Part 2
From Vicious Walk Model to Random Matrix Theory

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集中講義:東京大学大学院理学系研究科物理学専攻
「物理学特別講義 BXI:非平衡統計力学の最近の展開」

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1. Vicious Walker Model

- Let $(\{\mathbf{S}(t)\}_{t=0,1,2,\ldots}, \mathrm{P}^{\mathbf{x}})$ be the N-dimensional Markov chain starting from $\mathbf{x} = (x_1, x_2, \ldots x_N)$, such that the coordinates $S_j(t)$, $j = 1,2,\ldots,N$, are independent simple symmetric random walk on $\mathbf{Z}$.
- Take the starting point $\mathbf{x}$ from the set

$$\mathbf{Z}_<^N = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_N) \in (2\mathbf{Z})^N : x_1 < x_2 < \ldots < x_N \right\}.$$

- With a constant $T > 0$ we impose the **noncolliding condition** up to time T ;

$$S_1(t) < S_2(t) < \ldots.. < S_N(t), \qquad t = 1,2,\ldots..,T.$$

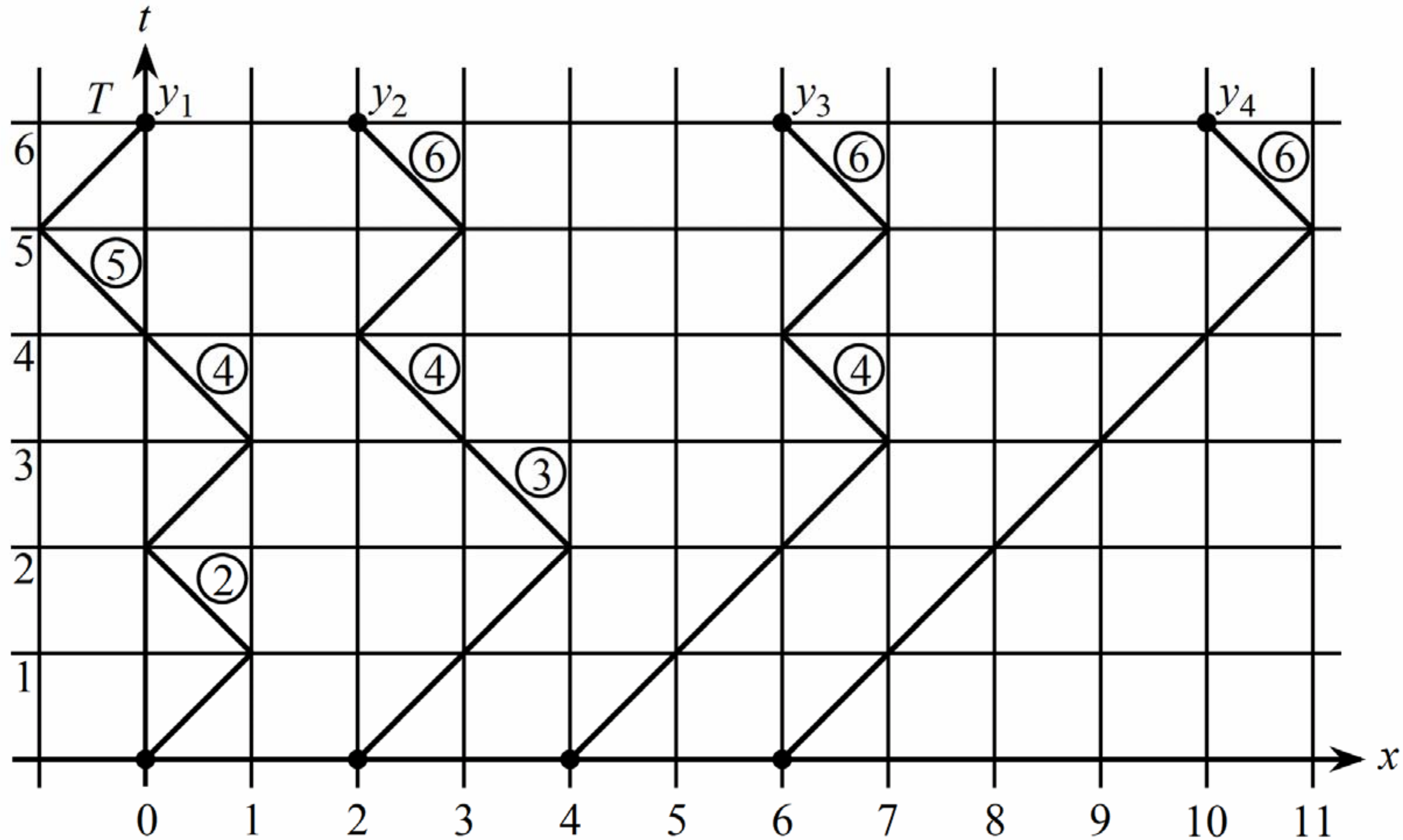
- We denote by $\mathrm{Q}_T^{\mathbf{x}}$ the conditional probability of $\mathrm{P}^{\mathbf{x}}$ under this noncolliding condition.

Michael Fisher called $\left(\{\mathbf{S}(t)\}_{t=0,1,2,3\ldots}, \mathrm{Q}_T^{\mathbf{x}}\right)$
a **vicious walker model** in his Boltzmann medal lecture.
(M. E. Fisher, *J. Statistical Physics* **34** (1984) 667-729.)

- We will assume that $S_j(0) = 2(j-1), \quad 1 \le j \le N.$
- Each realization of vicious walk is represented by an N-tuple of

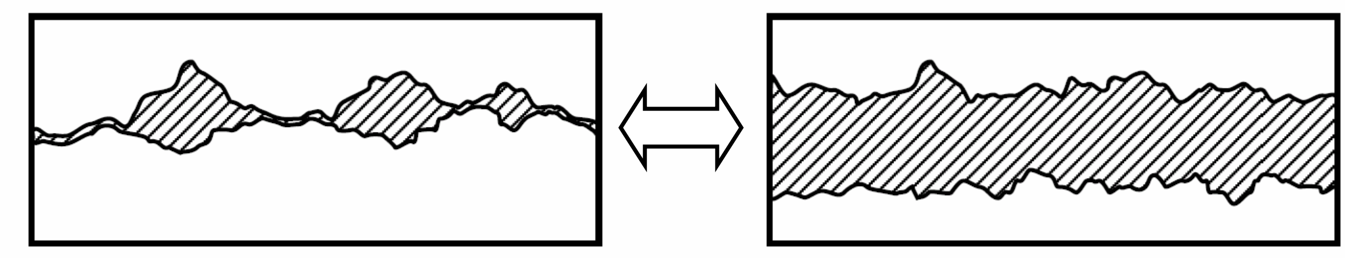
nonintersecting lattice paths on the 1+1 spatio-temporal plane, $\mathbf{Z} \times \{0,1,2,\ldots,T\}$.

An example is given by Figure 1 on the case ***N*=4 and *T*=6**.

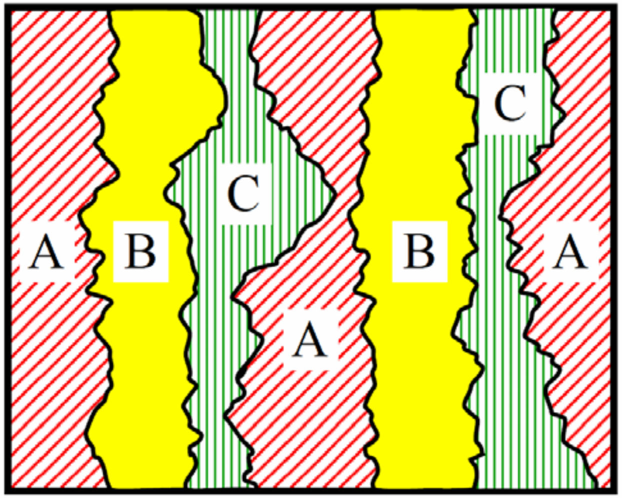


Physical Motivations to Study Vicious Walker Models

•As a model of **Wetting or Melting Transitions**
(Fisher (J. Statistical Physics 1984))

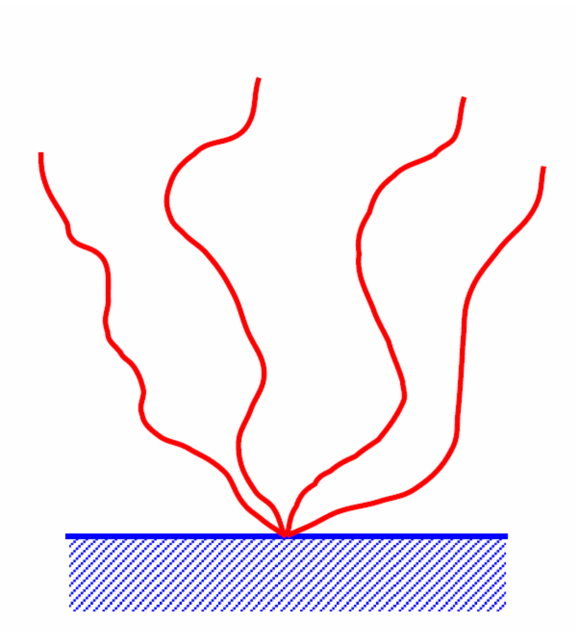


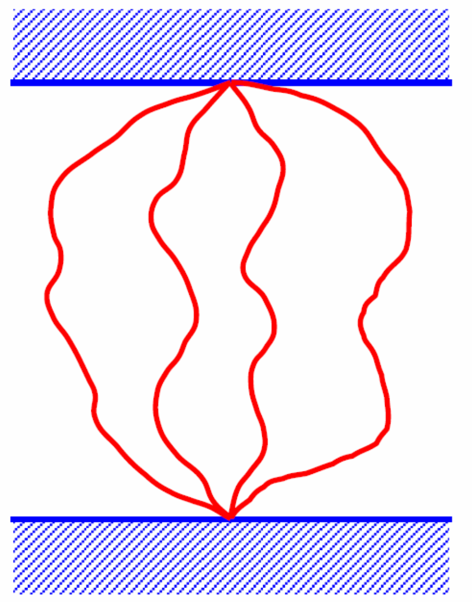
•As a model of **Commensurate-Incommensurate Transitions**
(Huse and Fisher (Physical Review B 1984))



•As a model of **Directed Polymer Networks**
(de Gennes (J. Chemical Phys. 1968),
Essam and Guttmann (Phys. Review E 1995))

(a) polymer with **star topology** (b) polymer with **watermelon topology**

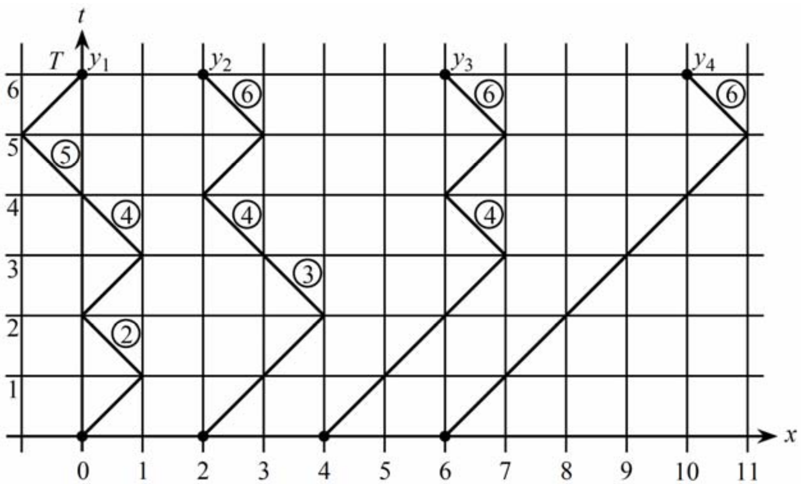


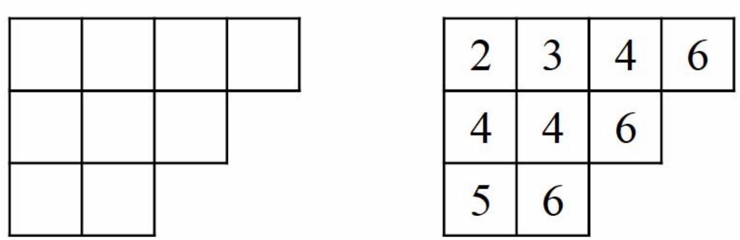


2. Young Diagrams, Young Tableaux and Schur Polynomials

A bijection between vicious walks and semistandard Young tableaux (SSYT).

[Guttmann, Owczarek and Viennot, J. Physics A (1998), Krattenthaler, Guttmann and Viennot, J. Physics A (2000)]





(a) (b)

(1) Let

L_j= the number of **leftward steps** among T steps of the j-th walker, for j =1,2,…., N.

Draw a collection of boxes with ***N* columns**,

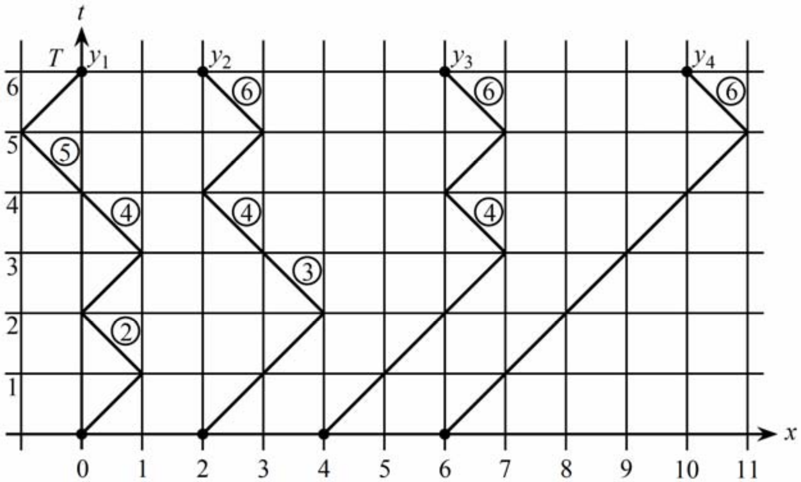
in which the number of boxes in the j-th column is L_j .

[e.g., $L = (L_1, L_2, L_3, L_4)$=(3, 3, 2, 1)]

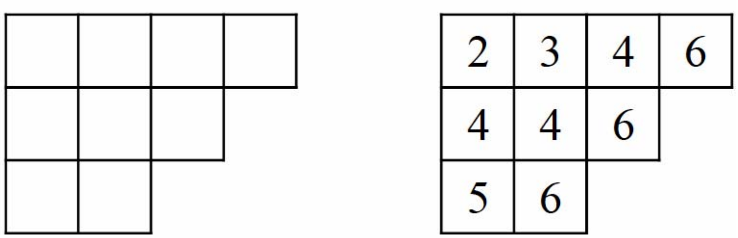
(2) For each walker, we **label** each leftward step by the integer 1,2,…,T, which is **the time when that leftward step is done**.

(3) Then for the j-th column of the collections of boxes, **fill the boxes** by the **labels of leftward steps** of the j-th walker, from the top to the bottom, j =1, 2,…, N.

Remark 1



Young diagram



(a) (b)

L_j = the number of **leftward steps** among T steps of the j-th walker, for j=1,2,…., N.

[*e.g.*, $\mathbf{L}=(L_1, L_2, L_3, L_4)=(3,3,2,1)$]

◎ The **noncolliding condition** of vicious walkers guarantees the equalities

$$L_1 \geq L_2 \geq .... \geq L_N \; . \qquad (2.1)$$

◎ Let

λ_k = the number of boxes in the k-th row.

Then the equalities (2.1) imply

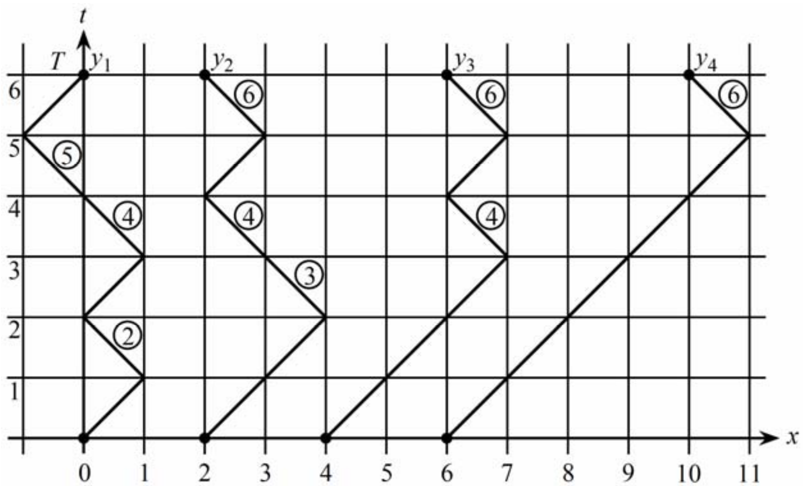
$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l \; . \qquad (2.2)$$

$$[\, e.g., \;\; \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (4,3,2) \;\,]$$

- The collections of boxes with conditions concerning the numbers of boxes in rows (2.2) (and in columns (2.1)) are called **Young diagram (YD).**
- The total number of rows in the YD is called the **length l of YD.**

[*e.g.*, length l = 3. In general, length l =0,1,2,…, T]

Remark 2



Let

$T(j, k)$= the integer in the box located in the j-th row and k-th column,

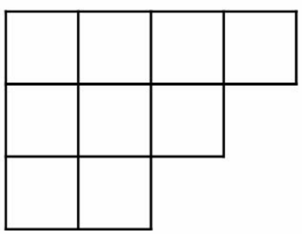
◎ The **noncolliding condition** of vicious walkers guarantees the equalities

$$T(j,k) < T(j+1,k)$$

strictly increasing in each column,

$$T(j,k) \leq T(j,k+1)$$

weakly increasing in each row.



(a)

Young Tableau

2	3	4	6
4	4	6	
5	6		

(b)

Young diagram with integers T=($T(j, k)$) satisfying the **above ordering** of $T(j, k)$'s are called **Semi-Standard Young Tableaux (SSYT).**

- YD with λ_k boxes in the k-th row, $k = 1,2, \ldots, l$, is said to be the YD of **shape** $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_l)$.
- Let L_j = the number of boxes in the j-th column.

  The YD with the shape $\mathbf{L} = (L_1, L_2, \ldots, L_N)$ is regarded as the **conjugate** of the YD with the shape $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_l)$.

  and denoted by

$$\mathbf{L} = \tilde{\boldsymbol{\lambda}} \quad \text{or} \quad \boldsymbol{\lambda} = \tilde{\mathbf{L}}.$$

- As shown in Figure 3, they are **mirror images with respect to the diagonal line**.

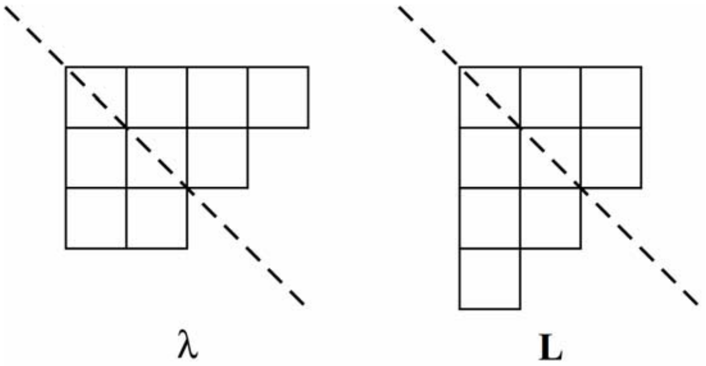


Figure 3

- The sequence of integers in the weakly decreasing order (representing the number of boxes in rows of YD)

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_T), \quad \lambda_k \in \mathbf{N}, \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_T$$

  is regarded as a **partition** of an integer $n = \sum_{k=1}^{T} \lambda_k$

- The number of parts l is equal to or less than T.

  We introduce a set of ***T*** **variables** $\mathbf{z} = (z_1, z_2, \ldots, z_T)$.

- For each SSYT, **T**=(*T*(*j*,*k*)) , define a monomial

$$\mathbf{z}^{\mathrm{T}} = \prod_{(j,k)} z_{T(j,k)}$$

$$= \prod_{m=1}^{T} z_m^{\#\text{ of times that the integer } m \text{ occurs in T}}$$

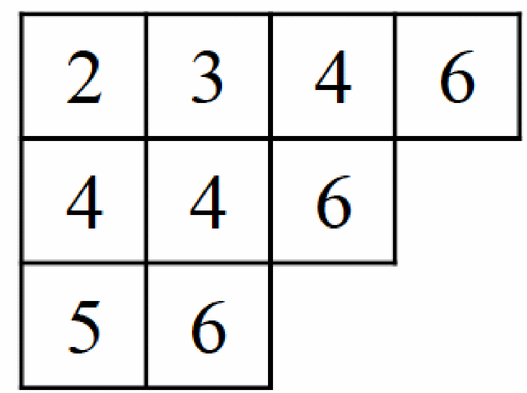
- e.g., for the SSYS shown in Figure 2 (b),

$$\mathbf{z}^{\mathrm{T}} = z_2 \times z_3 \times z_4 \times z_6$$

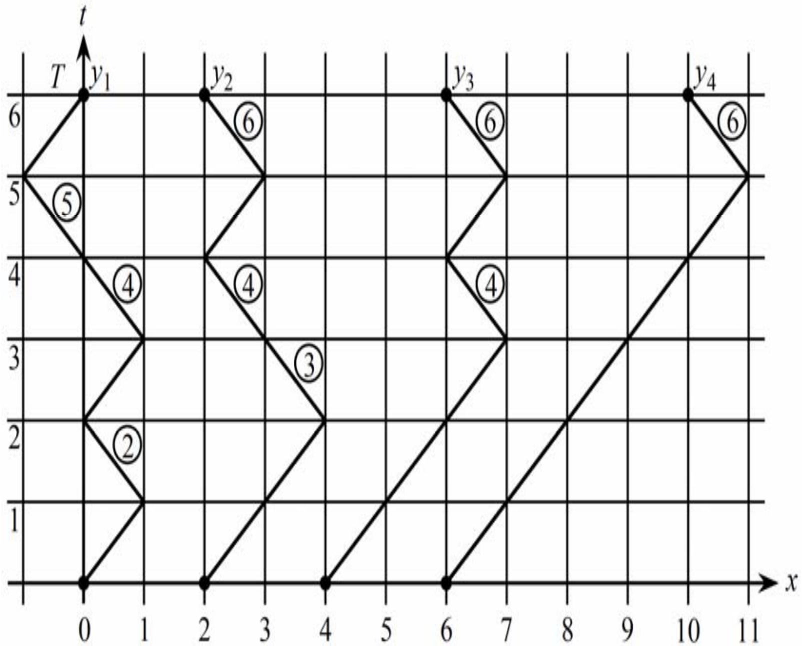
$$\times z_4 \times z_4 \times z_6$$

$$\times z_5 \times z_6$$

$$= z_2 z_3 z_4^3 z_5 z_6^3$$



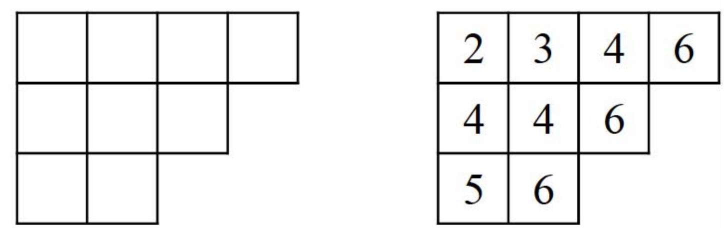
(b)



- Notice that for one **YD** with a given shape **λ**, there are **different ways of filling boxes with integers** to make **SSYT**.
- For **each YD** with shape **λ** , we define a **polynomial** of $\mathbf{z} = (z_1, z_2, \ldots, z_T)$

by summing the monomials **over all SSYT defined on that YD**:

$$s_\lambda(z_1, z_2, \ldots, z_T) = \sum_{\text{T:all SSYT with the same shape } \lambda} \mathbf{z}^{\mathrm{T}} \quad .$$

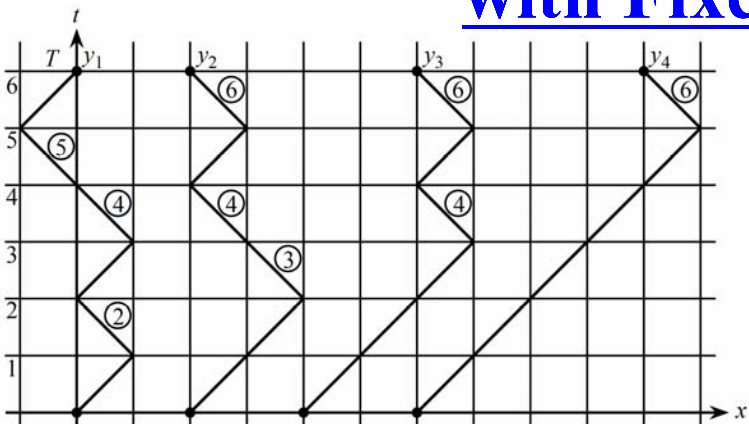


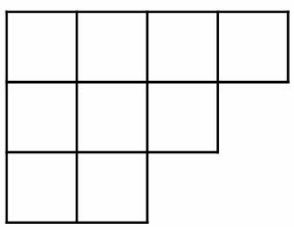
(a) (b)

This polynomial is called the **Schur Function** indexed by **λ** (the partition/YD with shape)

- **shape** of YD ⟷ the **final positions** of vicious walkers at $t = T$.
- variety of **SSYT** on a YD ⟷ variety of **different vicious walks** with the same final positions

3. Enumeration of Vicious Walks with Fixed Final Positions





2	3	4	6
4	4	6	
5	6		

- There is a simple relation between the number of leftward steps L_j, $j = 1, 2, \ldots, N$, and the final positions y_j, $j = 1, 2, \ldots, N$, as

$$L_j = (T - y_j)/2 + (j - 1), \quad j = 1, 2, \ldots, N,$$

or equivalently

$$y_j = T - 2L_j + 2(j - 1), \quad j = 1, 2, \ldots, N.$$

$\mathbf{y} \Leftrightarrow \mathbf{L}$ and $\lambda = \tilde{\mathbf{L}}$ (conjugate)

bijections

- **final positions y** of vicious walkers ⟺ **YD** with the shape λ
- **one realization of vicious walk** from $\mathbf{S}(0) = (2(j-1))$, $j = 1, \ldots, N$, to $\mathbf{S}(T) = \mathbf{y}$ ⟺ **one SSYT** $T = (T(j,k))$ with the shape λ
- **set of all vicious walk** from $\mathbf{S}(0) = (2(j-1))$, $j = 1, \ldots, N$, to $\mathbf{S}(T) = \mathbf{y}$ ⟺ **a Schur function** $s_\lambda(z_1, z_2, \ldots, z_T)$

Definition

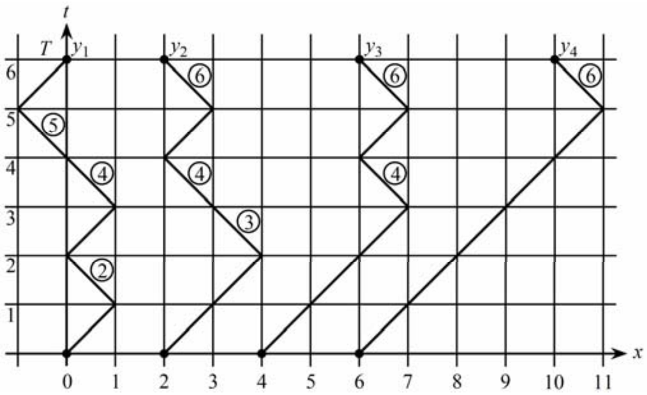
For $y_1 < y_2 < \dots < y_N$, define the number

$M_{N,T}(\mathbf{y})$ = total number of distinct vicious walks of N walkers from the initial positions

$$S_j(0) = 2(j-1)\ , \qquad j=1,2,\dots, N$$

to the final positions

$$S_j(T) = y_j\ , \qquad j=1,2,\dots, N\ .$$



The above **bijection** between vicious walks and Schur functions gives the following identity.

For $\mathbf{L} = (L_1, L_2, \dots, L_N) = (\ (T - y_j)/2 + (\ j-1)\ )$, set $\boldsymbol{\lambda} = \tilde{\mathbf{L}}$,
Then

$$M_{N,T}(\mathbf{y}) = s_{\boldsymbol{\lambda}}(z_1, z_2, \dots, z_T)\Big|_{z_1 = z_2 = \dots = z_T = 1}\ .$$

Jacobi-Trudi Formula

$$s_\lambda(z_1, z_2, \dots, z_T) = \frac{\det_{1\le j,k\le T}\left(z_j^{\lambda_k+T-k}\right)}{\det_{1\le j,k\le T}\left(z_j^{T-k}\right)},$$

Vandermonde determinant

$$\det_{1\le j,k\le T}\left(z_j^{T-k}\right) = \prod_{1\le j<k\le T}(z_k - z_j).$$

$$\begin{aligned} M_{N,T}(\mathbf{y}) &= \lim_{q\to 1} s_\lambda(1, q, q^2, \dots, q^{T-1}) \\ &= \lim_{q\to 1} \frac{\det_{1\le j,k\le T}\left(q^{(j-1)(\lambda_k+T-k)}\right)}{\det_{1\le j,k\le T}\left(q^{(j-1)(T-k)}\right)} \\ &= \lim_{q\to 1} q^{\sum_{m=1}^{T}(m-1)\lambda_m} \prod_{1\le j<k\le T} \frac{q^{\lambda_j-\lambda_k+k-j}-1}{q^{k-j}-1} \\ &= \prod_{1\le j<k\le T} \frac{\lambda_j-\lambda_k+k-j}{k-j}. \end{aligned}$$

Remark: This is nothing but the **dimension of the irreducible representation** R with the highest weight λ of the unitary group U(T)

Jacobi-Trudi formula

$$s_{\lambda}(z_1, z_2, \dots, z_T) = \det_{1\le j,k\le N}[e_{\tilde{\lambda}_k+(j-k)}(z_1, z_2, \dots, z_T)].$$

By definition of **elementary symmetric polynomials**

$$\prod_{j=1}^{T}(1+z_j\xi) = \sum_{k=0}^{T} e_k(z_1, z_2, \dots, z_T)\xi^k \;\Rightarrow\; \text{if all } z_j = 1, \text{ then } (1+\xi)^T = \sum_{k=0}^{T}\binom{T}{k}\xi^k,$$

$$\text{that is,} \quad e_k(1,1,\dots,1) = \binom{T}{k} \quad \text{(binomial coefficient)}$$

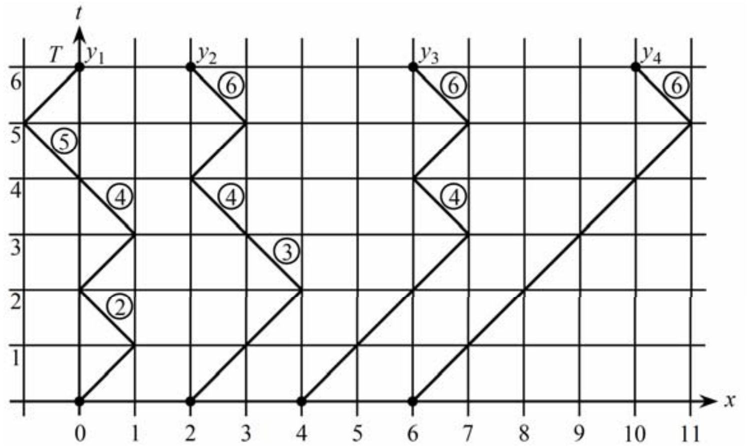
Since $\tilde{\boldsymbol{\lambda}} = \mathbf{L}$ and $L_k = (T-y_k)/2 + (j-1)$,

$$M_{N,T}(\mathbf{y}) = \det_{1\le j,k\le N}\left[\binom{T}{\tilde{\lambda}_k + j - k}\right]$$

$$= \det_{1\le j,k\le N}\left[\binom{T}{L_k + j - k}\right]$$

$$= \det_{1\le j,k\le N}\left[\binom{T}{(T+2(j-1)-y_k)/2}\right].$$

$$M_{N,T}(\mathbf{y}) = \det_{1 \le j,k \le N}\left[\binom{T}{(T+2(j-1)-y_k)/2}\right].$$



Since

$$\binom{T}{(T+2(j-1)-y_k)/2}$$ = the number of possible **single paths** on the spatio-temporal plane

from the position $2(j-1)$ at the initial time $t=0$

to the position y_k at the final time $t=T$,

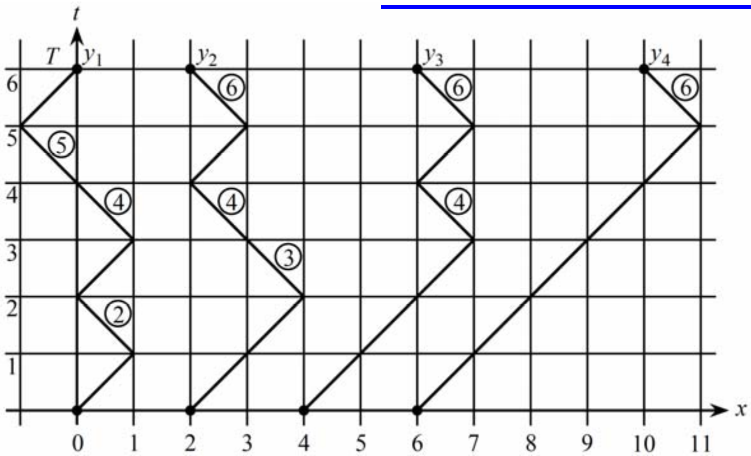
this **expression of binomial-coefficient determinant**

is regarded as a special case of

Karlin-McGregor formula known in **probability theory,** and

Lindstrom-Gessel-Viennot formula known in **combinatorics theory**.

4. Enumeration of Vicious Walks with All Possible Final Positions



Definition

$M_{N,T}$ = total number of distinct vicious walks of N walkers from the initial positions $S_j(0) = 2(j\text{-}1)$, $j=1,2,\ldots, N$ to any possible final positions at time T

$$M_{N,T} = \sum_{\mathbf{y}:y_1<y_2<\ldots<y_N} M_{N,T}(\mathbf{y})$$

$$= \sum_{\lambda:N\geq\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_T\geq 0}\ \prod_{1\leq j<k\leq T} \frac{\lambda_j-\lambda_k+k-j}{k-j} = \prod_{1\leq j<k\leq T} \frac{N+j+k-1}{j+k-1}.$$

Asymptotics of the Survival Probability of Vicious Walkers

$$P_{N,T} \equiv \frac{M_{N,T}}{2^{NT}} \approx T^{-\psi_N} \quad \text{with} \quad \psi_N = \frac{1}{4}N(N-1).$$

[Fisher (1984), Guttmann et al. (1998), Krattenthaler (2000)]

Summation Formula

$$\sum_{\lambda: N \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_T \geq 0} \prod_{1 \leq j < k \leq T} \frac{\lambda_j - \lambda_k + k - j}{k - j} = \prod_{1 \leq j < k \leq T} \frac{N + j + k - 1}{j + k - 1}.$$

This is obtained by $q \to 1$ limit of any of the following two equalities,

$$\sum_{\lambda: N \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_T \geq 0} s_\lambda\left(q^{(2T-1)/2}, q^{(2T-3)/2}, \ldots, q^{1/2}\right) = \prod_{i=1}^{T} \frac{1 - q^{(N+2i-1)/2}}{1 - q^{(2i-1)/2}} \prod_{1 \leq j < k \leq T} \frac{1 - q^{N+j+k-1}}{1 - q^{j+k-1}}$$

(MacMahon's conjecture)

$$\sum_{\lambda: N \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_T \geq 0} s_\lambda\left(1, q, q^2, \ldots, q^{T-1}\right) = \prod_{1 \leq j < k \leq T} \frac{1 - q^{N+j+k-1}}{1 - q^{j+k-1}}$$

(Bender - Knuth's conjecture)

More general summation formula is given in the textbook of Macdonald

$$\sum_{\boldsymbol{\lambda}:N\ge\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_T\ge 0} s_{\boldsymbol{\lambda}}(z_1,z_2,\ldots,z_T)=\frac{\det_{1\le i,j\le T}\left(z_i^{j-1}-z_i^{N+2T-j}\right)}{\det_{1\le i,j\le T}\left(z_i^{j-1}-z_i^{2T-j}\right)}.$$

Remark

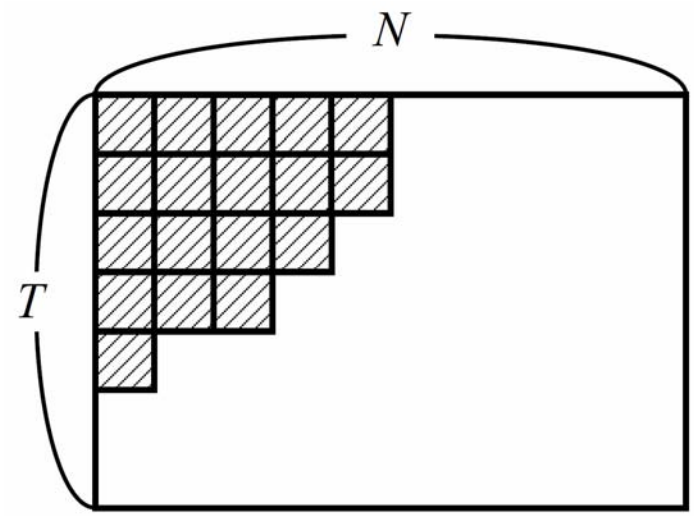
$$\{\boldsymbol{\lambda}: N\ge\lambda_1\ge\lambda_2\ge\ldots.\ge\lambda_T\ge 0\}$$

$$\Updownarrow$$

$$\left\{\begin{array}{l}\text{YD with the length of the first row } \lambda_1\le N\\ \quad\text{and with the number of parts } \ell(\boldsymbol{\lambda})\le T\end{array}\right\}$$

$$\Updownarrow$$

$$\{\text{YD included in an } N\times T \text{ rectangle}: \ \boldsymbol{\lambda}\in N^T\}$$



5. Diffusion Scaling Limit

- Remember that $(\{\mathbf{S}(t)\}_{t=0,1,2,\dots}, \mathrm{Q}^{\mathbf{x}})$ denotes the vicious walk with the noncolliding condition up to time T ; $S_1(t) < S_2(t) < \dots < S_N(t)$, $t = 0,1,2,\dots,T$.
- For $T > 0$, $\mathbf{x} \in \mathbf{Z}_<^N$, we consider probability measures

$$\mu_{L,T}^{\mathbf{x}}(\,.\,) = \mathrm{Q}_{L^2 T}^{\mathbf{x}}\left(\frac{1}{L}\mathbf{S}(L^2 t) \in .\,\right), \qquad L \ge 1, \text{ on the space of continuous paths } C([0,T] \to \mathbf{R}^N),$$

where $\mathbf{S}(t)$ is here considered to be the interpolation of the random walk $\mathbf{S}(t), t = 0,1,2,\dots$.

- Let $\mathbf{R}_<^N = \left\{ \mathbf{x} \in \mathbf{R}^N : x_1 < x_2 < \dots < x_N \right\}$ (Weyl chamber of type A_{N-1}).
- By vietue of the Karlin - McGregor formula,

$$\mathrm{f}_N(t, \mathbf{y} \mid \mathbf{x}) = \det_{1 \le j,k \le N}\left[\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{(x_j - y_k)^2}{2t}\right\}\right]$$

= the transition density of the absorbing Brownian motion in $\mathbf{R}_<^N$.

- The probability that the Brownian motion started at $\mathbf{x} \in \mathbf{R}_<^N$ does not hit the boundary of $\mathbf{R}_<^N$ up to time $\mathrm{t} > 0$ is given by

$$\mathrm{N}_N(t, \mathbf{x}) = \int_{\mathbf{R}_<^N} dy\, \mathrm{f}_N(t, \mathbf{y} \mid \mathbf{x}).$$

Theorem 1. For any fixed $\mathbf{x} \in \mathbf{Z}^N_<$ and $T > 0$, as $L \to \infty$, $\mu^{\mathbf{x}}_{L,T}(\,.\,)$ converges weakly to the law of the temporally inhomogeneous diffusion process

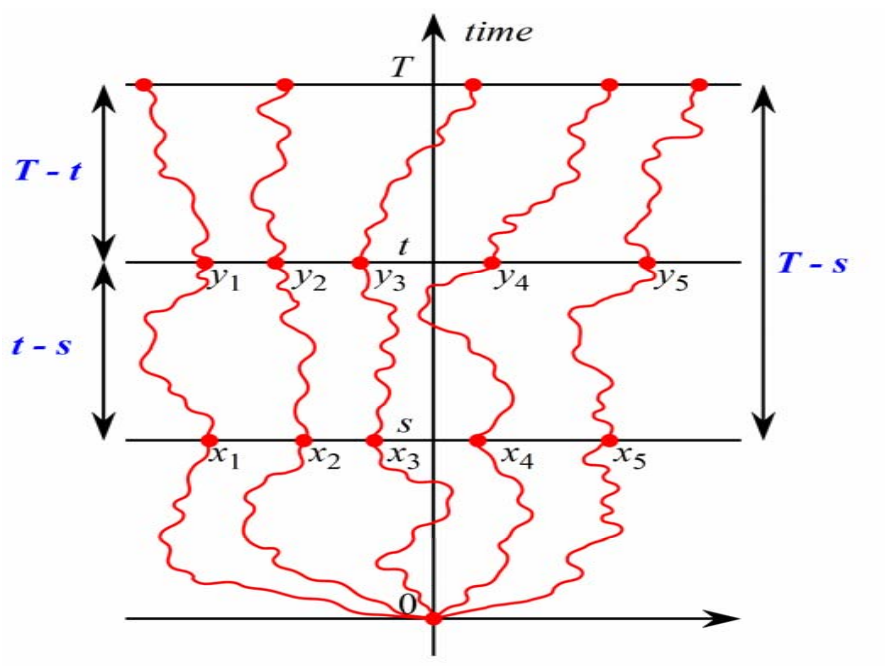
$$\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_N(t)), \quad t \in [0, T],$$

with the transition probability density

$$\mathrm{g}_{N,T}(0,\mathbf{0};t,\mathbf{y}) = c \times \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} \prod_{1 \le i < j \le N} (y_j - y_i)\, \mathrm{N}_N(T-t,\mathbf{y}),$$

$$\mathrm{g}_{N,T}(s,\mathbf{x};t,\mathbf{y}) = \frac{\mathrm{f}_N(t-s,\mathbf{y}\,|\,\mathbf{x})\, \mathrm{N}_N(T-t,\mathbf{y})}{\mathrm{N}_N(T-s,\mathbf{x})},$$

for $0 \le s < t \le T$, $\mathbf{x}, \mathbf{y} \in \mathbf{R}^N_<$, where $\mathbf{0} = (0,0,\ldots 0)$, $c = 2^{-N/2} T^{N(N-1)/4} t^{-N^2/2} / \prod_{j=1}^{N} \Gamma(j/2)$, $|\mathbf{y}|^2 = \sum_{j=1}^{N} y_j^2$.



Next we consider the case that the **noncolliding time-period** $T=T_L$ goes to **infinity**.

Corollary 2.

(i) Let T_L be an increasing function of L with $T_L \to \infty$ as $L \to \infty$.

For any fixed $\mathbf{x} \in \mathbf{Z}^N_<$ and $T>0$, as $L \to \infty$, $\mu^{\mathbf{x}}_{L,T}(\,.\,)$ converges weakly to the law of the temporally homogeneous diffusion process

$$\mathbf{Y}(t) = (Y_1(t), Y_2(t), \ldots, Y_N(t)), \quad t \in [0,\infty),$$

with the transition probability density

$$\mathrm{p}_N(0,\mathbf{0};t,\mathbf{y}) = c' \times \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} \mathrm{h}_N(\mathbf{y})^2,$$

$$\mathrm{p}_N(s,\mathbf{x};t,\mathbf{y}) = \frac{1}{\mathrm{h}_N(\mathbf{x})} \mathrm{f}_N(t-s,\mathbf{y}\,|\,\mathbf{x})\,\mathrm{h}_N(\mathrm{y}) \quad \text{with} \quad \mathrm{h}_N(\mathbf{x}) = \prod_{1\le j<k\le N}(x_k - x_j)$$

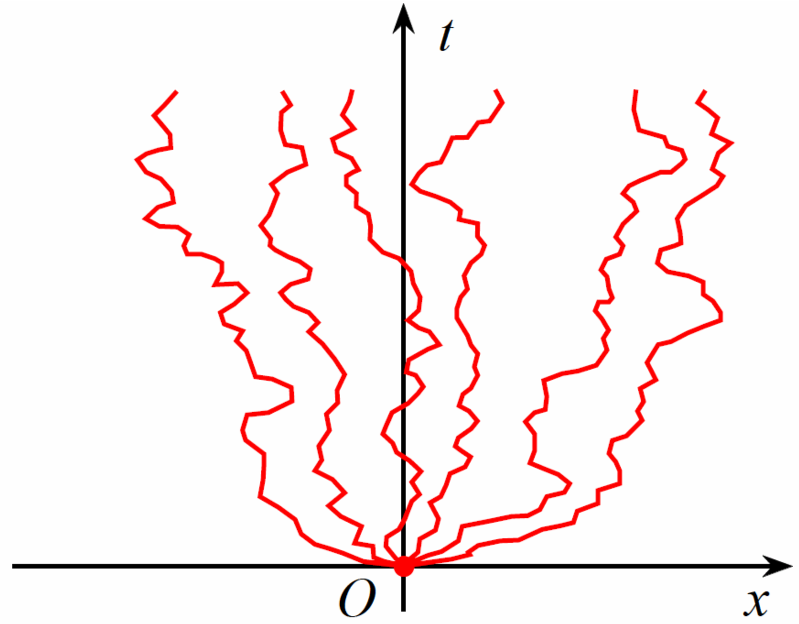
for $0 \le s < t \le T$, $\mathbf{x},\mathbf{y} \in \mathbf{R}^N_<$, where $c' = (2\pi)^{-N/2} t^{-N^2/2} / \prod_{j=1}^{N} \Gamma(j)$.

(ii) The diffusion process $\mathbf{Y}(t)$ solves the equations of Dyson's Brownian motion model:

$$dY_i(t) = dB_i(t) + \sum_{1\le j\le N,\, j\ne i} \frac{1}{Y_i(t) - Y_j(t)} dt, \quad t \in [0,\infty), \; i = 1,2,\ldots,N,$$

where $\{B_i(t)\}_{i=1}^N$ denote independent standard Brownian motions.

REMARKS: When all the particles are starting from the origin **0**,



Strong Repulsive Interactions

$$p_N(0,\mathbf{0};t,\mathbf{y}) \propto \prod_{k=1}^{N} \left\{ \frac{1}{\sqrt{2\pi t}} e^{-y_k^2/2t} \right\} \times \prod_{1 \le i < j \le N} (y_j - y_i)^2$$

Product of Independent Gaussian Distributions

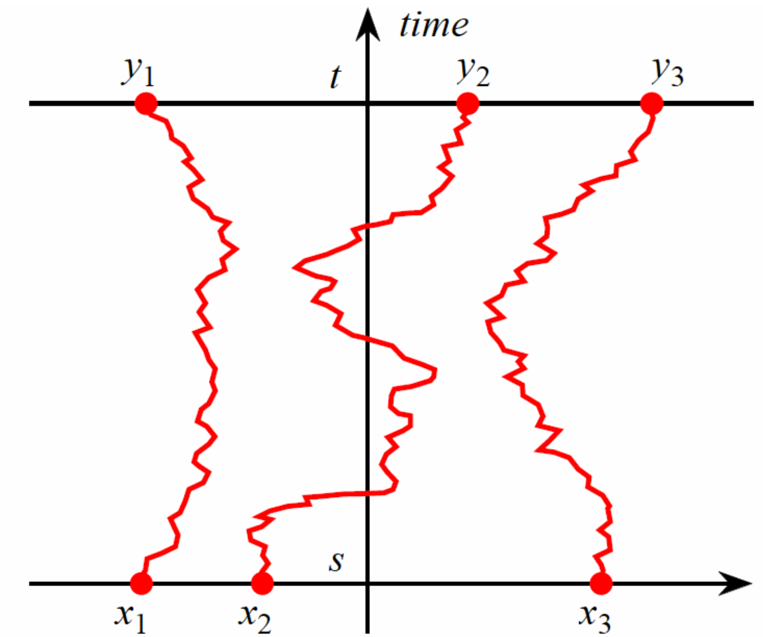
As $|y_j - y_i| \to 0, \quad p_N(0,\mathbf{0};t,\mathbf{y}) \to 0$.

The Process $\boldsymbol{Y}(t)$ = **NONCOLLIDING Browniam Motions**

REMARKS: Here we set $N=3$ as an example.

$$\mathrm{p}_N\left(s,(x_1,x_2,x_3);t,(y_1,y_2,y_3)\right)=\frac{\mathrm{h}_N(\mathbf{y})}{\mathrm{h}_N(\mathbf{x})}\times\det\begin{bmatrix}\mathrm{G}(t-s,y_1\,|\,x_1) & \mathrm{G}(t-s,y_2\,|\,x_1) & \mathrm{G}(t-s,y_3\,|\,x_1)\\ \mathrm{G}(t-s,y_1\,|\,x_2) & \mathrm{G}(t-s,y_2\,|\,x_2) & \mathrm{G}(t-s,y_3\,|\,x_2)\\ \mathrm{G}(t-s,y_1\,|\,x_3) & \mathrm{G}(t-s,y_2\,|\,x_3) & \mathrm{G}(t-s,y_3\,|\,x_3)\end{bmatrix}$$

***h*-transform in the sense of Doob**



Karlin-McGregor formula
Lindstrom-Gessel-Viennot formula

$$G(t-s,y\,|\,x)=\frac{1}{\sqrt{2\pi(t-s)}}\exp\left\{-\frac{(x-y)^2}{2(t-s)}\right\}$$

(heat kernel),

$$\mathrm{h}_N(x)=\prod_{1\le i<j\le N}(x_j-x_i).$$

The process *Y*(*t*) is identified with Dyson's Brownian motion model

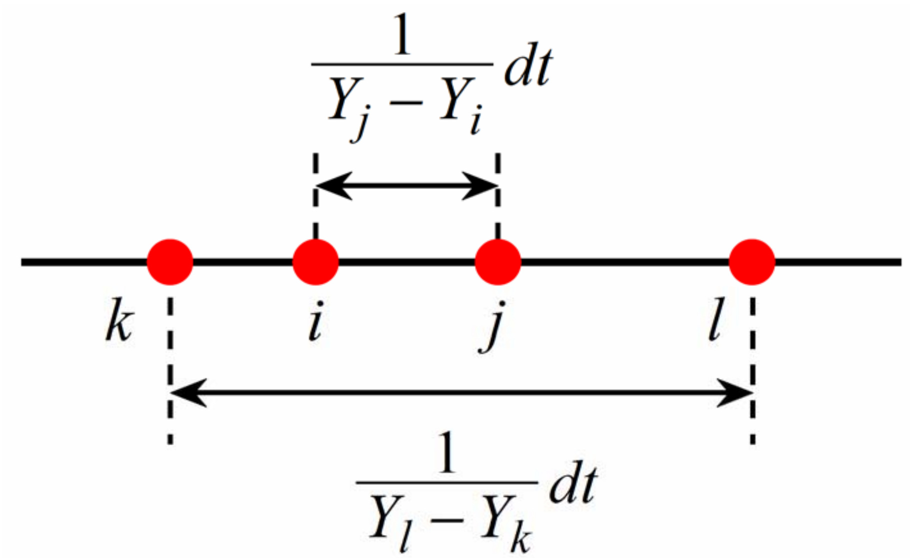
Dyson's Brownian motion model

1. For all $t > 0$, $\mathbf{Y}(t) \in \mathbf{R}^N_<$ with **Probability 1**.
2. The process is given as a solution of the stochastic differential equations,

$$dY_i(t) = dB_i(t) + \sum_{j:1\le j\le N,\, j\ne i} \frac{1}{Y_i(t) - Y_j(t)} dt \qquad 1 \le i \le N, t \in [0, \infty)$$

where $B_i(t)$ are independent standard Brownian motions $i = 1, 2, \ldots, N$.

- **Strong repulsive forces among any pair of particles** $\propto \dfrac{1}{\text{particle distance}}$



Remark. HERMITIAN MATRIX-VALUED PROCESS AND DYSON'S BROWNIAN MOTION MODEL

•Let $B_{ij}(t), \widetilde{B}_{ij}(t), 1 \le i, j \le N,$ be mutually independent (standard one-dim.) Brownian motions started from the origin. Define

$$s_{ij}(t) = \begin{cases} \frac{1}{\sqrt{2}} B_{ij}(t) & (i < j) \\ B_{ii}(t) & (i = j) \\ \frac{1}{\sqrt{2}} B_{ij}(t) & (i > j) \end{cases} \qquad a_{ij}(t) = \begin{cases} \frac{1}{\sqrt{2}} \widetilde{B}_{ij}(t) & (i < j) \\ 0 & (i = j) \\ -\frac{1}{\sqrt{2}} \widetilde{B}_{ij}(t) & (i > j) \end{cases}$$

•Consider the $N \times N$ **Hermitian Matrix-Valued** stochastic process

$$\Xi(t) = \left(\xi_{ij}(t)\right)_{1 \le i, j \le N} = \left(s_{ij}(t) + \sqrt{-1} a_{ij}(t)\right)_{1 \le i, j \le N}$$

That is,

$$\Xi(t) = \begin{pmatrix} s_{11}(t) & s_{12}(t) + \sqrt{-1} a_{12}(t) & s_{13}(t) + \sqrt{-1} a_{13}(t) & \dots & s_{1N}(t) + \sqrt{-1} a_{1N}(t) \\ s_{12}(t) - \sqrt{-1} a_{12}(t) & s_{22}(t) & s_{23}(t) + \sqrt{-1} a_{23}(t) & \dots & s_{2N}(t) + \sqrt{-1} a_{2N}(t) \\ s_{13}(t) - \sqrt{-1} a_{13}(t) & s_{23}(t) - \sqrt{-1} a_{23}(t) & s_{33}(t) & \dots & s_{3N}(t) + \sqrt{-1} a_{3N}(t) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ s_{1N}(t) - \sqrt{-1} a_{1N}(t) & s_{2N}(t) - \sqrt{-1} a_{2N}(t) & s_{3N}(t) - \sqrt{-1} a_{3N}(t) & \dots & s_{NN}(t) \end{pmatrix}$$

•Consider the variation of the matrix, $d\Xi(t) = \left(d\xi_{ij}(t)\right)_{1\le i,j\le N}$

It is clear that

$$\left\langle d\xi_{ij}(t)\right\rangle = 0 \qquad (1\le i,j\le N)$$

And by the previous observation, we find that

$$\left\langle \left(d\xi_{ij}(t)\right)^2\right\rangle = 0 \qquad (1\le i\ne j\le N)$$

$$\left\langle d\xi_{ij}(t)d\xi_{ji}(t)\right\rangle = \left\langle d\xi_{ij}(t)d\xi_{ij}(t)^*\right\rangle = dt \qquad (1\le i\ne j\le N)$$

$$\left\langle \left(d\xi_{ii}(t)\right)^2\right\rangle = dt \qquad (1\le i\le N)$$

They are summarized as

$$\left\langle d\xi_{ij}(t)d\xi_{kn}(t)\right\rangle = \delta_{in}\delta_{jk}dt \qquad (1\le i,j,k,n\le N)$$

- Since $\Xi(t)$ is a Hermitian matrix-valued process,

at each time t there is a Unitary Matrix $U(t) = (u_{ij}(t))_{1\le i,j\le N}$, such that

$$U(t)^+\Xi(t)U(t) = \Lambda(t) = \mathrm{diag}\left\{\lambda_1(t),\lambda_2(t),\ldots.,\lambda_N(t)\right\}$$

where the eigenvalues are in the increasing order

$$\lambda_1(t)\le\lambda_2(t)\le\ldots.\le\lambda_N(t), \qquad \forall t\in[0,\infty)$$

- We can regard

$$\boldsymbol{\lambda}(t)\equiv\left(\lambda_1(t),\lambda_2(t),\ldots.,\lambda_N(t)\right)\in\mathbf{R}^N$$

as an N-particle stochastic process in one dimension.

QUESTION

By the diagonalization of the matrix, what kind of interactions emerge among the N particles in the process $\boldsymbol{\lambda}(t)$?

•From now on we assume that

$$\lambda_1(0) < \lambda_2(0) < .... < \lambda_N(0)$$

•And we consider the following conditional configuration-space of one-dim. N particles,

$$\mathbf{W}_N^A = \{\mathbf{x} \in \mathbf{R}^N : x_1 < x_2 < .... < x_N\}$$

(This is called the Weyl chamber of type A_{N-1} .)

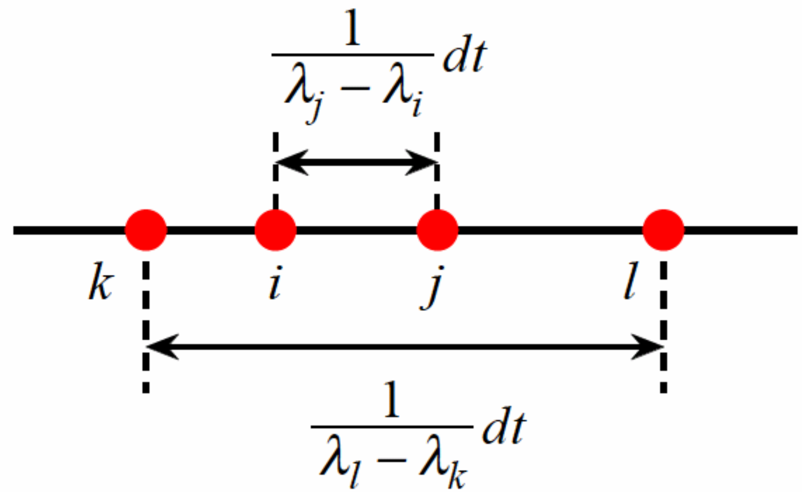
ANSWER 1 (by Dyson 1962)

1. For all $t > 0$, $\boldsymbol{\lambda}(t) \in \mathbf{W}_N^A$ with Probability 1.
2. The process is given as a solution of the stochastic differential equations,

$$d\lambda_i(t) = dB_i(t) + \sum_{j:1\le j\le N, j\ne i} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt \qquad 1 \le i \le N, t \in [0, \infty)$$

where $B_i(t)$ are independent standard one-dim. Brownian motions $(1 \le i \le N)$

- This process is called Dyson's Brownian motion model.
- Strong repulsive forces emerge among any pair of particles $\propto \dfrac{1}{\text{particle distance}}$



- Let
$$\begin{cases} \mathrm{h}(\mathbf{x}) = \prod_{1\le i<j\le N} (x_j - x_i) \quad \text{(product of differences)} \\ b_i(\mathbf{x}) = \sum_{j:1\le j\le N, j\ne i} \frac{1}{x_i - x_j} = \frac{\partial}{\partial x_i} \ln \mathrm{h}(\mathbf{x}) \ (1 \le i \le N), \ \mathbf{b}(\mathbf{x}) = (b_1(\mathbf{x}), b_2(\mathbf{x}), \dots, b_N(\mathbf{x})) \end{cases}$$

- Consider $\mathrm{p}(s,\mathbf{x};t,\mathbf{y}) = [$ transition probability density from $\boldsymbol{\lambda}(s) = \mathbf{x}$ to $\boldsymbol{\lambda}(t) = \mathbf{y}$ $]$,

  where $\mathbf{x} = (x_1, x_2, \dots, x_N)$, $\mathbf{y} = (y_1, y_2, \dots, y_N)$.

It solves the Fokker-Planck (FP) equation in the form

$$\frac{\partial}{\partial t}\mathrm{p}(s,\mathbf{x};t,\mathbf{y}) = \frac{1}{2}\Delta_{\mathbf{x}}\mathrm{p}(s,\mathbf{x};t,\mathbf{y}) + \mathbf{b}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\mathrm{p}(s,\mathbf{x};t,\mathbf{y})$$

ANSWER 2

Introduce a determinant

$$\mathrm{f}(t,\mathbf{y}\,|\,\mathbf{x}) = \det_{1\le i,j\le N}\left[\mathrm{G}(t,y_j\,|\,x_i)\right] \text{ with } \mathrm{G}(t,y_j\,|\,x_i) = \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-(x_i-y_j)^2/2t}$$

Then the solution of the FP equation is given by

$$\mathrm{p}(s,\mathbf{x};t,\mathbf{y}) = \frac{1}{\mathrm{h}(\mathbf{x})}\mathrm{f}(t-s,\mathbf{y}\,|\,\mathbf{x})\,\mathrm{h}(\mathbf{y}) \quad \text{for} \quad 0<s<t<\infty, \quad \mathbf{x},\mathbf{y}\in\mathbf{W}_N^A \ .$$

- If $\mathbf{x}\to\mathbf{0}=(0,0,\dots,0)$ at $s=0$, (all particles starting from the origin)

$$\mathrm{p}(0,\mathbf{0};t,\mathbf{y}) = \frac{t^{-N^2/2}}{C_1}\exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\}\mathrm{h}(\mathbf{y})^2 \quad \text{where } |\mathbf{y}|^2 = \sum_{i=1}^N y_i^2,\ C_1 = (2\pi)^{N/2}\prod_{i=1}^N \Gamma(i).$$

Remaks

Vicious Walker Model

Enumerative Combinatorics
Young Tableaux
Symmetric Functions

Diffusion Scaling limit

Noncolliding Diffusion Processes

Representation Theory of
Lie Algebra/Lie Groups

determinantal expressions
of correlation functions
$N \to \infty$ **limit**

Infinite Systems of
Noncolliding Diffusion Particles
Theory of Entire Functions

Random Matrix Theory
Matrix Models in Statistical Phys.
and
String Theory, etc…..

I hope …..

Statistical Physics
Probability Theory
(Infinite) Particle Systems

Noncolliding Processes

Other Fields of
Mathematics

6. Temporally Inhomogeneous Noncolliding Brownian Motions

The **temporally inhomogeneous diffusion process**

$$\mathbf{X}(t) = (X_1(t), X_2(t), \dots, X_N(t)), \quad t \in [0, T],$$

with the transition probability density

$$\mathrm{g}_{N,T}(0,\mathbf{0}; t, \mathbf{y}) = c \times \exp\left\{ -\frac{|\mathbf{y}|^2}{2t} \right\} \prod_{1 \le i < j \le N} (y_j - y_i)\, \mathrm{N}_N(T-t, \mathbf{y}),$$

$$\mathrm{g}_{N,T}(s, \mathbf{x}; t, \mathbf{y}) = \frac{\mathrm{f}_N(t-s, \mathbf{y} \mid \mathbf{x})\, \mathrm{N}_N(T-t, \mathbf{y})}{\mathrm{N}_N(T-s, \mathbf{x})},$$

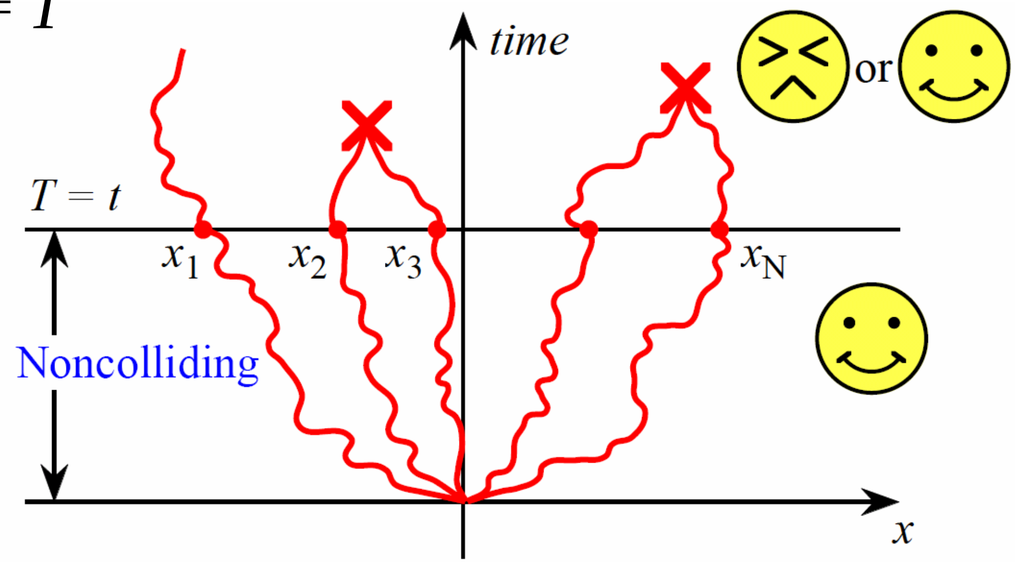
for $0 \le s < t \le T$, $\mathbf{x}, \mathbf{y} \in \mathbf{R}^N_<$, where $\mathbf{0} = (0,0,\dots 0)$,

$$c = 2^{-N/2} T^{N(N-1)/4} t^{-N^2/2} / \prod_{j=1}^{N} \Gamma(j/2), \quad |\mathbf{y}|^2 = \sum_{j=1}^{N} y_j^2 .$$

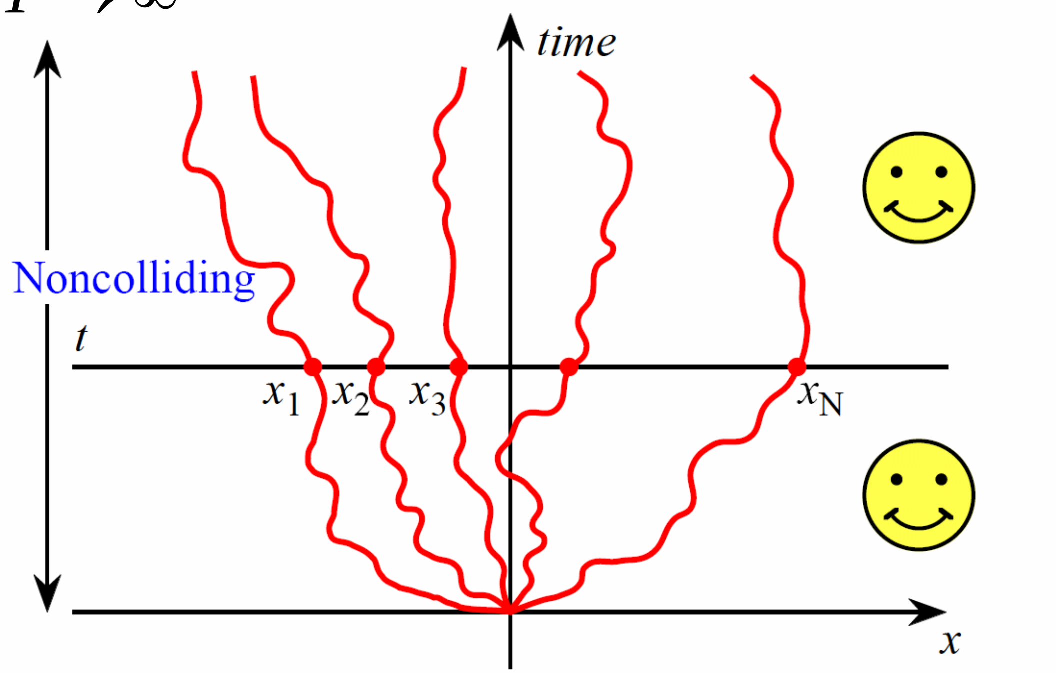
The temporally homogeneous diffusion process

$$\mathbf{Y}(t) = \lim_{T \to \infty} \mathbf{X}(t), \qquad t \in [0, \infty).$$

Case $t = T$



Case $T \to \infty$



Transition in Time of Particle Distribution

- **This observation implies that there occurs a transition.**

For a finite but large T

$$g_{N,T}(0,\mathbf{0};t,\mathbf{y}) \propto \exp\left\{-\frac{1}{2t}\sum_{i=1}^{N} y_i^2\right\} \prod_{1\le j<k\le N}(x_k - x_j)^2 \quad \text{for} \quad 0 < t << T$$

As the time t goes on from 0 to T

$$g_{N,T}(0,\mathbf{0};T,\mathbf{y}) \propto \exp\left\{-\frac{1}{2T}\sum_{i=1}^{N} y_i^2\right\} \prod_{1\le j<k\le N}(x_k - x_j) \quad \text{at} \quad t = T$$

Three Standard (Wigner-Dyson) Random Matrix Ensembles

[1] The distribution of Eigenvalues of $N \times N$ **Hermitian Matrices** in the **Gaussian Unitary Ensemble (GUE)** is given in the form

$$\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\lambda_i^2\right\}\prod_{1\le j<k\le N}(\lambda_k-\lambda_j)^2$$

[2] The distribution of Eigenvalues of $N \times N$ **Real Symmetric Matrices** in the **Gaussian Orthogonal Ensemble (GOE)** is given in the form

$$\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\lambda_i^2\right\}\prod_{1\le j<k\le N}(\lambda_k-\lambda_j)$$

[3] The distribution of Eigenvalues of $N \times N$ **Quternion Self-Dual Hermitian Matrices** in the **Gaussian Symplectic Ensemble (GSE)** is given in the form

$$\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\lambda_i^2\right\}\prod_{1\le j<k\le N}(\lambda_k-\lambda_j)^4$$

•Let $B_{ij}(t), \widetilde{B}_{ij}(t), 1 \le i, j \le N,$ be mutually **independent standard Brownian motions** started from the origin. Define

$$s_{ij}(t) = \begin{cases} \frac{1}{\sqrt{2}} B_{ij}(t) & (i < j) \\ B_{ii}(t) & (i = j) \\ \frac{1}{\sqrt{2}} B_{ij}(t) & (i > j) \end{cases} \qquad a_{ij}(t) = \begin{cases} \frac{1}{\sqrt{2}} \beta_{ij}(t) & (i < j) \\ 0 & (i = j) \\ -\frac{1}{\sqrt{2}} \beta_{ij}(t) & (i > j) \end{cases},$$

where $\beta_{ij}(t) = \widetilde{B}_{ij}(t) - \int_0^t \frac{\beta_{ij}(s)}{T-s} ds$, $\quad 1 \le i < j \le N,\ t \in [0,T].$ (Brownian bridges)

•Consider the $N \times N$ **Hermitian Matrix-Valued** stochastic process

$$\Xi_{N,T}(t) = \left(\xi_{ij}(t)\right)_{1 \le i,j \le N} = \left(s_{ij}(t) + \sqrt{-1}a_{ij}(t)\right)_{1 \le i,j \le N}$$

That is,

$$\Xi_{N,T}(t) = \begin{pmatrix} s_{11}(t) & s_{12}(t)+\sqrt{-1}a_{12}(t) & s_{13}(t)+\sqrt{-1}a_{13}(t) & \dots & s_{1N}(t)+\sqrt{-1}a_{1N}(t) \\ s_{12}(t)-\sqrt{-1}a_{12}(t) & s_{22}(t) & s_{23}(t)+\sqrt{-1}a_{23}(t) & \dots & s_{2N}(t)+\sqrt{-1}a_{2N}(t) \\ s_{13}(t)-\sqrt{-1}a_{13}(t) & s_{23}(t)-\sqrt{-1}a_{23}(t) & s_{33}(t) & \dots & s_{3N}(t)+\sqrt{-1}a_{3N}(t) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ s_{1N}(t)-\sqrt{-1}a_{1N}(t) & s_{2N}(t)-\sqrt{-1}a_{2N}(t) & s_{3N}(t)-\sqrt{-1}a_{3N}(t) & \dots & s_{NN}(t) \end{pmatrix}$$

- Since $\Xi(t)$ is a Hermitian matrix-valued process,

at each time *t* there is a **Unitary Matrix** $U(t)=(u_{ij}(t))_{1\le i,j\le N}$, such that

$$U(t)^{+}\Xi(t)U(t)=\Lambda(t)=\mathrm{diag}\{\lambda_1(t),\lambda_2(t),....,\lambda_N(t)\}$$

where the **eigenvalues** are in the **increasing order**

$$\lambda_1(t)\le\lambda_2(t)\le....\le\lambda_N(t),\qquad \forall t\in[0,\infty)$$

- We can regard $\boldsymbol{\lambda}(t)\equiv\left(\lambda_1(t),\lambda_2(t),....,\lambda_N(t)\right)\in\mathbf{R}^N$

as an ***N*-particle stochastic process in one dimension.**

We have proved the following theorem.

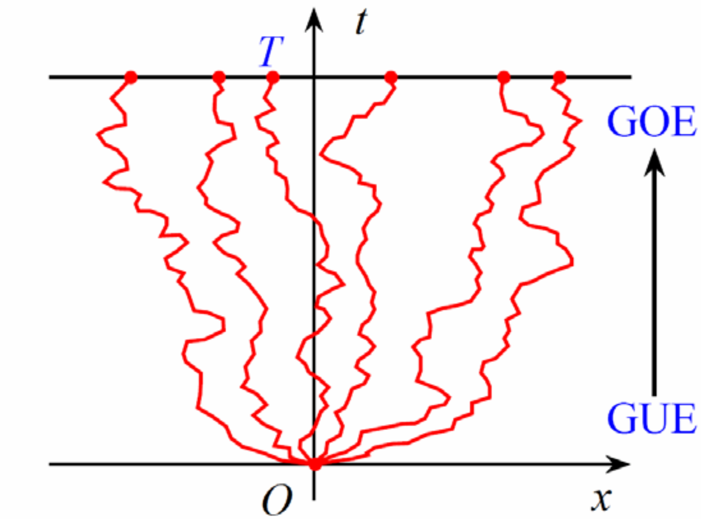
Theorem

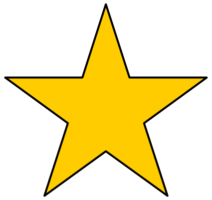
The eigenvalue process $\boldsymbol{\lambda}(t)\equiv\left(\lambda_1(t),\lambda_2(t),....,\lambda_N(t)\right)$ **is equivalent in distribution with the system of noncolliding Brownian motions X(*t*) with the initial condition X(0)=0 (i.e. all particles start from the origin), which are obtained as the scaling limit of vicious walker model.**

7. PATTERNS of NONCOLLIDING PATHS AND RANDOM MATRIX THEORIES

7.1 STAR CONFIGURATIONS

• There occurs a transition in distribution from GUE to GOE.





• This temporal transition can be decribed by the Two-Matrix Model of Pandey and Mehta, in which a Hermitian random matrix is coupled with a real symmetric random matrix.

See Katori and Tanemura, PRE **66** (2002) 011105/1-12.

• Techniques developed for multi-matrix models can be used to evaluate the dynamical correlation functions. Quaternion determinantal expressions are derived.

See Nagao, Katori and Tanemura, Phys. Lett. A307 (2003) 29-35.

• Using the exact correlation functions, we can discuss the scaling limits of infinite particles $N \to \infty$ and the infinite time-period $T \to \infty$.

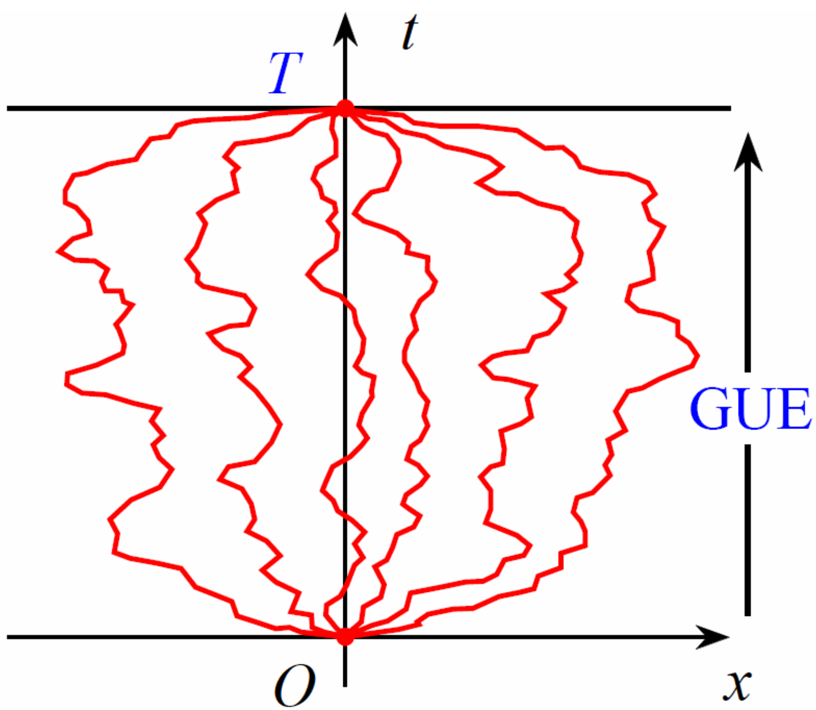
See Katori, Nagao and Tanemura, Adv.Stud.Pure Math. 39 (2004) 283-306.

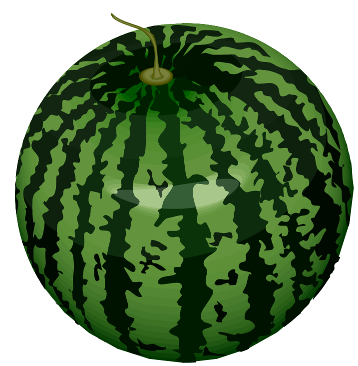
7.2 Watermelon Configurations

- Consider a finite time-period [0,T] and set
  y=**0** at the initial time t=0 and the final time t=T.
- The transition probability density is given as

$$q^{\text{watermelon}}(0,\mathbf{0};t,\mathbf{y}) = \frac{1}{C_1}\left\{t\left(1-\frac{t}{T}\right)\right\}^{-N^2/2} \exp\left\{-\frac{|y|^2}{2t(1-t/T)}\right\} h(\mathbf{y})^2$$

- The distribution is kept in the form of GUE.
- Only the variance changes as a function of t as $\sigma^2 = t\left(1-\frac{t}{T}\right)$.





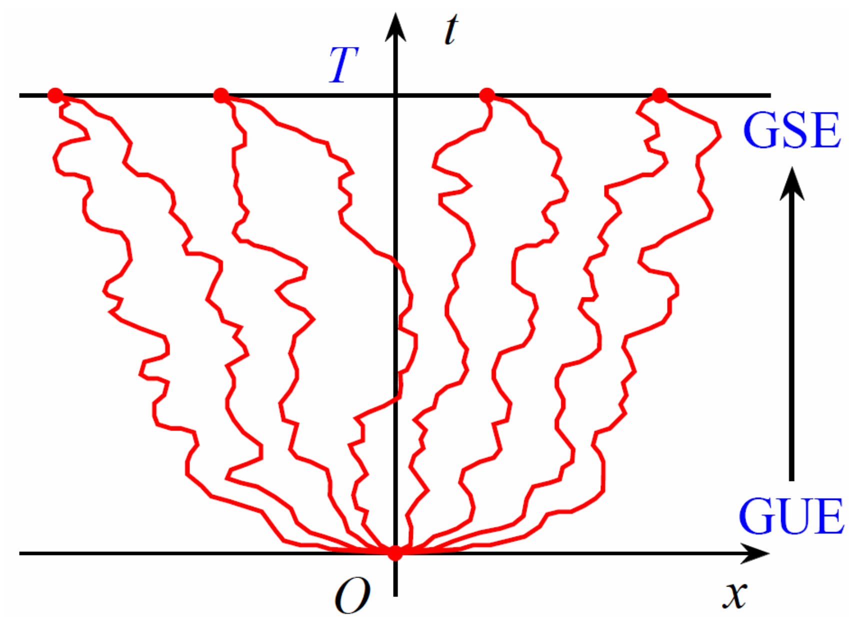
7.3 Banana Configurations

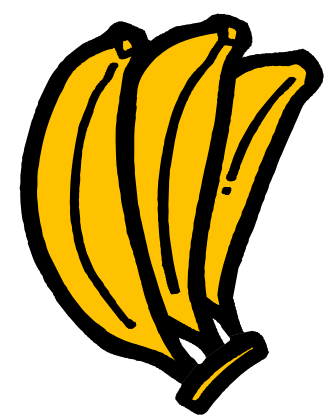
- Consider $2N$ particle system. Set $\mathbf{y}=\mathbf{0}$ at the initial time $t=0$.
  At the final time $t=T$, we assume the following Pairing of Particle Positions.
  $$y_1 = y_2 \,,\ y_3 = y_4 \,,\ \ldots.,\ y_{2N-1} = y_{2N} \quad \text{with} \quad y_1 < y_3 < \ldots. < y_{2N-1} \ .$$
- The transition probability density is given by

$$\mathrm{q}^{\text{banana}}(0,\mathbf{x};t,\mathbf{y}) = \frac{\mathrm{f}(t,\mathbf{y}\,|\,\mathbf{x})\,\mathrm{N}^{\text{banana}}(T-t,\mathbf{y})}{\mathrm{N}^{\text{banana}}(T,\mathbf{x})} \quad \text{for} \ \ 0 < s < t \le T, \ \ \mathbf{x},\mathbf{y} \in \mathbf{W}_{2N}^{A} \ ,$$

$$\text{where} \quad \mathrm{N}^{\text{banana}}(t,x) = \int_{\mathbf{W}_N^A} \det_{1\le i\le 2N,\,1\le j\le N} \left[\mathrm{G}(t,y_j\,|x_i) \quad \frac{x_i}{t}\mathrm{G}(t,y_j\,|x_i) \right] \ .$$

- As $t = 0 \rightarrow T$, there occurs a transition
  from the GUE distribution to the GSE distribution.

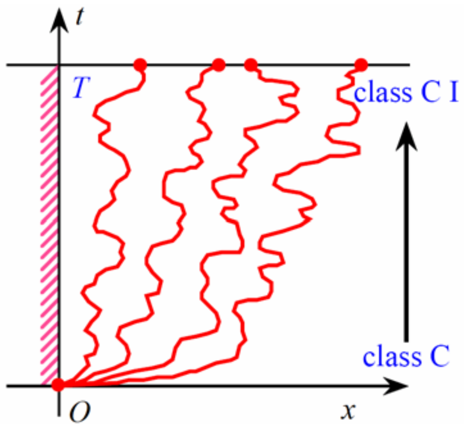




7.4 Star Configurations with Absorbing Wall

- Put an Absorbing Wall at the origin. Consider the N Brownian particles started from 0 conditioned never to collide with each other nor to collide with the wall.
- This is identified with the h-transform of the N-dim. Absorbing Brownian motion in

$$\mathbf{W}_N^C = \{x \in \mathbf{R}^N : 0 < x_1 < x_2 < .... < x_N\} \quad \text{(Weyl chamber of type } C_N\text{)}.$$



- For $T < \infty$, we can obtain a process showing a transition from the class C distribution of Altland and Zirnbauer (1996);

$$q^C(0,\mathbf{x};t,\mathbf{y}) \propto \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} \prod_{1\le i<j\le N}(y_j^2 - y_i^2)^2 \prod_{k=1}^{N} y_k^2 \quad \text{for } 0 < t << T$$

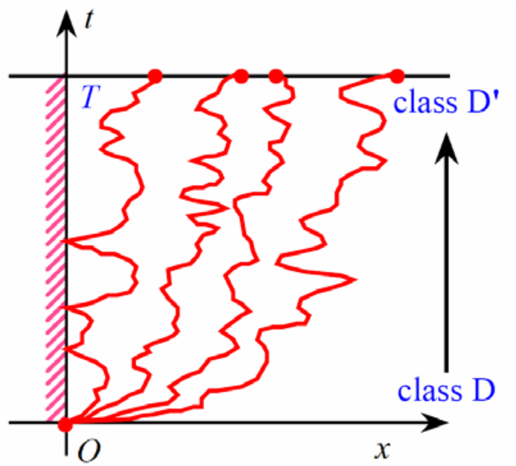
to the class CI distribution (studied for a theory of quantum dots)

$$q^C(0,\mathbf{x};T,\mathbf{y}) \propto \exp\left\{-\frac{|\mathbf{y}|^2}{2T}\right\} \prod_{1\le i<j\le N}(y_j^2 - y_i^2) \prod_{k=1}^{N} y_k \quad \text{at} \quad t = T.$$

7.5 Star Configurations with Reflection Wall

- Put a reflection wall at the origin. Consider the N Brownian particles started form 0 conditioned never to collide with each other.
- This is identified with the h-transform of the N-dim. Absorbing Brownian motion in

$$\mathbf{W}_N^D = \{x \in \mathbf{R}^N : |x_1| < x_2 < \dots < x_N\} \quad (\text{Weyl chamber of type } D_N).$$



For $T < \infty$, we can obtain a process showing a transition from the class D distribution of Altland and Zirnbauer (1996);

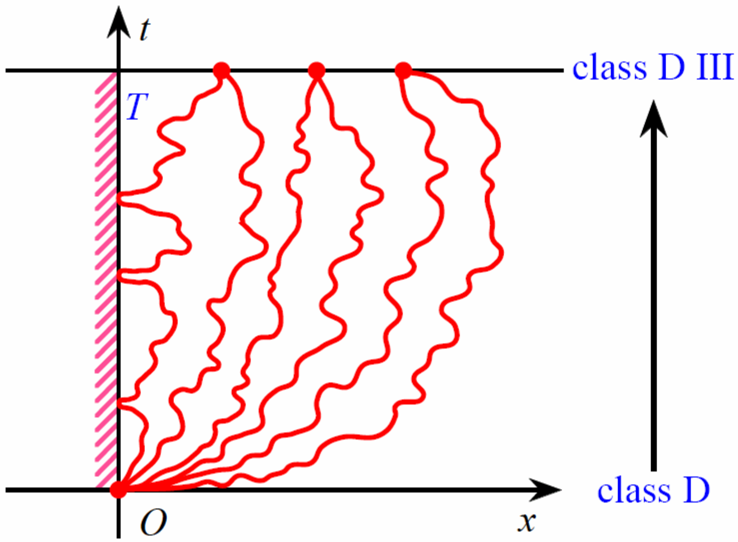
$$q^D(0,\mathbf{x};t,\mathbf{y}) \propto \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} \prod_{1\le i<j\le N} (y_j^2 - y_i^2)^2 \quad \text{for } 0 < t << T$$

to the ``real'' class D distribution

$$q^D(0,\mathbf{x};T,\mathbf{y}) \propto \exp\left\{-\frac{|\mathbf{y}|^2}{2T}\right\} \prod_{1\le i<j\le N} (y_j^2 - y_i^2) \quad \text{at} \quad t = T.$$

7.6 Banana Configurations with Reflection Wall

- Put a reflection wall at the origin.
- Consider the $2N$ Brownian particles started from 0 in Banana configurations.



- For $T < \infty$, we can obtain a process showing a transition from the class D distribution of Altland and Zirnbauer

$$q^{D,\text{ banana}}(0,\mathbf{x};t,\mathbf{y}) \propto \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} \prod_{1\le i<j\le N} (y_j^2 - y_i^2)^2 \quad \text{for } 0 < t << T$$

To the class DIII distribution.

$$q^{D,\text{ banana}}(0,\mathbf{x};T,\mathbf{y}) \propto \exp\left\{-\frac{|\mathbf{y}^{\text{odd}}|^2}{T}\right\} \prod_{1\le i<j\le N} (y_{2j-1}^2 - y_{2i-1}^2)^4 \prod_{k=1}^{N} y_{2k-1} \quad \text{at } t = T.$$

7.7 CONCLUDING REMARKS

- There are 10 CLASSES of Gaussian Random Matrix Theories.

Standard (Wigner-Dyson)

GUE — Star configurations
GOE — Star configurations
GUE — Banana configurations
GSE — Banana configurations

Nonstandard (chiral random matrices) Particle Physics of QCD

chGUE
chGOE
chGSE

Realized by Noncolliding Systems of 2D **Bessel processes** and **Generalized Meanders**

Nonstandard (Altland-Zirnbauer) Mesoscopic Physics with Superconductivity

class C — Star config. with Absorbing Wall
class CI — Star config. with Absorbing Wall
class D — Banana config. With Reflection Wall
class DIII — Banana config. With Reflection Wall

All of the 10 eigenvalue-distributions can be realized by the Noncolliding Diffusion Particle Systems (Vicious Walks).

See Katori and Tanemura, J.Math.Phys.(2004)

8. Remaks (again)

Vicious Walker Model ⟷ **Enumerative Combinatorics**
Young Tableaux
Symmetric Functions

Diffusion Scaling limit

Noncolliding Diffusion Processes

determinantal expressions
of correlation functions
$N \to \infty$ limit

Infinite Systems of
Noncolliding Diffusion Particles
Theory of Entire Functions

Representation Theory of
Lie Algebra/Lie Groups

Random Matrix Theory
Matrix Models in Statistical Phys.
and
String Theory, etc.....

I hope .....

Statistical Physics
Probability Theory
(Infinite) Particle Systems

Noncolliding Processes

Other Fields of
Mathematics

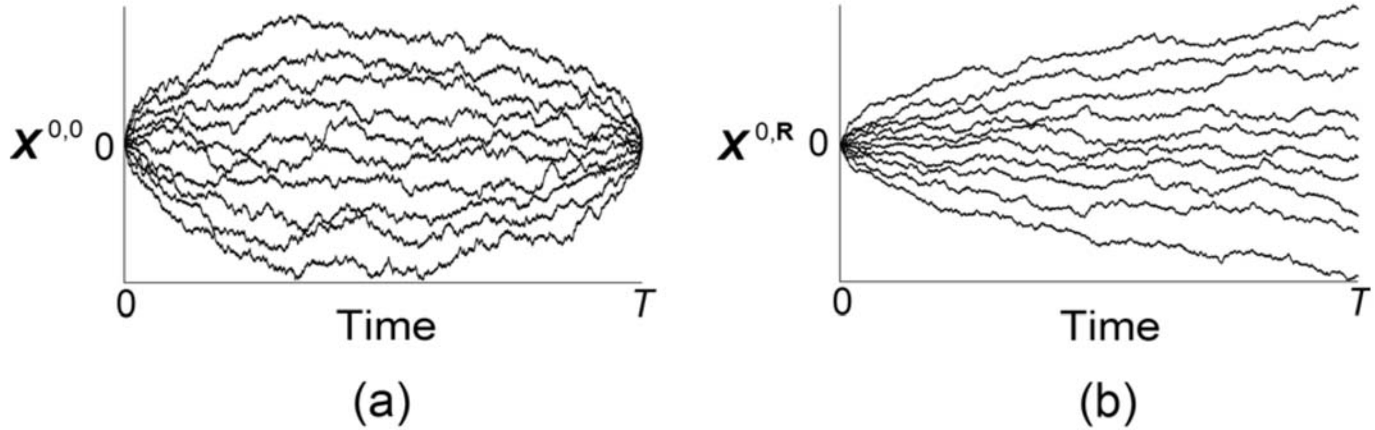


Figure 1. Samples of paths for (a) $\boldsymbol{X}^{\mathbf{0},0}(t)$ and (b) $\boldsymbol{X}^{\mathbf{0},\mathbb{R}}(t), t \in [0,T]$, generated by simulating the corresponding eigenvalue processes of random-matrix models.

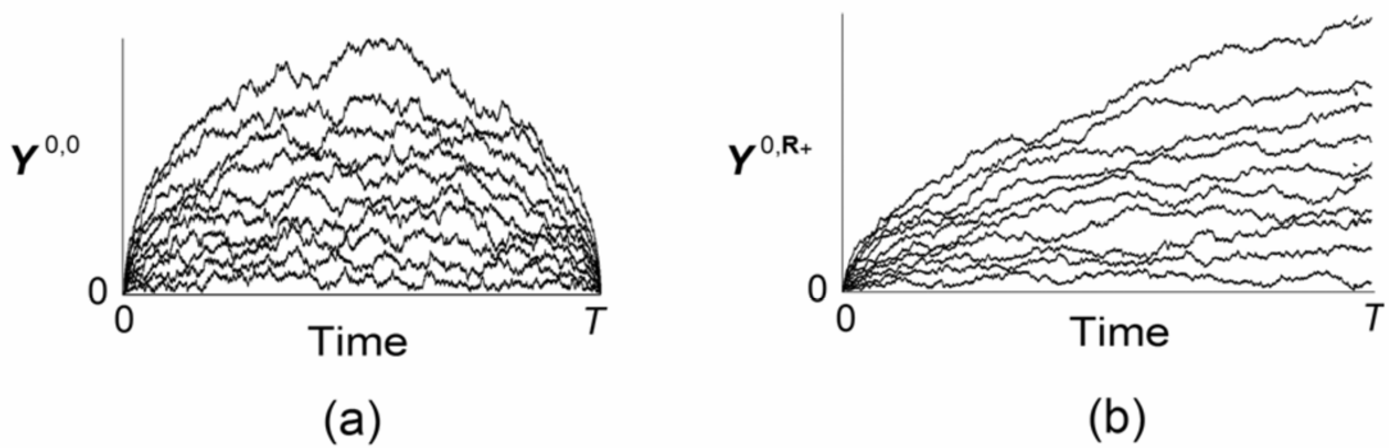


Figure 2. Samples of paths for (a) $\boldsymbol{Y}^{\mathbf{0},\mathbf{0}}(t)$ and (b) $\boldsymbol{Y}^{\mathbf{0},\mathbb{R}_+}(t), t \in [0,T]$.

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