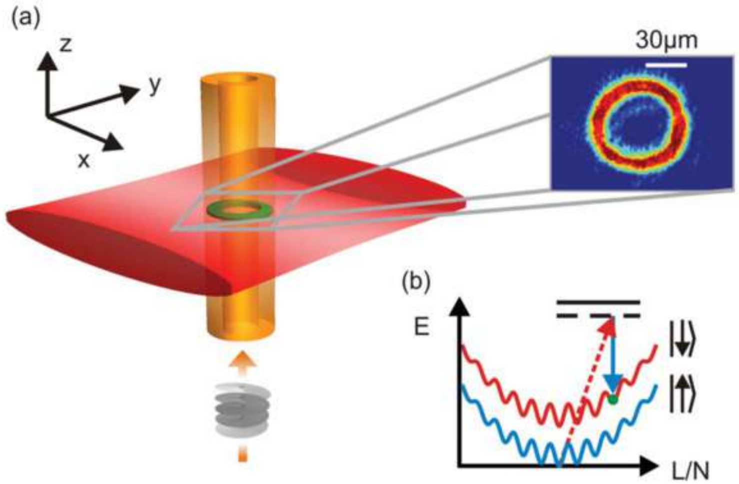
(a) Vortex locations
(b) atomic density profile

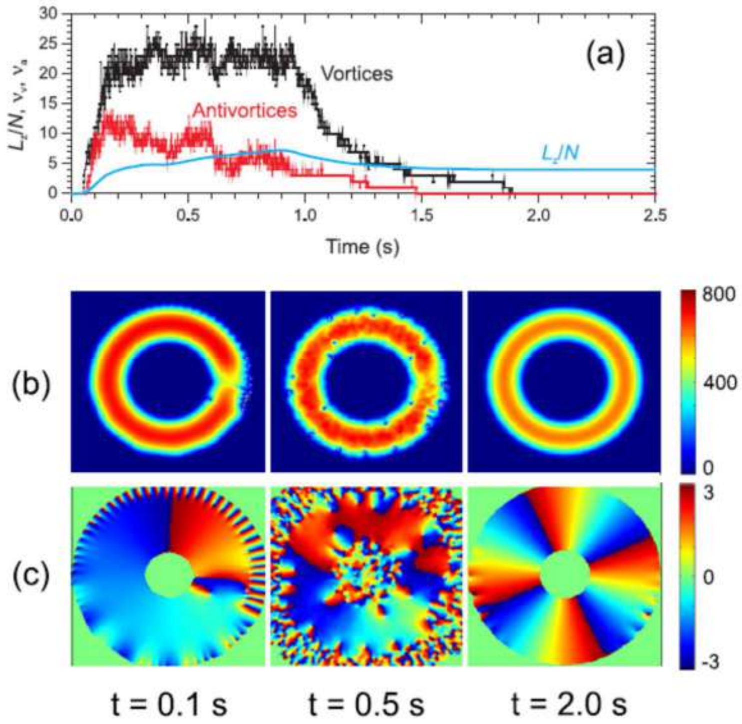
We can find the following papers ...

PHYSICAL REVIEW A **86**, 013629 (2012)

Quantized supercurrent decay in an annular Bose-Einstein condensate

Stuart Moulder, Scott Beattie, Robert P. Smith, Naaman Tammuz, and Zoran Hadzibabic





PHYSICAL REVIEW A **91**, 023607 (2015)

Vortex excitation in a stirred toroidal Bose-Einstein condensate

A. I. Yakimenko,[1] K. O. Isaieva,[1] S. I. Vilchinskii,[1,2] and E. A. Ostrovskaya[3]

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