Non-Commutative Probability Theory, Random Matrix Theory and their Applications

Institute of Statistical Mathematics

November, 8-10, 2023

Invited Speakers

- Nicolas Delporte (OIST)
- Syota Esaki (Fukuoka University)
- Tomohiro Hayase (Cluster Metaverse Lab)
- Kohei Noda (Kyusyu University)
- Ryosuke Sato (Chuo University)
- Hirohiko Shimada (National Institute of Technology, Tsuyama College)
- Jorge Garza Vargas (Caltech)
- Satoshi Yabuoku (National Institute of Technology, Kitakyushu College)
- Ping Zhong (University of Wyoming)

TBA

Nicolas Delporte (OIST)

Eigenvalues of the non-Hermitian matrix-valued stochastic processes and related topics

Syota Esaki (Fukuoka University)
Satoshi Yabuoku (National Institute of Technology, Kitakyushu College)

In this presentation, we consider stochastic processes associated with eigenvalues and eigenvectors of the non-Hermitian matrix-valued processes. A typical example of the non-Hermitian matrix-valued processes is the non-Hermitian matrix-valued Brownian motion (nHBM). The nHBM is a dynamical $N \times N$ random matrix model whose entries are given by i.i.d. complex Brownian motions. In addition, we also consider a one-parameter family of non-Hermitian matrix-valued processes, which can be regarded as a dynamical extension of Girko's ensemble interpolating the GUE and the Ginibre ensemble of random matrices. For each process in this family, the bi-orthogonality relation is imposed between the right and the left eigenvector processes, which allows for the scale-transformation invariance of the system. The eigenvalue process associated with the nHBM are related to the Ginibre ensemble. As matrix-valued Brownian motion, for example, the eigenvalue process for the dynamical extension of the GUE, which assumes the Hermitian symmetry, has been well studied. This process is known to be the solution of the SDE of the Dyson Brownian motion for $\beta = 2$. On the other hand, the eigenvalue process of the nHBM is difficult to give the SDE using only themselves. Each eigenvalue process is coupled with the eigenvector-overlap process, which is a Hermitian matrix-valued process with entries given by products of overlaps of the right and left eigenvectors. In this talk, we introduce SDE representations for the eigenvalues and the overlaps associated with the non-Hermitian matrix-valued processes. The Fuglede-Kadison (FK) determinants of the present matrix-valued processes are regularized by introducing an auxiliary complex variable. Then, associated with the regularized FK determinants, the time-dependent random fields are defined in the two-dimensional complex space and their stochastic partial differential equations (SPDEs) are derived. Time-dependent point processes of eigenvalues and their variations weighted by the diagonal elements of the eigenvector-overlap processes are related to the logarithmic derivatives of the regularized FK-determinant random-fields. We also discuss the PDEs obtained by averaging the SPDEs. The present talk is based on the joint work with Makoto KATORI (Chuo University).

Understanding MLP-Mixer as a wide and sparse MLP through Random Permutation Matrices

Tomohiro Hayase (Cluster Metaverse Lab)

Multi-layer perceptron (MLP) is a fundamental component of deep learning that has been extensively employed for various problems. However, recent empirical successes in MLP-based architectures, particularly the progress of the MLP-Mixer, suggest that our understanding of how MLPs achieve better performance remains limited and there is an underlying mechanism. In this research, we reveal that the MLP-Mixer effectively behaves as a wide MLP with sparse weights. Initially, we clarify that the mixing layer of the Mixer has an effective expression as a wider MLP whose weights are sparse and represented by the tensor product. It is also regarded as an approximation of Monarch matrices. Next, we confirmed similarities between the mixer and the unstructured sparse-weight MLP in hidden features and performance when adjusting sparsity and width. To verify the similarity in much wider cases, we introduced the Randomly-Permuted-Mixer (RP-Mixer), a more memory-efficient alternative to the unstructured sparse-weight MLP. Then we verified similar tendencies between the MLP-Mixer and the RP-Mixer, confirming that the MLP-Mixer behaves as a sparse and wide MLP, and that its better performance is from its extreme wideness. Notably, when the number of connections is fixed and the width of hidden layers is increased, sparsity rises, leading to improved performance, consistent with the hypothesis by Golubeva, Neyshabur and Gur-Ari (2021). Particularly, maximizing the width enables us to quantitatively determine the optimal mixing layer's size.

Pfaffian strucutre of the weighted multi-point intensity of the on-diagonal ovelap for the Ginbre symplectic ensemble

Kohei Noda (Kyusyu University)

In this talk, we will discuss the Pfaffian structure of the weighted multi-point intensity associated with the on-diagonal overlap conditioned on the real line for the Ginibre symplectic ensemble. To demonstrate our results, we need to construct a family of skew-orthogonal polynomials (SOPs) associated with the overlap weight function. However, the planar orthogonal polynomials with respect to our weight function do not satisfy the classical three-term recurrence, such as Hermite, Laguerre, or Jacobi polynomials. This implies that we cannot apply the method for constructing SOPs established by Akemann, Ebke, and Parra in 2022 to our specific case. Instead, we will introduce a novel approach to construct SOPs, from which we will derive the finite-N skew pre-kernel associated with the overlap weight function and the limiting pre-kernel using the ODE method. As a consequence, we will extend the asymptotic behavior of the on-diagonal overlap conditioned on the origin, which was originally established by Dubach in 2019, to apply to any real line. In addition to the overlap of non-Hermitian random matrices, we emphasize that our result is remarkable and interesting from the perspective of the analysis of local statistics for point processes of the two-dimensional symplectic ensembles. This talk is based on an on-going collaboration with Gernot Akemann (Bielefeld University) and Sung-Soo Byun (Seoul National University).

On the interaction between determinantal point processes and operator algebras

Ryosuke Sato (Chuo University)

In this talk, we will discuss the interaction between operator algebras and point processes. We are particularly interested in point processes whose correlations of particles are given in the form of determinants, known as determinantal point processes. Determinantal point processes appear in diverse fields of mathematics and mathematical physics, including random matrix theory, combinatorics, statistical mechanics, and representation theory. Furthermore, it is known that a certain class of determinantal point processes can be realized by non-commutative probability spaces, which consist of pairs of (operator) algebras and states. The purpose of my talk is to extend such a relationship between determinantal point processes and non-commutative probability to encompass dynamical aspects. Specially, within the framework of operator algebras, we discuss how dynamics on operator algebras give rise to stochastic dynamics on determinantal point processes.

Bootstrap, Eigenvalues, and Schrödinger Invariant Correlation Functions

Hirohiko Shimada (National Institute of Technology, Tsuyama College)

The non-relativistic conformal field theory (NRCFT) describes a wide range of non-equilibrium phase transitions with a dynamical exponent $z = 2$, where the scale invariance is extended to Schrödinger symmetry. We study the basis of the NRCFT bootstrap, highlighting the role of the asymptotic expansions and the global behavior of the correlation functions when swapping between short temporal and spatial intervals. In particular, the path-integral is used to derive the exact 4-point functions in the field theory of the Calogero model in $1 + 1$ dimensions. We identify the Schrödinger block, a non-relativistic analogue of the conformal block, derived as a function of three generalized cross-ratios. Confluent functions, including the Humbert functions Ψ_2, emerge as characteristic features of the $z = 2$ theory. In a $0 + 1$-dimensional limit, corresponding to the time-like syzygy configuration, a reduction to ${}_4F_3$ hypergeometric function can be observed, which would be of use in the bootstrap. This is based on arXiv:2107.07770 with Hidehiko Shimada (YITP). With regard to random matrix theory, our equal-time correlation functions can be related to the generalized Bessel functions associated with the A-type root systems, as well as to the Harish Chandra-Itzykson-Zuber integral for the beta ensembles. In contrast, correlation functions for generic configurations may entail alterations in the matrix size. In the Dyson Brownian motion setting, continuing the path beyond the lines of a "uniform instance of time" and assigning a proper time to each eigenvalue could reveal the hidden Schrödinger symmetry.

Spectra of periodic operators on universal covering trees

Jorge Garza Vargas (Caltech)

Periodic operators on universal covering trees generalize discrete one-dimensional periodic Schrodinger operators (which originated in mathematical physics), while retaining many of their fascinating spectral properties. On the other hand, periodic operators on universal covering trees govern the spectral behavior of large random lifts of finite graphs, which have been studied in the context of expanders (of interest in computer science). In fact, these operators are a random matrix limit and can be viewed as an operator-valued matrix with entries in the reduced C*-algebra of the free group. In these lectures I will survey the main results about these objects, emphasizing their connection to free probability. I will mention joint works with Jess Banks, Jonathan Breuer, Archit Kulkarni, Satyaki Mukherjee, Eyal Seelig, and Barry Simon.

Brown measure of a sum of two free random variables and deformed random matrix models

Ping Zhong (University of Wyoming)

In the 1980s, L. Brown introduced an analogue of eigenvalue distribution in the framework of operator algebras, now called Brown measure. The Brown measure is a fascinating object that has spectacular applications in operator algebras and non-Hermitian random matrix theory. Random variables in Voiculescu's free probability theory can model the limits of suitable random matrices. It is desirable to calculate the Brown measures of free random variables as they can often predict the limiting eigenvalue distributions of non-Hermitian random matrices.

I will discuss main ingredients in some recent works on the Brown measure of a sum of two free random variables, one of which is circular, elliptic, or R-diagonal. Some ideas were implicitly used in earlier works of Haagerup-Schultz, Belinschi-Sniady-Speicher, and some physics papers. It is shown that subordination functions that appear in the study of free additive convolution can detect information about the Brown measure. In many cases, this leads to an explicit calculation of Brown measures. These Brown measures are related to the limiting eigenvalue distributions of the full rank perturbations of i.i.d., Wigner, elliptic, and single ring random matrix models. The talk is based on my recent works arXiv:2108.09844, arXiv:2209.11823 (joint with Belinschi and Yin), and arXiv:2209.12379 (joint with Bercovici).

If time permits, I will compare the subordination approach with the PDE methods (introduced by Driver-Kemp-Hall) designed for the free additive Brownian motions taken from some works of Hall, Ho and myself.

Students Speakers

- Ayana Ezoe (Chuo University)
- Rina Hirota (Ochanomizu University)
- Saori Morimoto (Chuo University)
- Yuya Tanaka (Chuo University)

Switching particle systems for foraging ants showing phase transitions in path selections

Ayana Ezoe (Chuo University)

Switching interacting particle systems studied in probability theory are the stochastic processes of hopping particles on a lattice consisting of slow particles and fast particles, where the switching between these two types of particles happens randomly with a given transition rate. In the present talk, we show that such stochastic processes of many particles are useful in modeling group behaviors of ants. Recently the situation-depending switching between two distinct types of primarily relied cues for ants in selecting foraging paths has been experimentally studied by the research group of Nishimori (Meiji). We propose a discrete-time interacting random-walk model on a square lattice, in which two kinds of hopping rules are included. We report the numerical simulation results which exhibit the global changes in selected homing paths from trailing paths of the 'pheromone road' to nearly-optimal-paths depending on the switching parameters. By introducing two kinds of order parameters characterizing the switching-parameter dependence of homing-duration distributions, we discuss such global changes as phase transitions realized in path selections of ants. Critical phenomena associated with the continuous phase transitions are also studied. The present talk is based on the joint work with S. Morimoto, Y. Tanaka, M. Katori (Chuo), and H. Nishimori (Meiji).

An Application of Kernel Random Matrices to the optimization of the Parameter for the RBF Kernel

Rina Hirota (Ochanomizu University)

In recent years, significant progress has been made in the development and application of kernel methods for many practical machine learning and data analysis. The kernel methods are used for a range of problems such as classification, regression, and unsupervised learning. The most significant feature of kernel methods is their ability to solve nonlinear problems using linear techniques. This approach avoids complex optimization problems and enables the analysis of large datasets. In this talk, it will be given how to determine the optimal kernel parameter and the number of clusters by applying the theory of random matrices to kernel matrices. Namely regarding a kernel random matrix as Wishart matrix, the eigenvalue distribution corresponding to the noise component in kernel matrix can be approximated by the Marchenko-Pastur distribution of parameter 1. It also leverages the relation of the power-law between the number of clusters and the distance between clusters.

Eigenvalue processes generated by the cyclic permutation matrix and by the shift matrix with perturbation

Saori Morimoto (Chuo University)

Motivated by the recent study of non-Hermitian random-matrix dynamics, we consider the dynamical processes of $n \times n$ matrices generated by multiplication of a cyclic permutation matrix and of a shift matrix. In the former case, all eigenvalues are on a unit circle. We prove that the degeneracy and configuration of eigenvalues at time $m \leq n$ is determined by the ratio n/m. In the latter case, we add a constant ε to all matrix elements at time m as perturbation. We have found that the time evolution of the eigenvalues on the complex plane seems to be very complicated in the numerical results. In the present talk, we will show the movies of the eigenvalue processes obtained by numerical calculation and give some exact and analytic results. We will also discuss the case that perturbation is random, which is given by adding an independent Gaussian random variable to each matrix element. The present talk is based on the joint work with M. Katori and T. Shirai (Kyushu, IMI).

Numerical study of eigenvector-overlaps and the regularized FK determinants of the non-Hermitian matrix-valued stochastic processes

Yuya Tanaka (Chuo University)

We made Python programs to simulate the eigenvalue processes, the eigenvector-overlap processes, and the logarithmic versions of the Fuglede-Kadison (FK) determinant random fields. We have numerically studied the time evolutionary processes of the FK fields and compare their singular behaviors with the point processes of eigenvalues weighted by the diagonal elements of the eigenvector-overlap matrix-valued processes. We will show the movies of our computer simulation results. The present talk is based on the joint work with S. Esaki (Fukuoka), M. Katori (Chuo), and S. Yabuoku (Kitakyushu).