# Symmetries and Structures of Matrix-Valued Stochastic Processes and Noncolliding Diffusion Processes

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23-27 May 2011

## Part 1

Conférence aux Houches

"Vicious walkers and random matrices"

du 16 au 27 mai 2011

# 1st day: Bessel process and the Dyson model

## 1.1 Vector-valued and matrix-valued Brownian motions and their "radial parts"

- $B(t) \in \mathbb{R}$ : one-dimensional standard Brownian motion (BM) starting from 0.

$$B(0) = 0, \quad dB(t)^2 = dt, \quad t \geq 0.$$

- $D = 2, 3, 4, \dots,$
  $D$ dimensional BM $\equiv$ $D$-component vector-valued BM

$$\boldsymbol{B}(t) = (B_1(t), B_2(t), \dots, B_D(t)) \in \mathbb{R}^D.$$

  independent one-dim. standard BMs

$$dB_j(t)dB_k(t) = \delta_{jk}dt, \quad 1 \leq j, k \leq D.$$

- Consider the radial part of $\boldsymbol{B}(t)$.

$$\begin{aligned} X(t) &\equiv |\boldsymbol{B}(t)| \\ &= \sqrt{B_1(t)^2 + B_2(t)^2 + \cdots + B_D(t)^2}. \end{aligned}$$

  That is, consider a function of $D$ variables, $x_1, x_2, \dots, x_D$,

$$F(x_1, x_2, \dots, x_D) = \sqrt{x_1^2 + x_2^2 + \cdots + x_D^2},$$

  then put random variables $B_j(t)$ to $x_j, 1 \leq j \leq D$,

$$X(t) = \sqrt{B_1(t)^2 + B_2(t)^2 + \cdots + B_D(t)^2}.$$

- Taylor-Maclaurin expansion

$$dF(x_1, \ldots, x_D) = \sum_{j=1}^{D} \frac{\partial F}{\partial x_j} dx_j + \frac{1}{2} \sum_{j=1}^{D} \sum_{k=1}^{D} \frac{\partial^2 F}{\partial x_j \partial x_k} dx_j dx_k + \cdots$$

$$\frac{1}{2} \frac{2x_j}{\sqrt{x_1^2 + \cdots + x_D^2}} = \frac{x_j}{F} \qquad\qquad \frac{\delta_{jk}}{F} - \frac{x_j x_k}{F^3}$$

$$dX(t) = \sum_{j=1}^{D} \frac{B_j(t)}{X(t)} dB_j(t) + \frac{1}{2} \sum_{j=1}^{D} \sum_{k=1}^{D} \left( \frac{\delta_{jk}}{X(t)} - \frac{B_j(t) B_k(t)}{X(t)^3} \right) dB_j(t) dB_k(t) + \cdots$$

$dB_j(t)dB_k(t) \to \delta_{jk} dt$

fluctuation martingale term

second term of Taylor-MacLaurin expansion but first order term of $dt$

others are higher orders in $dt$ neglected

**Itô's formula**

$$\text{(2nd term)} = \frac{1}{2}\sum_{j=1}^{D}\left\{\frac{1}{X(t)} - \frac{B_j(t)^2}{X(t)^3}\right\}dt$$

$$= \frac{1}{2}\frac{1}{X(t)}\left\{\sum_{j=1}^{D} 1 - \frac{1}{X(t)^2}\sum_{j=1}^{D} B_j(t)^2\right\}dt$$

$$= \frac{D-1}{2}\frac{1}{X(t)}dt.$$

$X(t)^2$

$D$

$\delta_{jk}dt$

$$\text{(1st term)}^2 = \sum_{j=1}^{D}\frac{B_j(t)}{X(t)}dB_j(t) \times \sum_{k=1}^{D}\frac{B_k(t)}{X(t)}dB_k(t)$$

$$= \frac{1}{X(t)^2}\sum_{j=1}^{D} B_j(t)^2 dt = dt$$

$X(t)^2$

$$\text{(1st term)} \overset{\mathrm{d}}{=} dB(t)$$

$B(t)$ is another BM different from any $B_j(t), 1 \le j \le D$.

Stochastic Differential Equation (SDE) of $X(t)$

$$dX(t) = dB(t) + \frac{D-1}{2} \frac{1}{X(t)} dt$$

$X(t) = 0$ singular

Let $X(0) = x > 0$ (initial point)

$$X(t) = x + B(t) + \frac{D-1}{2} \int_0^t \frac{ds}{X(s)}$$

$B^x(t)$ BM starting from $x > 0$

| |
|---|
| $D$-dimensional Bessel process $\mathrm{BES}^{(D)}$ |

- $N = 2, 3, 4, \dots$ Consider $N \times N$ hermitian matrix

$$M = \Big(M_{jk}\Big)_{1 \le j,k \le N} = \Big(s_{jk} + i a_{jk}\Big)_{1 \le j,k \le N},$$

$$i = \sqrt{-1}$$

$$\begin{array}{lll} s_{jk} = s_{kj} & \text{symmetric} \in \mathbb{R} & \\ a_{jk} = -a_{kj} & (a_{jj} = 0) & \text{anti-symmetric} \in \mathbb{R}. \end{array}$$

$$\begin{aligned} \mathrm{tr}(M^\dagger M) &\equiv \sum_{j=1}^{N} \Big(M^\dagger M\Big)_{jj} \\ &= \sum_{j=1}^{N} s_{jj}^2 + 2 \sum_{1 \le j < k \le N} (s_{jk}^2 + a_{jk}^2) \end{aligned}$$

- Here prepare the following independent BMs

$$\begin{array}{ll} B_{jj}(t) & 1 \le j \le N \\ B_{jk}(t) & 1 \le j < k \le N \\ \widetilde{B}_{jk}(t) & 1 \le j < k \le N \end{array}$$

and set

$$s_{jk} = \begin{cases} B_{jj}(t) & 1 \le j = k \le N \\ \dfrac{1}{\sqrt{2}} B_{jk}(t) & 1 \le j < k \le N \end{cases}$$

$$a_{jk} = \begin{cases} 0 & 1 \le j = k \le N \\ \dfrac{1}{\sqrt{2}} \widetilde{B}_{jk}(t) & 1 \le j < k \le N \end{cases}$$

$$\begin{array}{rr} & N \\ & N(N-1)/2 \\ +) & N(N-1)/2 \\ \hline & N^2 \end{array}$$

Then we have defined

$$\widehat{\mathsf{B}}(t) : N \times N \text{ hermitian-matrix-valued BM, s.t.}$$

$$\mathrm{tr}\Big(\widehat{\mathsf{B}}(t)^\dagger \widehat{\mathsf{B}}(t)\Big) = \sum_{j=1}^{N} B_{jj}(t)^2 + \sum_{1 \le j < k \le N} \Big(B_{jk}(t)^2 + \widetilde{B}_{jk}(t)^2\Big).$$

- $\mathcal{H}(N) =$ the space of $N \times N$ hermitian matrices

$$\mathcal{H}(N) \simeq \mathbb{R}^{N^2}$$

$$\widehat{\mathsf{B}}(t) \simeq \text{BM in } \mathbb{R}^{N^2} \ (N^2\text{-dim. Euclidean space})$$

"radial part" eigenvalue process

radial part $= \mathrm{tr}\left(\widehat{\mathsf{B}}(t)^{\dagger}\widehat{\mathsf{B}}(t)\right) = N^2$-dim. Bessel process

- $\forall t \geq 0, \quad {}^{\exists!}U(t) \quad N \times N$ unitary matrix, s.t.

$$U(t)^{\dagger}\widehat{\mathsf{B}}(t)U(t) = \Lambda(t) = \begin{pmatrix} \lambda_1(t) & 0 & \cdots & 0 \\ 0 & \lambda_2(t) & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & \lambda_N(t) \end{pmatrix},$$

$\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_N(t)$.

- Note that

$$\lambda_j(t) = \text{functionals of } \left\{B_{jj}(t), B_{jk}(t), \widetilde{B}_{jk}(t)\right\}.$$

$$d\lambda_j(t) = ???$$

Itô's formula $\longrightarrow$ **generalized Bru's theorem**

Results (Dyson 1962)

$$d\lambda_j(t) = dB_j(t) + \sum_{1 \le k \le N: k \ne j} \frac{dt}{\lambda_j(t) - \lambda_k(t)}, \qquad 1 \le j \le N, \quad t \in [0, \infty).$$

- **Remark.**
  Dyson's BM model = one-parameter family with parameter $\beta > 0$

$$d\lambda_j(t) = dB_j(t) + \frac{\beta}{2} \sum_{1 \le k \le N: k \ne j} \frac{dt}{\lambda_j(t) - \lambda_k(t)}, \qquad 1 \le j \le N, \quad t \in [0, \infty).$$

$\beta = 2$ GUE

$\beta = 1$ GOE

$\beta = 4$ GSE

Here we simply call the $\beta = 2$ case the Dyson model.

- **Remark.**
  Compare with $\text{BES}^{(D)}$

$$dX(t) = dB(t) + \frac{D-1}{2} \frac{dt}{X(t)}$$

$$D - 1 \iff \beta$$

$D = 3 \iff \beta = 2$

$D = 2 \iff \beta = 1$ ???

$D = 5 \iff \beta = 4$ ???

# 1.2 Modified Bessel function for BES and Karlin-McGregor-LGV determinant for the Dyson model

SDE for diffusion process
Kolmogorov equation for transition probability density (t.p.d.)

- $\text{BES}^{(D)}$

$$dX(t) = dB(t) + \frac{D-1}{2}\frac{1}{X(t)}dt$$

$$q^{(D)}(t, y|x) \equiv \text{t.p.d. of BES}^{(D)} \text{ from } x \geq 0 \text{ to } y \geq 0 \text{ during time period } t \geq 0$$

$$\frac{\partial}{\partial t}q^{(D)}(t,y|x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}q^{(D)}(t,y|x) + \frac{D-1}{2}\frac{1}{x}\frac{\partial}{\partial x}q^{(D)}(t,y|x)$$

diffusion term　　drift term

partial differential equation (PDE) as a function of $t$ and $x$

starting point

**Kolmogorov backward equation**

- **Problem 1**
  Solve the PDE

$$\frac{\partial}{\partial t} f(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t,x) + \frac{D-1}{2}\frac{1}{x}\frac{\partial}{\partial x} f(t,x), \quad t \geq 0, \quad x \geq 0,$$

under the initial condition

$$f(0,x) = \delta(x-y), \quad y \geq 0.$$

- An easy answer may be ...
  Assume the form

$$f(t,x) = \frac{y^{\nu+1}}{x^\nu}\frac{1}{t}\exp\left\{-\frac{1}{2t}(x^2+y^2)\right\} g(z)$$

with

$$\nu = \frac{D-2}{2}, \qquad z = \frac{xy}{t}.$$

Then the above equation becomes

$$g''(z) + \frac{1}{z}g'(z) - \left(1+\frac{\nu^2}{z^2}\right) g(z) = 0$$

modified Bessel diff.eq.

The modified Bessel function with index $\nu$

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)}\left(\frac{z}{2}\right)^{2n+\nu}$$

solves this equation.

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du, \quad \Re z > 0 \quad \text{(Gamma function)}$$

$$q^{(D)}(t,y|x) = \frac{y^{\nu+1}}{x^\nu}\frac{1}{t} e^{-(x^2+y^2)/2t} I_\nu\left(\frac{xy}{t}\right), \qquad x,y \geq 0, \quad t \geq 0.$$

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z = \frac{e^z - e^{-z}}{\sqrt{2\pi z}}$$

- **Remark.**
  In particular, if $D = 3$,

$$q^{(3)}(t, y|x) = \frac{y}{x} \Big\{ p(t, y|x) - p(t, -y|x) \Big\},$$

where

$$\begin{aligned} p(t, y|x) &= \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t} \quad \text{heat kernel} \\ &= \text{t.p.d. of one-dim. standard BM} \end{aligned}$$

- the Dyson model

SDE $\quad d\lambda_j(t) = dB_j(t) + \sum_{1 \le k \le N : k \ne j} \frac{dt}{\lambda_j(t) - \lambda_k(t)}, \qquad 1 \le j \le N, \quad t \in [0, \infty).$

$p_N(t, \boldsymbol{y}|\boldsymbol{x}) \;=\;$ t.p.d. of the Dyson model
from $\boldsymbol{x} = (x_1, x_2, \dots, x_N), x_1 \le x_2 \le \cdots \le x_N$
to $\boldsymbol{y} = (y_1, y_2, \dots, y_N), y_1 \le y_2 \le \cdots \le y_N$
during time period $t \ge 0$

$$\frac{\partial}{\partial t} p_N(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} p_N(t, \boldsymbol{y}|\boldsymbol{x}) + \sum_{1 \le j \le N} \sum_{1 \le k \le N : k \ne j} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j} p_N(t, \boldsymbol{y}|\boldsymbol{x})$$

- **Problem 2**
Solve the above $(N+1)$-variate PDE under the initial condition

$$p_N(0, \boldsymbol{y}|\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{y}) \equiv \prod_{j=1}^{N} \delta(x_j - y_j).$$

Karlin-McGregor-LGV determinant

**Solution**

$$p_N(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})} \det_{1 \le j,k \le N} \left[ p(t, y_j | x_k) \right]$$

with

$$h_N(\boldsymbol{x}) = \prod_{1 \le j < k \le N} (x_k - x_j) = \det_{1 \le j,k \le N} \left[ x_j^{k-1} \right].$$

product of differences = Vandermonde determinant

- Correspondence

$$\mathrm{BES}^{(3)} \quad \Longleftrightarrow \quad \text{the Dyson model } (\beta = 3 \text{ Dyson's BM model})$$

$$q^{(3)}(t, y|x) = \frac{y}{x}\Big\{p(t, y|x) - p(t, -y|x)\Big\}$$

$$p_N(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})} \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} p(t, y_{\sigma(j)}|x_j)$$

- $h_1(x) \equiv x$

  harmonic

$$\frac{d^2}{dx^2} h_1(t) = 0$$
$$h(0) = 0$$

  The origin 0 is the boundary of the semi-infinite interval $(0, \infty)$.

- $h_N(x) = \prod_{1 \le j < k \le N} (x_k - x_j)$

  harmonic

$$\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} h_N(\boldsymbol{x}) = 0$$

$$h_N(\boldsymbol{x}) = 0 \qquad \text{if} \quad x_j = x_k, \quad j \neq k$$

  boundary of the Weyl chamber of type $\mathrm{A}_{N-1}$

$$\mathbb{W}_N^{\mathrm{A}} = \Big\{\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\Big\}.$$