

Symmetries and Structures of Matrix-Valued Stochastic Processes and Noncolliding Diffusion Processes

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Part 1

Conférence aux Houches

"Vicious walkers and random matrices"

du 16 au 27 mai 2011

1st day: Bessel process and the Dyson model

1.1 Vector-valued and matrix-valued Brownian motions and their “radial parts”

- $B(t) \in \mathbb{R}$: one-dimensional standard Brownian motion (BM) starting from 0.

$$B(0) = 0, \quad dB(t)^2 = dt, \quad t \geq 0.$$

- $D = 2, 3, 4, \dots$,
 D dimensional BM $\equiv D$ -component vector-valued BM

$$\mathbf{B}(t) = (B_1(t), B_2(t), \dots, B_D(t)) \in \mathbb{R}^D.$$

independent one-dim. standard BMs

$$dB_j(t)dB_k(t) = \delta_{jk}dt, \quad 1 \leq j, k \leq D.$$

- Consider the radial part of $B(t)$.

$$\begin{aligned} X(t) &\equiv |\mathbf{B}(t)| \\ &= \sqrt{B_1(t)^2 + B_2(t)^2 + \dots + B_D(t)^2}. \end{aligned}$$

That is, consider a function of D variables, x_1, x_2, \dots, x_D ,

$$F(x_1, x_2, \dots, x_D) = \sqrt{x_1^2 + x_2^2 + \dots + x_D^2},$$

then put random variables $B_j(t)$ to $x_j, 1 \leq j \leq D$,

$$X(t) = \sqrt{B_1(t)^2 + B_2(t)^2 + \dots + B_D(t)^2}.$$

- Taylor-Maclaurin expansion

$$dF(x_1, \dots, x_D) = \sum_{j=1}^D \frac{\partial F}{\partial x_j} dx_j + \frac{1}{2} \sum_{j=1}^D \sum_{k=1}^D \frac{\partial^2 F}{\partial x_j \partial x_k} dx_j dx_k + \dots$$

$$\frac{1}{2} \frac{2x_j}{\sqrt{x_1^2 + \dots + x_D^2}} = \frac{x_j}{F} \qquad \frac{\delta_{jk}}{F} - \frac{x_j x_k}{F^3}$$

$$dX(t) = \sum_{j=1}^D \frac{B_j(t)}{X(t)} dB_j(t) + \frac{1}{2} \sum_{j=1}^D \sum_{k=1}^D \left(\frac{\delta_{jk}}{X(t)} - \frac{B_j(t)B_k(t)}{X(t)^3} \right) dB_j(t) dB_k(t) + \dots$$

fluctuation
martingale term

second term of Taylor-MacLaurin expansion
but first order term of dt

$\delta_{jk} dt$

others are higher orders in dt
neglected

Itô's formula

$$\begin{aligned}
 \text{(2nd term)} &= \frac{1}{2} \sum_{j=1}^D \left\{ \frac{1}{X(t)} - \frac{B_j(t)^2}{X(t)^3} \right\} dt \\
 &= \frac{1}{2} \frac{1}{X(t)} \left\{ \sum_{j=1}^D 1 - \frac{1}{X(t)^2} \sum_{j=1}^D B_j(t)^2 \right\} dt \\
 &= \frac{D-1}{2} \frac{1}{X(t)} dt.
 \end{aligned}$$

$X(t)^2$

D

$$\begin{aligned}
 \text{(1st term)}^2 &= \sum_{j=1}^D \frac{B_j(t)}{X(t)} dB_j(t) \times \sum_{k=1}^D \frac{B_k(t)}{X(t)} dB_k(t) \\
 &= \frac{1}{X(t)^2} \sum_{j=1}^D B_j(t)^2 dt = dt
 \end{aligned}$$

$\delta_{jk} dt$

$X(t)^2$

$$\text{(1st term)} \stackrel{d}{=} dB(t)$$

$B(t)$ is another BM different from any $B_j(t), 1 \leq j \leq D$.

Stochastic Differential Equation (SDE) of $X(t)$

$$dX(t) = dB(t) + \frac{D-1}{2} \frac{1}{X(t)} dt$$

$X(t) = 0$ ← singular

Let $X(0) = x > 0$ (initial point)

$$X(t) = x + B(t) + \frac{D-1}{2} \int_0^t \frac{ds}{X(s)}$$

$B^x(t)$ BM starting from $x > 0$

D -dimensional Bessel process $\text{BES}^{(D)}$

- $N = 2, 3, 4, \dots$ Consider $N \times N$ hermitian matrix

$$M = \left(M_{jk} \right)_{1 \leq j, k \leq N} = \left(s_{jk} + ia_{jk} \right)_{1 \leq j, k \leq N},$$

$$i = \sqrt{-1}$$

$$s_{jk} = s_{kj} \quad \text{symmetric} \in \mathbb{R}$$

$$a_{jk} = -a_{kj} \quad (a_{jj} = 0) \quad \text{anti-symmetric} \in \mathbb{R}.$$

$$\begin{aligned} \text{tr}(M^\dagger M) &\equiv \sum_{j=1}^N \left(M^\dagger M \right)_{jj} \\ &= \sum_{j=1}^N s_{jj}^2 + 2 \sum_{1 \leq j < k \leq N} (s_{jk}^2 + a_{jk}^2) \end{aligned}$$

- Here prepare the following independent BMs

$$\begin{aligned} B_{jj}(t) & \quad 1 \leq j \leq N \\ B_{jk}(t) & \quad 1 \leq j < k \leq N \\ \tilde{B}_{jk}(t) & \quad 1 \leq j < k \leq N \end{aligned}$$

and set

$$s_{jk} = \begin{cases} B_{jj}(t) & 1 \leq j = k \leq N \\ \frac{1}{\sqrt{2}} B_{jk}(t) & 1 \leq j < k \leq N \end{cases}$$

$$a_{jk} = \begin{cases} 0 & 1 \leq j = k \leq N \\ \frac{1}{\sqrt{2}} \tilde{B}_{jk}(t) & 1 \leq j < k \leq N \end{cases}$$

Then we have defined

$\hat{\mathbf{B}}(t) : N \times N$ hermitian-matrix-valued BM, s.t.

$$\text{tr} \left(\hat{\mathbf{B}}(t)^\dagger \hat{\mathbf{B}}(t) \right) = \sum_{j=1}^N B_{jj}(t)^2 + \sum_{1 \leq j < k \leq N} \left(B_{jk}(t)^2 + \tilde{B}_{jk}(t)^2 \right).$$

$$\begin{aligned} & N \\ & N(N-1)/2 \\ & N(N-1)/2 \\ +) & \hline & N^2 \end{aligned}$$

- $\mathcal{H}(N) =$ the space of $N \times N$ hermitian matrices

$$\mathcal{H}(N) \simeq \mathbb{R}^{N^2}$$

$\widehat{\mathbf{B}}(t) \simeq \text{BM in } \mathbb{R}^{N^2}$ (N^2 -dim. Euclidean space)

“radial part”
eigenvalue process

radial part = $\text{tr}(\widehat{\mathbf{B}}(t)^\dagger \widehat{\mathbf{B}}(t)) = N^2$ -dim. Bessel process

- $\forall t \geq 0, \exists! U(t)$ $N \times N$ unitary matrix, s.t.

$$U(t)^\dagger \widehat{\mathbf{B}}(t) U(t) = \Lambda(t) = \begin{pmatrix} \lambda_1(t) & 0 & \cdots & 0 \\ 0 & \lambda_2(t) & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & \lambda_N(t) \end{pmatrix},$$

$$\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_N(t).$$

- Note that

$$\lambda_j(t) = \text{functionals of } \{B_{jj}(t), B_{jk}(t), \widetilde{B}_{jk}(t)\}.$$

$$d\lambda_j(t) = ???$$

Itô's formula \longrightarrow

generalized Bru's theorem

Results (Dyson 1962)

$$d\lambda_j(t) = dB_j(t) + \sum_{1 \leq k \leq N: k \neq j} \frac{dt}{\lambda_j(t) - \lambda_k(t)}, \quad 1 \leq j \leq N, \quad t \in [0, \infty).$$

- **Remark.**

Dyson's BM model = one-parameter family with parameter $\beta > 0$

$$d\lambda_j(t) = dB_j(t) + \frac{\beta}{2} \sum_{1 \leq k \leq N: k \neq j} \frac{dt}{\lambda_j(t) - \lambda_k(t)}, \quad 1 \leq j \leq N, \quad t \in [0, \infty).$$

$\beta = 2$ GUE

$\beta = 1$ GOE

$\beta = 4$ GSE

Here we simply call the $\beta = 2$ case the Dyson model.

- **Remark.**

Compare with $BES^{(D)}$

$$dX(t) = dB(t) + \frac{D-1}{2} \frac{dt}{X(t)}$$

$$D-1 \iff \beta$$

$D = 3 \iff \beta = 2$

$D = 2 \iff \beta = 1 \quad ???$

$D = 5 \iff \beta = 4 \quad ???$

1.2 Modified Bessel function for BES and Karlin-McGregor-LGV determinant for the Dyson model

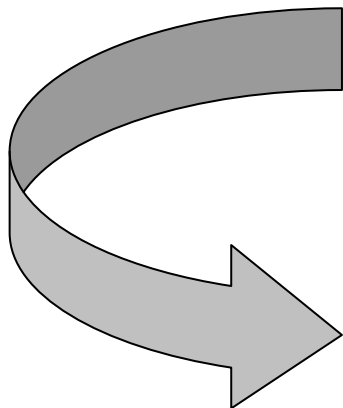


SDE for diffusion process

Kolmogorov equation for transition probability density (t.p.d.)

- BES^(D)

$$dX(t) = dB(t) + \frac{D-1}{2} \frac{1}{X(t)} dt$$



$q^{(D)}(t, y|x) \equiv$ t.p.d. of BES^(D) from $x \geq 0$ to $y \geq 0$
during time period $t \geq 0$

$$\frac{\partial}{\partial t} q^{(D)}(t, y|x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} q^{(D)}(t, y|x) + \frac{D-1}{2} \frac{1}{x} \frac{\partial}{\partial x} q^{(D)}(t, y|x)$$

diffusion term

drift term

partial differential equation (PDE) as a function of t and x

Kolmogorov backward equation

starting point

- **Problem 1**

Solve the PDE

$$\frac{\partial}{\partial t} f(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} f(t, x) + \frac{D-1}{2} \frac{1}{x} \frac{\partial}{\partial x} f(t, x), \quad t \geq 0, \quad x \geq 0,$$

under the initial condition

$$f(0, x) = \delta(x - y), \quad y \geq 0.$$

- An easy answer may be ...

Assume the form

$$f(t, x) = \frac{y^{\nu+1}}{x^\nu} \frac{1}{t} \exp \left\{ -\frac{1}{2t} (x^2 + y^2) \right\} g(z)$$

with

$$\nu = \frac{D-2}{2}, \quad z = \frac{xy}{t}.$$

Then the above equation becomes

$$g''(z) + \frac{1}{z} g'(z) - \left(1 + \frac{\nu^2}{z^2} \right) g(z) = 0$$

modified Bessel diff.eq.

The modified Bessel function with index ν

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + \nu + 1)} \left(\frac{z}{2} \right)^{2n + \nu}$$

solves this equation.

$$\Gamma(z) = \int_0^{\infty} e^{-u} u^{z-1} du, \quad \Re z > 0 \quad (\text{Gamma function})$$

$$q^{(D)}(t, y|x) = \frac{y^{\nu+1}}{x^\nu} \frac{1}{t} e^{-(x^2 + y^2)/2t} I_\nu \left(\frac{xy}{t} \right), \quad x, y \geq 0, \quad t \geq 0.$$

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z = \frac{e^z - e^{-z}}{\sqrt{2\pi z}}$$

- **Remark.**

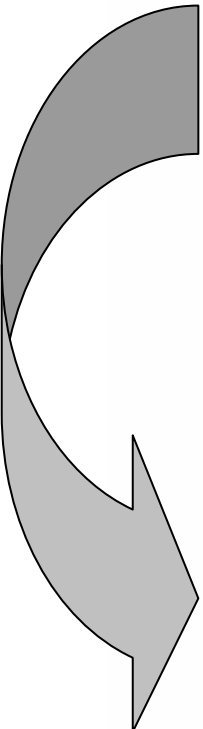
In particular, if $D = 3$,

$$q^{(3)}(t, y|x) = \frac{y}{x} \left\{ p(t, y|x) - p(t, -y|x) \right\},$$

where

$$\begin{aligned} p(t, y|x) &= \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t} && \text{heat kernel} \\ &= \text{t.p.d. of one-dim. standard BM} \end{aligned}$$

- the Dyson model



SDE
$$d\lambda_j(t) = dB_j(t) + \sum_{1 \leq k \leq N: k \neq j} \frac{dt}{\lambda_j(t) - \lambda_k(t)}, \quad 1 \leq j \leq N, \quad t \in [0, \infty).$$

$p_N(t, \mathbf{y}|\mathbf{x})$ = t.p.d. of the Dyson model
 from $\mathbf{x} = (x_1, x_2, \dots, x_N), x_1 \leq x_2 \leq \dots \leq x_N$
 to $\mathbf{y} = (y_1, y_2, \dots, y_N), y_1 \leq y_2 \leq \dots \leq y_N$
 during time period $t \geq 0$

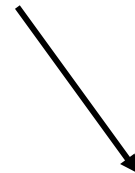
$$\frac{\partial}{\partial t} p_N(t, \mathbf{y}|\mathbf{x}) = \frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} p_N(t, \mathbf{y}|\mathbf{x}) + \sum_{1 \leq j \leq N} \sum_{1 \leq k \leq N: k \neq j} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j} p_N(t, \mathbf{y}|\mathbf{x})$$

- **Problem 2**

Solve the above $(N + 1)$ -variate PDE under the initial condition

$$p_N(0, \mathbf{y}|\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y}) \equiv \prod_{j=1}^N \delta(x_j - y_j).$$

Karlin-McGregor-LGV determinant

**Solution**

$$p_N(t, \mathbf{y} | \mathbf{x}) = \frac{h_N(\mathbf{y})}{h_N(\mathbf{x})} \det_{1 \leq j, k \leq N} [p(t, y_j | x_k)]$$

with

$$h_N(\mathbf{x}) = \prod_{1 \leq j < k \leq N} (x_k - x_j) = \det_{1 \leq j, k \leq N} [x_j^{k-1}].$$

product of differences = Vandermonde determinant

- Correspondence

BES⁽³⁾ \iff the Dyson model ($\beta = 3$ Dyson's BM model)

$$q^{(3)}(t, y|x) = \frac{y}{x} \left\{ p(t, y|x) - p(t, -y|x) \right\}$$

$$p_N(t, \mathbf{y}|\mathbf{x}) = \frac{h_N(\mathbf{y})}{h_N(\mathbf{x})} \sum_{\sigma \in S_N} \text{sgn}(\sigma) \prod_{j=1}^N p(t, y_{\sigma(j)}|x_j)$$

- $h_1(x) \equiv x$
harmonic

$$\begin{aligned} \frac{d^2}{dx^2} h_1(t) &= 0 \\ h(0) &= 0 \end{aligned}$$

The origin 0 is the boundary of the semi-infinite interval $(0, \infty)$.

- $h_N(x) = \prod_{1 \leq j < k \leq N} (x_k - x_j)$

harmonic

$$\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} h_N(\mathbf{x}) = 0$$

$$h_N(\mathbf{x}) = 0 \quad \text{if } x_j = x_k, \quad j \neq k$$

boundary of the Weyl chamber of type A_{N-1}

$$\mathbb{W}_N^A = \left\{ \mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \dots < x_N \right\}.$$