Symmetries and Structures of Matrix-Valued Stochastic Processes and Noncolliding Diffusion Processes

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Part 2

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2^{nd} day: Generalized Bru's Theorem, BM on Lie Algebras and their Eigenvalue Processes

2.1 Generalized Bru's Theorem

- $\xi_{jk}(t), 1 \le j, k \le N$: complex-valued continuous semi-martingales (martingale part + bounded variation part), s.t.

$$\xi_{kj}(t)^* = \xi_{jk}(t), \qquad 1 \le j \le k \le N.$$

$$\Xi(t) = \Big(\xi_{jk}(t)\Big)_{1 \le j,k \le N} \qquad N \times N \text{ hermitian-matrix-valued diffusion process}$$

- For each $t \ge 0$, let $\boldsymbol{\lambda}(t) = (\lambda_1(t), \lambda_2(t), \dots, \lambda_N(t))$ be the vector, s.t. $\{\lambda_j(t)\}$ are eigenvalues of $\Xi(t)$ satisfying

$$\lambda_1(t) \le \lambda_2(t) \le \cdots \le \lambda_N(t).$$

${}^{\exists!}U(t), t \ge 0$, s.t.

$$U(t)^\dagger \Xi(t) U(t) = \Lambda(t) = \begin{pmatrix} \lambda_1(t) & 0 & \cdots & 0 \\ 0 & \lambda_2(t) & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & \lambda_N(t) \end{pmatrix},$$

$U(t)$: unitary-matrix-valued process.

- $d\Xi(t) = \Big(d\xi_{jk}(t)\Big)_{1 \leq j,k \leq N}$

$$
\begin{aligned}
\Gamma_{jk,\ell m}(t)dt &\equiv \Big(U(t)^\dagger d\Xi(t) U(t)\Big)_{jk} \Big(U(t)^\dagger d\Xi(t) U(t)\Big)_{\ell m} \\
d\Upsilon_j(t) &\equiv \text{the bounded variation part of } \Big(U(t)^\dagger d\Xi(t) U(t)\Big)_{jj}.
\end{aligned}
$$

Theorem 2.1 [generalized Bru's theorem]
The eigenvalue process satisfies the following SDEs;

$$
d\lambda_j(t) = dM_j(t) + dJ_j(t), \quad 1 \leq j \leq N, \quad t \geq 0,
$$

where

$\boldsymbol{M}(t) = (M_1(t), M_2(t), \dots, M_N(t))$

martingales with

$$
dM_j(t)dM_k(t) = \{(U(t)^\dagger d\Xi(t)U(t))_{jj} - d\Upsilon_j(t)\}\{(U(t)^\dagger d\Xi(t)U(t))_{kk} - d\Upsilon_k(t)\}
$$

$\boldsymbol{J}(t) = (J_1(t), J_2(t), \dots, J_N(t))$

processes with bounded variation given by

$$
dJ_j(t) = \sum_{1 \leq k \leq N: k \neq j} \frac{1}{\lambda_j(t) - \lambda_k(t)} \mathbf{1}(\lambda_j(t) \neq \lambda_k(t)) \Gamma_{jk,kj}(t)dt + d\Upsilon_j(t), \quad 1 \leq j \leq N.
$$

2.2 Two simple examples

(A) GUE(Gaussian unitary ensemble)-process

indep. one-dim. standard BMs

$$
\begin{aligned}
\xi_{jk}(t) &= s_{jk}(t) + i a_{jk}(t), \quad i = \sqrt{-1} \\
s_{jk}(t) &= \begin{cases} \dfrac{1}{\sqrt{2}} B_{jk}(t) & j < k \\ B_{jj}(t) & j = k \\ \dfrac{1}{\sqrt{2}} B_{kj}(t) & j > k \end{cases} \\
a_{jk}(t) &= \begin{cases} \dfrac{1}{\sqrt{2}} \widetilde{B}_{jk}(t) & j < k \\ 0 & j = k \\ -\dfrac{1}{\sqrt{2}} \widetilde{B}_{kj}(t) & j > k \end{cases}
\end{aligned}
$$

$$d\xi_{jk}(t) = ds_{jk}(t) + i da_{jk}(t)$$

martingale

no bounded variation parts

$$d\Upsilon_j \equiv 0.$$

- $1 \le j < k \le N$

$$\begin{aligned} d\xi_{jk}(t)d\xi_{kj}(t) &= d\xi_{jk}(t)d\xi_{jk}(t)^* \\ &= \left(\frac{1}{\sqrt{2}}dB_{jk}(t) + \frac{i}{\sqrt{2}}d\widetilde{B}_{jk}(t)\right)\left(\frac{1}{\sqrt{2}}dB_{jk}(t) - \frac{i}{\sqrt{2}}d\widetilde{B}_{jk}(t)\right) \\ &= \frac{1}{2}\Big(dB_{jk}(t)\Big)^2 + \frac{1}{2}\Big(d\widetilde{B}_{jk}(t)\Big)^2 = \frac{1}{2}dt + \frac{1}{2}dt = dt. \end{aligned}$$

- $1 \le j \le N$

$$\Big(d\xi_{jj}(t)\Big)^2 = \Big(dB_{jj}(t)\Big)^2 = dt.$$

- otherwise

$$d\xi_{jk}(t)d\xi_{\ell m}(t) = 0$$

$$d\xi_{jk}(t)d\xi_{\ell m}(t) = \delta_{jm}\delta_{k\ell}dt, \quad 1 \le j,k,\ell,m \le N.$$

$$\begin{aligned} \Gamma_{jk,kj}(t)dt &= (U(t)^\dagger d\Xi(t)U(t))_{jk}(U(t)^\dagger d\Xi(t)U(t))_{kj} \\ &= \sum_{\ell}\sum_{m}(U(t)^\dagger)_{j\ell}d\xi_{\ell m}(t)(U(t))_{mk} \times \sum_{p}\sum_{q}(U(t)^\dagger)_{kp}d\xi_{pq}(t)(U(t))_{qj} \\ &= \sum_{\ell}\sum_{m}\sum_{p}\sum_{q}(U(t)^\dagger)_{j\ell}(U(t))_{mk}(U(t)^\dagger)_{kp}(U(t))_{qj} \times \underbrace{d\xi_{\ell m}(t)d\xi_{pq}(t)}_{\delta_{\ell q}\delta_{mp}dt} \\ &= \sum_{\ell}\sum_{m}(U(t)^\dagger)_{j\ell}(U(t))_{mk}(U(t)^\dagger)_{km}(U(t))_{\ell j}dt \\ &= \sum_{\ell}(U(t)^\dagger)_{j\ell}(U(t))_{\ell j}\sum_{m}(U(t)^\dagger)_{km}(U(t))_{mk}dt \\ &= \delta_{jj}\delta_{kk}dt = dt \end{aligned}$$

$$\boxed{U(t)^\dagger U(t) = I_N \quad (N \times N \text{ unit matrix})}$$

In this case

$$d\Upsilon_j(t) = 0$$

and

$$\begin{aligned} dM_j(t)dM_k(t) &= \Gamma_{jj,kk}(t)dt \\ &= \left(\sum_{\ell} (U(t)^{\dagger})_{j\ell}(U(t))_{\ell k} \right)^2 dt = \delta_{jk}dt \end{aligned}$$

Then

$$dM_j(t) \stackrel{\mathrm{d}}{=} dB_j(t), \quad 1 \leq j \leq N.$$

$$\begin{aligned} d\lambda_j(t) &= dM_j(t) + dJ_j(t) \\ &= dB_j(t) + \sum_{1 \leq k \leq N : k \neq j} \frac{1}{\lambda_j(t) - \lambda_k(t)} \mathbf{1}(\lambda_j(t) \neq \lambda_k(t)) \underbrace{\Gamma_{jk,kj}(t)dt}_{dt} + \underbrace{d\Upsilon_j(t)}_{0} \\ &= dB_j(t) + \sum_{1 \leq k \leq N : k \neq j} \frac{1}{\lambda_j(t) - \lambda_k(t)} dt, \quad 1 \leq j \leq N. \end{aligned}$$

(B) GOE(Gaussian orthogonal ensemble)-process

$\Longrightarrow$ $U(t)$ is a real orthogonal matrix

$$\xi_{jk}(t) = s_{jk}(t)$$

If $j \neq k, \ell \neq m$,

$$\begin{aligned} d\xi_{jk}(t) d\xi_{\ell m}(t) &= ds_{jk}(t) ds_{\ell m}(t) \\ &= \frac{1}{\sqrt{2}} dB_{jk}(t) \frac{1}{2} dB_{\ell m}(t) \\ &= \frac{1}{2}(\delta_{j\ell}\delta_{km} + \delta_{jm}\delta_{k\ell}) dt. \end{aligned}$$

$$\frac{1}{2}(\delta_{\ell p}\delta_{mq} + \delta_{\ell q}\delta_{mp})$$

$$\begin{aligned} \Gamma_{jk,kj}(t) dt &= \sum_{\ell}\sum_{m}\sum_{p}\sum_{q} (U(t)^{\dagger})_{j\ell}(U(t))_{mk}(U(t)^{\dagger})_{kp}(U(t))_{qj} \times d\xi_{\ell m}(t) d\xi_{pq}(t) \\ &= \frac{1}{2}\Bigg\{ \sum_{\ell}\sum_{m} (U(t)^{\dagger})_{j\ell}(U(t))_{mk}(U(t)^{\dagger})_{k\ell}(U(t))_{mj} \\ &\qquad + \sum_{\ell}\sum_{m} (U(t)^{\dagger})_{j\ell}(U(t))_{mk}(U(t)^{\dagger})_{km}(U(t))_{\ell j} \Bigg\} dt \\ &= \frac{1}{2}(\delta_{jk}\delta_{jk} + \delta_{jj}\delta_{kk}) dt = \frac{1}{2}(\delta_{jk} + 1) dt \end{aligned}$$

$$\boxed{({}^{\mathrm{T}}U(t))_{jm}}$$

$$\boxed{({}^{\mathrm{T}}U(t))_{j\ell}}$$

$$\boxed{(U(t)_{\ell k})^{*} = (U(t))_{\ell k}}$$

In this case

$$d\Upsilon_j(t) = 0$$

and

$$dM_j(t) dM_k(t) = \delta_{jk} dt \quad \Longrightarrow \quad dM_j(t) \stackrel{\mathrm{d}}{=} dB_j(t), \quad 1 \leq j \leq N.$$

$$\boxed{\begin{aligned} d\lambda_j(t) &= dM_j(t) + dJ_j(t) \\ &= dB_j(t) + \sum_{1 \leq k \leq N: k \neq j} \frac{1}{\lambda_j(t) - \lambda_k(t)} \mathbf{1}(\lambda_j(t) \neq \lambda_k(t)) \Gamma_{jk,kj}(t) dt + d\Upsilon_j(t) \\ &= dB_j(t) + \sum_{1 \leq k \leq N: k \neq j} \frac{1}{\lambda_j(t) - \lambda_k(t)} \frac{1}{2} (\delta_{jk} + 1) dt \\ &= dB_j(t) + \frac{1}{2} \sum_{1 \leq k \leq N: k \neq j} \frac{1}{\lambda_j(t) - \lambda_k(t)} dt \quad 1 \leq j \leq N. \end{aligned}}$$

2.3 Proof of the theorem

By Itô's formula

$$
\begin{aligned}
d\Lambda(t) &= d(U(t)^{\dagger}\Xi(t)U(t)) \\
&= U(t)^{\dagger}d\Xi(t)U(t) \\
&\quad +\Big\{U(t)^{\dagger}\Xi(t)dU(t) + dU(t)^{\dagger}\Xi(t)U(t)\Big\} \\
&\quad +\Big\{U(t)^{\dagger}d\Xi(t)dU(t) + dU(t)^{\dagger}d\Xi(t)U(t)\Big\} \\
&\quad +dU(t)^{\dagger}\Xi(t)dU(t) \\
&= U(t)^{\dagger}d\Xi(t)U(t) \\
&\quad +\Big\{U(t)^{\dagger}\Xi(t)U(t)U(t)^{\dagger}dU(t) + dU(t)^{\dagger}U(t)U(t)^{\dagger}\Xi(t)U(t)\Big\} \\
&\quad +\Big\{U(t)^{\dagger}d\Xi(t)dU(t) + dU(t)^{\dagger}d\Xi(t)U(t)\Big\} \\
&\quad +dU(t)^{\dagger}\Xi(t)dU(t) \\
&= U(t)^{\dagger}d\Xi(t)U(t) \\
&\quad +\Big\{\Lambda(t)U(t)^{\dagger}dU(t) + dU(t)^{\dagger}U(t)\Lambda(t)\Big\} \\
&\quad +\Big\{U(t)^{\dagger}d\Xi(t)dU(t) + dU(t)^{\dagger}d\Xi(t)U(t)\Big\} \\
&\quad +dU(t)^{\dagger}\Xi(t)dU(t)
\end{aligned}
$$

[1] [2]

[3] [4]

[5]

On the other hand, from $U(t)^\dagger U(t) = I_N$ for any $t \geq 0$,

$$\begin{aligned}
0 &= d(U(t)^\dagger U(t)) \\
&= dU(t)^\dagger U(t) + U(t)^\dagger dU(t) + dU(t)^\dagger dU(t) \\
&= U(t)^\dagger dU(t) + \frac{1}{2} dU(t)^\dagger dU(t) \\
&\quad + dU(t)^\dagger U(t) + \frac{1}{2} dU(t)^\dagger dU(t) \\
&= \left(U(t)^\dagger dU(t) + \frac{1}{2} dU(t)^\dagger dU(t) \right) \\
&\quad + \left(U(t)^\dagger dU(t) + \frac{1}{2} dU(t)^\dagger dU(t) \right)^\dagger \\
&\equiv dA(t) + dA(t)^\dagger .
\end{aligned}$$

$$dA(t)^\dagger = -dA(t) \qquad \text{anti-hermitian}$$

$$\begin{aligned}
-dA(t)dA(t) &= dA(t)^\dagger dA(t) \\
&= dU(t)^\dagger U(t)U(t)^\dagger dU(t) + \text{higher infinitesimals} \\
&= dU(t)^\dagger dU(t)
\end{aligned}$$

$$\begin{aligned}
U(t)dA(t) &= U(t)\left\{U(t)^\dagger dU(t) + \frac{1}{2}dU(t)^\dagger dU(t)\right\} \\
&= U(t)U(t)^\dagger dU(t) + \frac{1}{2}U(t)dU(t)^\dagger dU(t) \\
&= dU(t) - \frac{1}{2}U(t)dA(t)dA(t).
\end{aligned}$$

$$dU(t) = U(t)\left(dA(t) + \frac{1}{2}dA(t)dA(t)\right).$$

$$\begin{aligned}
[1] &= \Lambda(t)dA(t) + \frac{1}{2}\Lambda(t)dA(t)dA(t) \\
[2] &= [1]^\dagger = dA(t)^\dagger\Lambda(t) + \frac{1}{2}dA(t)dA(t)\Lambda(t) \\
[3] &= U(t)^\dagger d\Xi(t)U(t)dA(t) \\
[4] &= [3]^\dagger \\
[5] &= -dA(t)\Lambda(t)dA(t)
\end{aligned}$$

$$\begin{aligned} d\Lambda(t) &= U(t)^\dagger d\Xi(t) U(t) + \Lambda(t) dA(t) + (\Lambda(t) dA(t))^\dagger \\ &\quad + \frac{1}{2}\Lambda(t) dA(t) dA(t) + \frac{1}{2}(\Lambda(t) dA(t) dA(t))^\dagger \\ &\quad + U(t)^\dagger d\Xi(t) U(t) dA(t) + (U(t)^\dagger d\Xi(t) U(t) dA(t))^\dagger \\ &\quad - dA(t)\Lambda(t) dA(t) \qquad \textbf{Eq.(A.1)} \end{aligned}$$

Let

$$\begin{aligned} d\gamma_{jk}(t) &\equiv \left(\frac{1}{2} dA(t) dA(t)\right)_{jk} = \frac{1}{2}\sum_{\ell} d\alpha_{j\ell}(t) d\alpha_{\ell k}(t) \\ &= d\gamma_{kj}(t)^* \\ d\phi_{jk}(t) &\equiv (U(t)^\dagger d\Xi(t) U(t) dA(t))_{jk} \\ &= \sum_{\ell,m,n} u^*_{\ell j}(t) d\xi_{\ell m}(t) u_{mn}(t) d\alpha_{nk}(t) \\ d\psi_{jk}(t) &\equiv (dA(t)^\dagger \Lambda(t) dA(t))_{jk} \\ &= \sum_{\ell} d\alpha^*_{\ell j}(t) \lambda_\ell(t) d\alpha_{\ell k}(t) \\ &= -\sum_{\ell} d\alpha_{j\ell}(t) \lambda_\ell(t) d\alpha_{\ell k}(t). \end{aligned}$$

$\boxed{\text{bounded variation parts}}$

The diagonal elements of **Eq.(A.1)**: (j,j)-elements

$$\begin{aligned} d\lambda_j(t) &= \sum_{k,\ell} u^*_{kj}(t) d\xi_{k\ell}(t) u_{\ell j}(t) \\ &\quad + \Big\{ 2\lambda_j(t) d\gamma_{jj}(t) + d\phi_{jj}(t) + d\phi^*_{jj}(t) + d\psi_{jj}(t) \Big\} \qquad \textbf{Eq.(A.2)} \end{aligned}$$

The off-diagonal elements of **Eq.(A.1)**; (j,m)-elements, $\quad j \neq m$

$$\begin{aligned} 0 &= \sum_{k,\ell} u^*_{k\ell}(t) d\xi_{k\ell}(t) u_{\ell m}(t) \\ &\quad + \lambda_j(t) d\alpha_{jm}(t) + \lambda_m(t) d\alpha^*_{mj}(t) \\ &\quad + \lambda_j(t) d\gamma_{jm}(t) + \lambda_m(t) d\gamma^*_{mj}(t) \\ &\quad + d\phi_{jm}(t) + d\phi^*_{mj}(t) + d\psi_{jm}(t) \\ &= \sum_{k,\ell} u^*_{k\ell}(t) d\xi_{k\ell}(t) u_{\ell m}(t) \\ &\quad + \lambda_j(t) d\alpha_{jm}(t) - \lambda_m(t) d\alpha_{jm}(t) \\ &\quad + \lambda_j(t) d\gamma_{jm}(t) + \lambda_m(t) d\gamma_{jm}(t) \\ &\quad + d\phi_{jm}(t) + d\phi^*_{mj}(t) + d\psi_{jm}(t) \qquad \textbf{Eq.(A.3)} \end{aligned}$$

[The first term of **Eq.(A.2)** $-d\Upsilon_j(t)$] is martingale, $dM_j(t)$.

$$
\begin{aligned}
(\mathbf{Eq.(A.3)}) \quad = \quad & \sum_{k,\ell} u^*_{k\ell}(t) u_{\ell m}(t) d\xi_{k\ell}(t) \\
& +(\lambda_j(t) - \lambda_m(t)) d\alpha_{jm}(t) \\
& +(\lambda_j(t) + \lambda_m(t)) d\gamma_{jm}(t) \\
& +d\phi_{jm}(t) + d\phi^*_{mj}(t) + d\psi_{jm}(t) = 0 \quad (j \neq m).
\end{aligned}
$$

Then

$$
\begin{aligned}
& \sum_{k,\ell} u^*_{k\ell}(t) u_{\ell m}(t) d\xi_{k\ell}(t) \\
= \quad & (\lambda_m(t) - \lambda_j(t)) d\alpha_{jm}(t) \\
& -(\lambda_j(t) + \lambda_m(t)) d\gamma_{jm}(t) - d\phi_{jm}(t) - d\phi^*_{mj}(t) - d\psi_{jm}(t) \qquad \mathbf{Eq.(A.4)}
\end{aligned}
$$

$$
d\alpha_{jm}(t) = \frac{\mathbf{1}(\lambda_m(t) \neq \lambda_j(t))}{\lambda_m(t) - \lambda_j(t)} \sum_{k,\ell} u^*_{kj}(t) u_{\ell m}(t) d\xi_{k\ell}(t) + \text{higher infinitesimals}
$$

Eq.(A.5)

Using **Eq.(A.4)**, $d\phi_{jk}(t)$ is rewritten as

$$
\begin{aligned}
d\phi_{jk}(t) &\equiv \sum_{\ell,m,n} u^*_{\ell j}(t) u_{mn}(t) d\xi_{\ell m}(t) d\alpha_{nk}(t) \\
&= \sum_n \Big\{ (\lambda_n(t) - \lambda_j(t)) d\alpha_{jn}(t) - (\lambda_j(t) + \lambda_n(t)) d\gamma_{jn}(t) \\
&\qquad -d\phi_{jn}(t) - d\phi^*_{nj}(t) - d\psi_{jn}(t) \Big\} d\alpha_{nk}(t) \\
&= \sum_n (\lambda_n(t) - \lambda_j(t)) d\alpha_{jn}(t) d\alpha_{nk}(t) + \text{higher infinitesimals}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
dB_j(t) &\equiv \text{the second term of } \textbf{Eq.(A.2)} \\
&= 2\lambda_j(t) d\gamma_{jj}(t) + d\phi_{jj}(t) + d\phi^*_{jj}(t) + d\psi_{jj}(t) \\
&= 2\lambda_j(t) \frac{1}{2} \sum_k d\alpha_{jk}(t) d\alpha_{kj}(t) \\
&\quad + \sum_n (\lambda_n(t) - \lambda_j(t)) d\alpha_{jn}(t) d\alpha_{nj}(t) \\
&\quad + \sum_n (\lambda_n(t) - \lambda_j(t)) d\alpha^*_{nj}(t) d\alpha^*_{jn}(t) \\
&\quad - \sum_\ell d\alpha_{j\ell}(t) \lambda_\ell d\alpha_{\ell j}(t) \\
&= \sum_n (\lambda_n(t) - \lambda_j(t)) d\alpha_{jn}(t) d\alpha_{nj}(t).
\end{aligned}
$$

By **Eq.(A.5)**,

$$\begin{aligned}
dB_j(t) &= \sum_n (\lambda_n(t) - \lambda_j(t)) \mathbf{1}(\lambda_n(t) \neq \lambda_j(t)) \\
&\quad \times \frac{1}{\lambda_n(t) - \lambda_j(t)} \sum_{k,\ell} u^*_{kj}(t) u_{\ell n}(t) d\xi_{k\ell}(t) \\
&\quad \times \frac{1}{\lambda_j(t) - \lambda_n(t)} \sum_{p,q} u^*_{pn}(t) u_{qj}(t) d\xi_{pq}(t) \\
&= \sum_n \frac{\mathbf{1}(\lambda_n(t) \neq \lambda_j(t))}{\lambda_j(t) - \lambda_n(t)} \sum_{k,\ell} \sum_{p,q} u^*_{kj}(t) d\xi_{k\ell}(t) u_{\ell n}(t) u^*_{pn}(t) d\xi_{pq}(t) u_{qj}(t) \\
&= \sum_n \frac{\mathbf{1}(\lambda_n(t) \neq \lambda_j(t))}{\lambda_j(t) - \lambda_n(t)} (U(t)^\dagger d\Xi(t) U(t))_{jn} (U(t)^\dagger d\Xi(t) U(t))_{nj}.
\end{aligned}$$

We set

$$dJ_j(t) = dB_j(t) + d\Upsilon_j(t).$$

2.4 BM on Lie Algebras and their Eigenvalue Processes

1 A_{N-1} type (unitary group)

- Lie group

$$U(N) = \left\{ g \in GL(N, \mathbb{C}) : g^\dagger g = I_N \right\}$$

- Lie algebra

$$\mathfrak{u}(N) = \left\{ X \in \mathfrak{gl}(N, \mathbb{C}) : X^\dagger + X = O \right\}$$

- **Remark.** $\sqrt{-1}\mathfrak{u}(N) = \mathcal{H}(N) = \{N \times N \text{ Hermitian matrix}\}$
- maximally Abelian subspace of $\mathfrak{u}(N)$

$$\mathfrak{t}_0 = \{Y(\theta_1, \cdots, \theta_N) \equiv \mathrm{diag}(\sqrt{-1}\theta_1, \cdots, \sqrt{-1}\theta_N) : \theta_1, \cdots, \theta_N \in \mathbb{R}\}$$

- root system

$$\begin{aligned} \Delta &= \Delta(\mathfrak{u}(N), \mathfrak{t}_0) \\ &= \{\varepsilon_j - \varepsilon_k : j \neq k\} \end{aligned}$$

- dimensions

$$\dim \mathfrak{u}(N) = \dim \mathfrak{t}_0 + \dim \Delta = N + N(N-1) = N^2$$

- root decomposition

$$\begin{aligned} \mathfrak{u}(N) &= \mathfrak{t}_0 + \sum_{j \neq k} \mathbb{C} E_{jk} \\ &= \mathfrak{t}_0 + \sum_{1 \leq j < k \leq N} (\mathbb{C} E_{jk} - \overline{\mathbb{C}} E_{kj}), \end{aligned}$$

where

$$\begin{aligned} E_{jk} \;&\equiv\; N \times N \text{ matrix s.t.} \\ &\quad (j,k)\text{-element}=1, \quad \text{other element} = 0. \end{aligned}$$

- $\mathfrak{u}(N)$-valued BM

$$M^{\mathrm{A}}(t) = \sqrt{-1}\sum_{j=1}^{N} E_{jj}B_j(t) + \sum_{1\le j<k\le N}\left[\hat{B}_{jk}(t)E_{jk} - \overline{\hat{B}_{jk}}(t)E_{kj}\right],$$

where

$$\begin{aligned} &B_j(t), 1 \le j \le N : \text{one-dim. standard. BM} \\ &\hat{B}_{jk}(t) = \frac{1}{\sqrt{2}}(B^{\mathrm{R}}_{jk}(t) + \sqrt{-1}B^{\mathrm{I}}_{jk}(t)), 1 \le j < k \le N : \text{complex BM.} \end{aligned}$$

- diagonalization: For each $t \ge 0$, $^\exists U(t) \in SU(N)$, s.t.

$$\begin{aligned} U(t)^\dagger M^{\mathrm{A}}(t)U(t) &= Y(\theta_1(t), \dots, \theta_N(t)) \\ &= \mathrm{diag}(\sqrt{-1}\theta_1(t), \cdots, \sqrt{-1}\theta_N(t)). \end{aligned}$$

- Application of the generalized Bru's theorem

$$\Gamma_{ij,j\ell}(t) = \delta_{i\ell}\delta_{kj}.$$

$$\begin{aligned} \Gamma_{ii,jj}(t) &= \delta_{ij} \\ \Gamma_{ij,ji}(t) &= 1 \end{aligned}$$

$$dM_i(t)dM_j(t) = \delta_{ij}dt$$

$$d\theta_i(t) = dB_i(t) + \sum_{1\le j\le N, j\ne i}\frac{1}{\theta_i(t)-\theta_j(t)}dt, \quad 1 \le i \le N$$

- noncolliding BM ($T = \infty$)
  = the Dyson model ($\beta = 2$)

2 B_N type ($n = 2N+1$ orthogonal group)

- Lie group

$$SO(2N+1) = \left\{ g \in SL(2N+1, \mathbb{R}) : {}^t g g = I_{2N+1} \right\}$$

- Lie algebra

$$\mathfrak{so}(2N+1) = \left\{ X \in \mathfrak{sl}(2N+1, \mathbb{R}) : {}^t X + X = O \right\}$$

- complexification of $\mathfrak{so}(2N+1)$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2N+1, \mathbb{C}) = \{ X \in \mathfrak{gl}(2N+1, \mathbb{C}) : {}^t X = -X \}$$

Let

$$c = \frac{1}{\sqrt{2}} (I_{2N+1} - \sqrt{-1} I'_{2N+1}), \quad c^{-1} = c^\dagger = \overline{c} = \frac{1}{\sqrt{2}} (I_{2N+1} + \sqrt{-1} I'_{2N+1})$$

where

$$I_{2N+1} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}, \qquad I'_{2N+1} = \begin{pmatrix} 0 & & 1 \\ & \cdots & \\ 1 & & 0 \end{pmatrix}.$$

- adjoint action

$$\begin{aligned} \mathfrak{g}'_{\mathbb{C}} &\equiv \mathrm{Ad}(c^{-1}) \mathfrak{g}_{\mathbb{C}} = \{ Y = c^{-1} X c : X \in \mathfrak{g}_{\mathbb{C}} \} \\ &= \{ X \in \mathfrak{gl}(2N+1, \mathbb{C}) : I'_{2N+1} {}^t X I'_{2N+1} = -X \} \end{aligned}$$

- root system

$$\Delta = \{\varepsilon_j - \varepsilon_k : j + k \leq 2N + 1, j \neq k\}.$$

Since $\varepsilon_{2N+2} = 0, \varepsilon_{2N+2-j} = -\varepsilon_j, 1 \leq j \leq N$,

$$\begin{aligned} \Delta &= \{\pm(\varepsilon_j - \varepsilon_k) : 1 \leq j < k \leq N\} \\ &\quad \cup \{\pm\varepsilon_j : 1 \leq j \leq N\} \\ &\quad \cup \{\pm(\varepsilon_j + \varepsilon_k) : 1 \leq j < k \leq N\} \end{aligned}$$

- dimensions

$$\begin{aligned} |\Delta| &= N(N-1) + 2N + N(N-1) \\ &= 2N^2, \\ \dim \mathfrak{t}' &= N, \\ \dim \mathfrak{g}' &= N + 2N^2 = N(2N+1). \end{aligned}$$

- root decomposition

$$\mathfrak{g}'_{\mathbb{C}} = \bigoplus_{j+k \leq 2N+1} \mathbb{C}(E_{jk} - E_{2N+2-k,2N+2-j})$$

- $\mathfrak{g}'_{\mathbb{C}} \cap \mathfrak{u}(2N+1)$-valued BM

$$\mathrm{Ad}(c^{-1})\mathfrak{so}(2N+1,\mathbb{C}) \cap \mathfrak{u}(2N+1) = \{X \in \mathfrak{gl}(2N+1,\mathbb{C}) : X^\dagger = -X, I'_{2N+1}{}^tXI'_{2N+1} = -X\}$$

$$\begin{aligned} M^{\mathrm{B}}(t) &= \sqrt{-1}\sum_{j=1}^{N}(E_{jj} - E_{2N+2-j,2N+2-j})B_j(t) \\ &\quad + \sum_{1\leq j<k\leq 2N+1, j+k\leq 2N+1}\Big\{(E_{jk} - E_{2N+2-k,2N+2-j})\hat{B}_{jk}(t) - (E_{kj} - E_{2N+2-j,2N+2-k})\overline{\hat{B}_{jk}(t)}\Big\} \end{aligned}$$

- diagonalization: For each $t \geq 0$

$$^\exists U(t) \in \{g \in SU(2N+1) : {}^tgI'_{2N+1}g = I'_{2N+1}\}$$

s.t.

$$\begin{aligned} U(t)^\dagger M^{\mathrm{B}}(t)U(t) &= \begin{pmatrix} \sqrt{-1}\theta_1(t) & & & & & & 0 \\ & \ddots & & & & & \\ & & \sqrt{-1}\theta_N(t) & & & & \\ & & & 0 & & & \\ & & & & -\sqrt{-1}\theta_N(t) & & \\ & & & & & \ddots & \\ 0 & & & & & & -\sqrt{-1}\theta_1(t) \end{pmatrix} \\ &= Y(\theta_1(t),\dots,\theta_N(t),0,-\theta_N(t),\dots,\theta_1(t)). \end{aligned}$$

- Application of the generalized Bru's theorem

$$\Gamma_{ij,k\ell}(t) = \delta_{i\ell}\delta_{kj} - \delta_{i,2N+2-k}\delta_{j,2N+2-\ell}$$

$$\begin{aligned}\Gamma_{ii,jj}(t) &= \delta_{ij} - \delta_{i,2N+2-j} \\ \Gamma_{ij,ji}(t) &= 1 - \delta_{i,2N+2-j}\end{aligned}$$

$$dM_i(t)dM_j(t) = \delta_{ij}dt, 1 \le i,j \le N.$$

$$d\theta_i(t) = dB_i(t) + \left[\frac{1}{\theta_i(t)} + \sum_{1\le j\le N, j\ne i}\left\{\frac{1}{\theta_i(t)-\theta_j(t)} + \frac{1}{\theta_i(t)+\theta_j(t)}\right\}\right]dt, \quad 1 \le i \le N$$

- noncolliding BES$^{(3)}$ = noncolliding absorbing BM

  cf: the class C ensemble of Altland-Zirnbauer

3 C_N type (symplectic group)

- Lie group: Let

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0. \end{pmatrix}$$

Then

$$Sp(N, \mathbb{C}) = \{g \in GL(2N, \mathbb{C}) : {}^t g J g = J\}$$

is called the complex symplectic group.

Here we consider its subgroup called the compact symplectic group;

$$Sp(N) = Sp(N, \mathbb{C}) \cap U(2N) = \{g \in U(2N) : {}^t g J g = J\}$$

- Lie algebra

$$\begin{aligned} \mathfrak{g} &= \mathfrak{sp}(N) = \mathfrak{g}_{\mathbb{C}} \cap \mathfrak{u}(2N) \\ &= \{X \in \mathfrak{u}(2N) : {}^t X J + J X = 0\} \\ &= \left\{ X = \begin{pmatrix} A & B \\ B^\dagger & -{}^t A \end{pmatrix} : A \in \mathfrak{u}(N), B \in \mathfrak{gl}(N, \mathbb{C}), {}^t B = B, {}^t C = C \right\}. \end{aligned}$$

- maximally Abelian subgroup

$$\mathfrak{t}_0 = \left\{ Y(\theta_1, \cdots, \theta_N, -\theta_1, \cdots, -\theta_N) : \theta_1, \cdots, \theta_N \in \mathbb{R} \right\}$$

- root system

$$\begin{aligned} \Delta &= \{\pm(\varepsilon_j - \varepsilon_k) : 1 \le j < k \le N\} \\ &\quad \cup\{\pm 2\varepsilon_j : 1 \le j \le N\} \\ &\quad \cup\{\pm(\varepsilon_j + \varepsilon_k) : 1 \le j < k \le N\} \end{aligned}$$

- dimensions

$$\begin{aligned} |\Delta| &= N(N-1) + 2N + N(N-1) = 2N^2, \\ \dim \mathfrak{t}_0 &= N, \\ \dim \mathfrak{sp}(N) &= N + 2N^2 = N(2N+1). \end{aligned}$$

- root decomposition

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{j=1}^{N}\bigoplus_{k=1}^{N} \mathbb{C}(E_{jk} - E_{N+k,N+j}) \oplus \bigoplus_{1\le j\le k\le N} \mathbb{C}(E_{j,N+k} + E_{k,N+j}) \oplus \bigoplus_{1\le j\le k\le N} \mathbb{C}(E_{N+j,k} + E_{N+k,j})$$

- $\mathfrak{sp}(N)$-valued BM

$$
\begin{aligned}
M^{\mathrm{C}}(t) &= \sqrt{-1}\sum_{j=1}^{N}(E_{jj}-E_{N+j,N+j})B_j(t) \\
&\quad + \sum_{1\le j<k\le N}\Big\{(E_{jk}-E_{N+k,N+j})\hat{B}_{jk}(t)-(E_{kj}-E_{N+j,N+k})\overline{\hat{B}_{jk}(t)}\Big\} \\
&\quad + \sum_{j=1}^{N}\sqrt{2}\Big\{E_{j,N+j}\tilde{B}_{jj}(t)-E_{N+j,j}\overline{\tilde{B}_{jj}(t)}\Big\} \\
&\quad + \sum_{1\le j<k\le N}\Big\{(E_{j,N+k}+E_{k,N+j})\tilde{B}_{jk}(t)-(E_{N+j,k}-E_{N+k,j})\overline{\tilde{B}_{jk}(t)}\Big\}
\end{aligned}
$$

- diagonalization: For each $t \geq 0$

$$
{}^{\exists}U(t) \in Sp(N) = \{g \in U(2N) : {}^{t}gJg = J\}
$$

s.t.

$$
\begin{aligned}
U(t)^{\dagger}M^{\mathrm{C}}(t)U(t) &= \begin{pmatrix}
\sqrt{-1}\theta_1(t) & & & & & 0 \\
& \ddots & & & & \\
& & \sqrt{-1}\theta_N(t) & & & \\
& & & -\sqrt{-1}\theta_1(t) & & \\
& & & & \ddots & \\
0 & & & & & -\sqrt{-1}\theta_N(t)
\end{pmatrix} \\
&= Y(\theta_1(t),\cdots,\theta_N(t),-\theta_1(t),\cdots,-\theta_N(t)).
\end{aligned}
$$

- Application of the generalized Bru's theorem

$$\Gamma_{ij,k\ell}(t) = \delta_{i\ell}\delta_{kj} - J_{ik}J_{j\ell}$$

$$\begin{aligned}\Gamma_{ii,jj}(t) &= \delta_{ij} - \{(J)_{ij}\}^2 \\ \Gamma_{ij,ji}(t) &= 1 - (J)_{ij}(J)_{ji}\end{aligned}$$

$$dM_i(t)dM_j(t) = \delta_{ij}dt, \quad 1 \leq i,j \leq N$$

$$d\theta_i(t) = dB_i(t) + \left[\frac{1}{\theta_i(t)} + \sum_{1\leq j\leq N, j\neq i}\left\{\frac{1}{\theta_i(t)-\theta_j(t)} + \frac{1}{\theta_i(t)+\theta_j(t)}\right\}\right]dt, \quad 1 \leq i \leq N$$

- This is the same as the B_N type.

4 D_N type ($n = 2N$ orthogonal group)

- Lie group

$$SO(2N) = \left\{ g \in SL(2N, \mathbb{R}) :{}^t gg = I_{2N} \right\}$$

- Lie algebra

$$\mathfrak{so}(2N) = \left\{ X \in \mathfrak{sl}(2N, \mathbb{R}) :{}^t X + X = O \right\}$$

- adjoint action For the complexificated version

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2N, \mathbb{C}) = \{X \in \mathfrak{gl}(2N, \mathbb{C}) :{}^t X = -X\},$$

Let

$$\begin{aligned} \mathfrak{g}'_{\mathbb{C}} &= \mathrm{Ad}(c^{-1})\mathfrak{g}_{\mathbb{C}} = \{Y = c^{-1}Xc : X \in \mathfrak{g}_{\mathbb{C}}\} \\ &= \{X \in \mathfrak{gl}(2N, \mathbb{C}) : I'_{2N}\, {}^t X I'_{2N} = -X\} \end{aligned}$$

- root system

$$\begin{aligned} \Delta &= \{\varepsilon_j - \varepsilon_k : j + k \leq 2N, j \neq k\} \\ &= \{\pm(\varepsilon_j - \varepsilon_k) : 1 \leq j < k \leq N\} \\ &\quad \cup \{\pm(\varepsilon_j + \varepsilon_k) : 1 \leq j < k \leq N\} \end{aligned}$$

- dimensions

$$\begin{aligned} |\Delta| &= N(N-1) + N(N-1) = 2N(N-1), \\ \dim \mathfrak{t}' &= N, \\ \dim \mathfrak{g}' &= N + 2N(N-1) = N(2N-1). \end{aligned}$$

- root decomposition

$$\mathfrak{g}'_{\mathbb{C}} = \bigoplus_{j+k \leq 2N} \mathbb{C}(E_{jk} - E_{2N+1-k, 2N+1-j})$$

- $\mathfrak{g}'_{\mathbb{C}} \cap \mathfrak{u}(2N)$-valued BM

$$\mathrm{Ad}(c^{-1})\mathfrak{so}(2N,\mathbb{C}) \cap \mathfrak{u}(2N) = \{X \in \mathfrak{gl}(2N,\mathbb{C}) : X^\dagger = -X, I'_{2N}{}^tX I'_{2N} = -X\}$$

$$\begin{aligned} M^{\mathrm{D}}(t) &= \sqrt{-1}\sum_{j=1}^{N}(E_{jj} - E_{2N+2-j,2N+2-j})B_j(t) \\ &+ \sum_{1\le j<k\le 2N, j+k\le 2N} \left\{(E_{jk} - E_{2N+1-k,2N+1-j})\hat{B}_{jk}(t) - (E_{kj} - E_{2N+1-j,2N+1-k})\overline{\hat{B}_{jk}(t)}\right\} \end{aligned}$$

- diagonalization: For each $t \ge 0$

$$^{\exists}U(t) \in \{g \in SU(2N) : {}^tg I'_{2N} g = I'_{2N}\}$$

s.t.

$$\begin{aligned} U(t)^\dagger M^{\mathrm{D}}(t)U(t) &= \begin{pmatrix} \sqrt{-1}\theta_1(t) & & & & & 0 \\ & \ddots & & & & \\ & & \sqrt{-1}\theta_N(t) & & & \\ & & & -\sqrt{-1}\theta_1(t) & & \\ & & & & \ddots & \\ 0 & & & & & -\sqrt{-1}\theta_N(t) \end{pmatrix} \\ &= Y(\theta_1(t), \cdots, \theta_N(t), -\theta_1(t), \cdots, -\theta_N(t)). \end{aligned}$$

- Application of the generalized Bru's theorem

$$\Gamma_{ij,k\ell}(t) = \delta_{i\ell}\delta_{kj} - \delta_{i,2N+1-k}\delta_{j,2N+1-\ell}$$

$$\begin{aligned}\Gamma_{ii,jj}(t) &= \delta_{ij} - \delta_{i,2N+1-j} \\ \Gamma_{ij,ji}(t) &= 1 - \delta_{i,2N+1-j}\end{aligned}$$

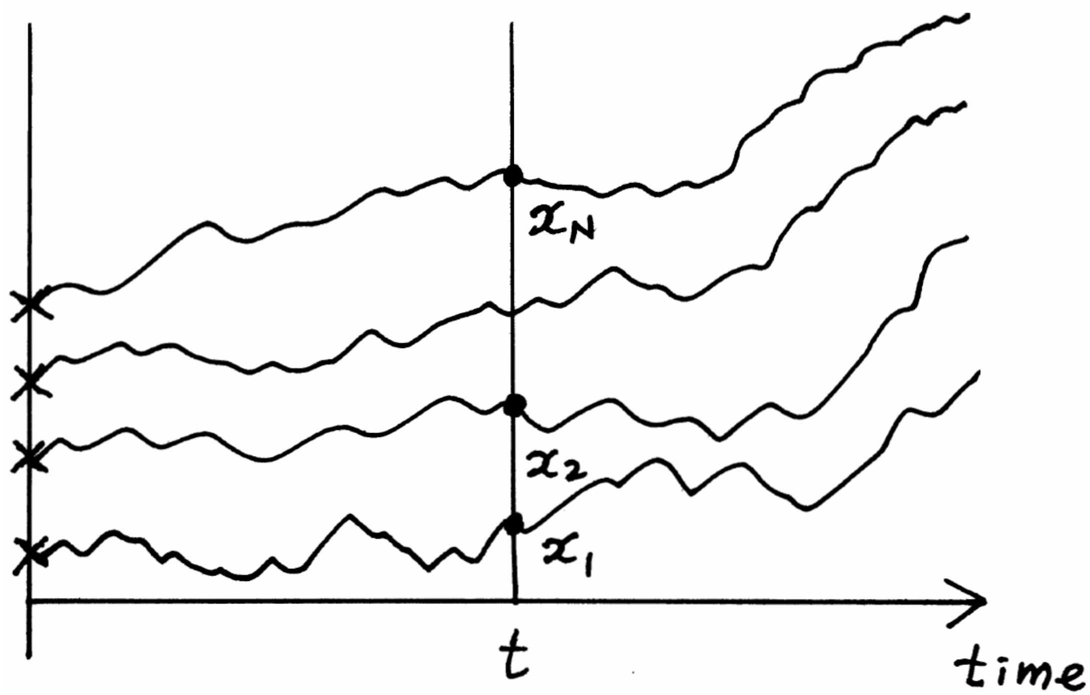
$$dM_i(t)dM_j(t) = \delta_{ij}dt, \quad 1 \le i,j \le N$$

$$d\theta_i(t) = dB_i(t) + \sum_{1\le j\le N, j\ne i} \left\{ \frac{1}{\theta_i(t)-\theta_j(t)} + \frac{1}{\theta_i(t)+\theta_j(t)} \right\} dt, \quad 1 \le i \le N$$

- noncolliding $\mathrm{BES}^{(1)}$ = noncolliding reflecting BM

  cf: the class D ensemble of Altland-Zirnbauer

$$\begin{aligned} \text{class B process } (n = 2N+1) &= \text{clas C process } (n = 2N) \\ &= \text{noncolliding BES}^{(3)} \\ &= \text{noncolliding absorbing BM} \end{aligned}$$



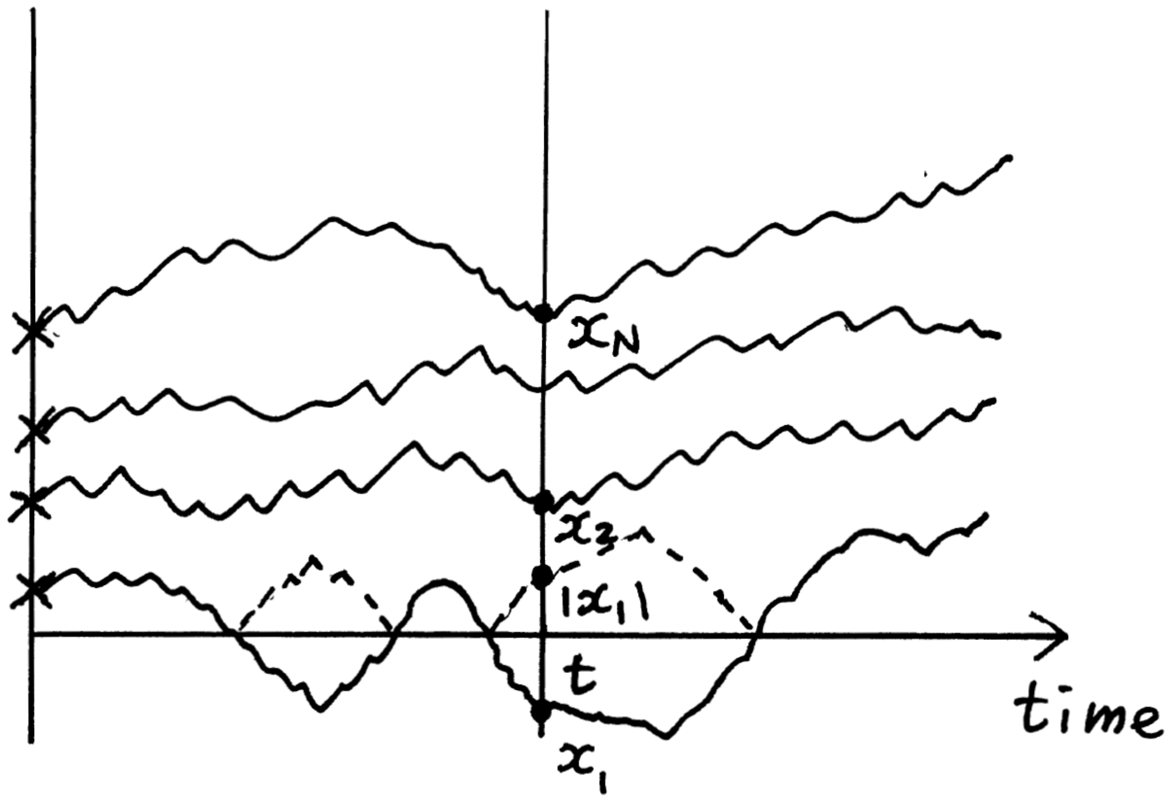
(h-transform of)

absorbing BM in the Weyl chamber of type C

$$\mathbb{W}_N^{\mathrm{C}} = \left\{ \mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N : 0 < x_1 < \cdots < x_N \right\}$$

class D process ($n = 2N + 1$) $=$ noncolliding BES$^{(1)}$

$=$ noncolliding reflecting BM



(h-transform of)

absorbing BM in the Weyl chamber of type D

$$\mathbb{W}_N^{\mathrm{D}} = \left\{ \mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N : |x_1| < x_2 < \cdots < x_N \right\}$$