Symmetries and Structures of Matrix-Valued Stochastic Processes and Noncolliding Diffusion Processes

Makoto Katori (Chuo Univ., Tokyo)

23-27 May 2011

Part 3

Conférence aux Houches

"Vicious walkers and random matrices"

du 16 au 27 mai 2011

3[rd] day: Temporally Inhomogeneous Processes and their homogeneous limits

3.1 Noncolliding BM with duration T

- Weyl chamber of type A_{N-1}

$$\mathbb{W}_N^{\mathrm{A}} = \left\{ \boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N \right\}$$

Boundary

$$\partial \mathbb{W}_N^{\mathrm{A}} = \left\{ \boldsymbol{x} \in \mathbb{R}^N : ^{\exists} j \neq k \quad \text{s.t.} \quad x_j = x_k \right\}.$$

- N-dim. absorbing BM in $\mathbb{W}_N^{\mathrm{A}}$
  in which absorbing walls are put at $\partial \mathbb{W}_N^{\mathrm{A}}$

- Karlin-McGregor-LGV formula

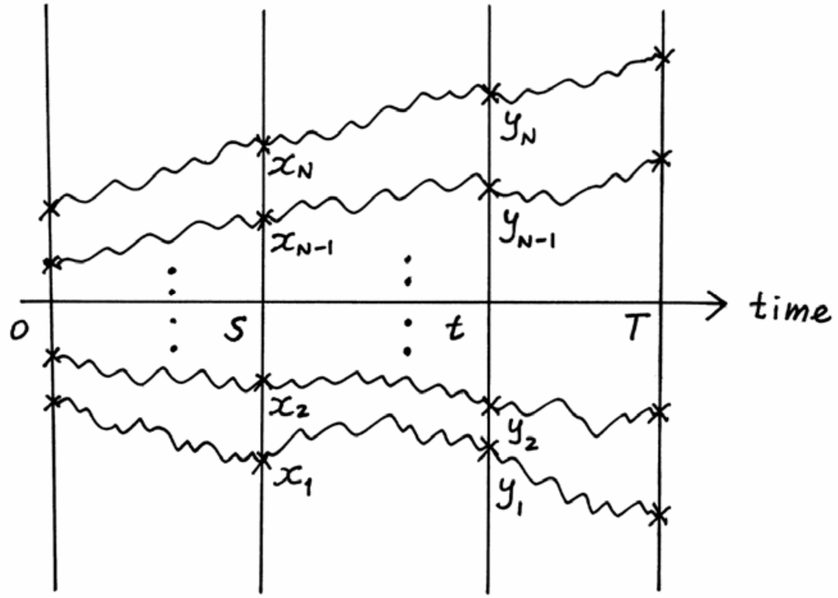
surviving

$$\mathbb{P}\Big((s, \boldsymbol{x}) \rightsquigarrow (t, \boldsymbol{y}) \Big| (s, \boldsymbol{x}) \Big) = \det_{1 \le j,k \le N} \Big[p(t-s, y_j | x_k) \Big] \equiv f_N(t-s, \boldsymbol{y}|\boldsymbol{x})$$

$\in \mathbb{W}_N^{\mathrm{A}}$ $\in \mathbb{W}_N^{\mathrm{A}}$

$$\begin{aligned} \mathcal{N}_N(t, \boldsymbol{x}) &= \mathbb{P}\Big(\text{not absorbed in } [0,t] \Big| (0, \boldsymbol{x}) \Big) \\ &= \int_{\mathbb{W}_N^{\mathrm{A}}} d\boldsymbol{y}\, f_N(t, \boldsymbol{y}|\boldsymbol{x}) \qquad \Big(\text{where} \quad d\boldsymbol{y} = \prod_{j=1}^{N} dy_j \Big) \\ &= \text{survival probability during time period } [0,t] \\ &\qquad \text{starting from } \boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}} \\ &= \text{noncolliding probability during time period } [0,t] \\ &\qquad \text{starting from } \boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}} \end{aligned}$$

- Let $0 < T < \infty$.
  Consider the system of N BMs conditioned never to collide with each other up to time T.



- Under this condition

$$g_N^T(s, \boldsymbol{x}; t, \boldsymbol{y}) = \text{t.p.d. from } (s, \boldsymbol{x}) \rightsquigarrow (t, \boldsymbol{y})$$
$$0 < s < t < T, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N^{\mathrm{A}}$$
$$\Downarrow$$
$$g_N^T(s, \boldsymbol{x}; t, \boldsymbol{y}) = \frac{f_N(t-s, \boldsymbol{y}|\boldsymbol{x})\mathcal{N}_N(T-t, \boldsymbol{y})}{\mathcal{N}_N(T-s, \boldsymbol{x})}.$$

- Markov process $\Longrightarrow$ indep. of the process before time s

- It depends not only $t-s$, but also $T-t, T-s$:
  temporally inhomogeneous

3.2 Schur function expansion and asymptotics

- Karlin-McGregor-LGV determinant

$$\begin{aligned} f_N(t, \boldsymbol{y}|\boldsymbol{x}) &= \det_{1 \le j,k \le N} \left[\frac{1}{\sqrt{2\pi t}} e^{-(x_j - y_k)^2/2t} \right] \\ &= \det_{1 \le j,k \le N} \left[\frac{1}{(2\pi t)^{1/2}} e^{-x_j^2/2t} e^{x_j y_k/t} e^{-y_k^2/2t} \right] \end{aligned}$$

by multilinearlity of det.

$$\begin{aligned} &= \frac{1}{(2\pi t)^{N/2}} \prod_{j=1}^{N} e^{-x_j^2/2t} \times \prod_{k=1}^{N} e^{-y_k^2/2t} \times \det_{1 \le j,k \le N} \left[e^{x_j y_k/t} \right] \\ &= \frac{1}{(2\pi t)^{N/2}} \exp\left\{ -\frac{1}{2t}(|\boldsymbol{x}|^2 + |\boldsymbol{y}|^2) \right\} \det_{1 \le j,k \le N} \left[e^{x_j/\sqrt{t} \times y_k/\sqrt{t}} \right]. \end{aligned}$$

$$\frac{x_j}{\sqrt{t}} \to x_j \qquad \frac{y_j}{\sqrt{t}} \to y_j$$

Here let

$$F_N(\boldsymbol{x},\boldsymbol{y}) \equiv \det_{1\le j,k\le N}\left[e^{x_j y_k}\right]$$

$$= \det_{1\le j,k\le N}\left[\sum_{n=0}^{\infty}\frac{1}{n!}(x_j y_k)^n\right]$$ Taylor expansion **in** a determinant

by multilinearlity of det.

$$= \sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\frac{1}{n_1!\cdots n_N!}\det_{1\le j,k\le N}\left[(x_j y_k)^{n_j}\right]$$

symmetric function of $\boldsymbol{n}=(n_1,\dots,n_N)$

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^N}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{N}_0^N}\prod_{j=1}^{N}\frac{1}{n_j!}\frac{1}{N!}\sum_{\sigma\in S_N}\det_{1\le j,k\le N}\left[(x_j y_k)^{n_{\sigma(j)}}\right]$$

$$\sum_{0\le n_1<n_2<\cdots<n_N}$$

$$\det_{1\le j,k\le N}\left[x_j^{n_k}\right]\times\det_{1\le \ell,m\le N}\left[y_\ell^{n_m}\right]$$

anti-sym. in $\boldsymbol{n}$

$= 0$ if $^\exists j\neq k, n_j=n_k$

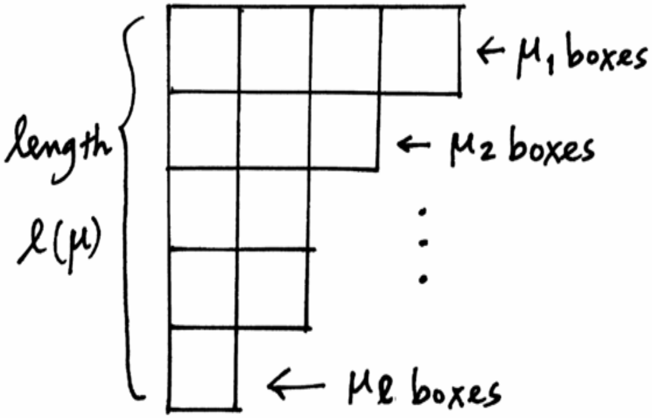
anti-sym. in $\boldsymbol{n}$

sym. in $\boldsymbol{n}$

$$F_N(\boldsymbol{x}, \boldsymbol{y}) = \sum_{0 \le n_1 < n_2 < \cdots < n_N} \prod_{j=1}^{N} \frac{1}{n_j!} \det_{1 \le j,k \le N} \left[x_j^{n_k} \right] \det_{1 \le \ell,m \le N} \left[y_\ell^{n_m} \right]$$

- For a partition μ with length $\ell(\mu) \le N$

$$\mu = (\mu_1, \mu_2, \dots, \mu_N), \quad \mu_1 \ge \mu_2 \ge \cdots \ge \mu_N \ge 0, \quad \mu_j \in \mathbb{N}$$



Young diagram

Introduce N variables $\boldsymbol{x} = (x_1, x_2, \dots, x_N)$

symmetric polynomial of $x_1, x_2, \dots, x_N$

$$s_\mu(\boldsymbol{x}) = \frac{\det_{1 \le j,k \le N}[x_j^{\mu_k + N - k}]}{\det_{1 \le j,k \le N}[x_j^{N-k}]} \qquad \text{Schur function}$$

Here we assume

$$s_\mu(\mathbf{0}) \equiv \begin{cases} 1 & \text{if } \mu = \mathbf{0} \\ 0 & \text{if } \mu \ne \mathbf{0} \end{cases}$$

$$\mathbf{0} \equiv (0, 0, \dots, 0)$$

- Note the correspondence

$$0 \le n_1 < n_2 < \cdots < n_N \quad \Longleftrightarrow \quad \mu_1 \ge \mu_2 \ge \cdots \ge \mu_N \ge 0$$

if $\qquad \mu_j = n_j - N + j,\ 1 \le j \le N$

$$F_N(\boldsymbol{x},\boldsymbol{y}) = \sum_{\mu_1\geq\cdots\geq\mu_N\geq 0}\prod_{j=1}^{N}\frac{1}{(\mu_j+N-j)!} \times \det_{1\leq j,k\leq N}[x_j^{N-k}] \times s_\mu(\boldsymbol{x}) \times \det_{1\leq \ell,m\leq N}[y_\ell^{N-m}] \times s_\mu(\boldsymbol{y}).$$

$$\det_{1\leq j,k\leq N}[x_j^{N-k}] = \prod_{1\leq j<k\leq N}(x_j-x_k) = (-1)^{N(N-1)/2}\prod_{1\leq j<k\leq N}(x_k-x_j) = (-1)^{N(N-1)/2}h_N(\boldsymbol{x})$$

$$F_N(\boldsymbol{x},\boldsymbol{y}) \equiv \det_{1\leq j,k\leq N}[e^{x_j y_k}]$$

$$= \det_{1\leq j,k\leq N}\left[\sum_{n=0}^{\infty}\frac{1}{n!}(x_j y_k)^n\right]$$

Taylor expansion **in** a determinant

$$= h_N(\boldsymbol{x})h_N(\boldsymbol{y}) \times \sum_{\mu:\ell(\mu)\leq N}\prod_{j=1}^{N}\frac{1}{(\mu_j+N-j)!}s_\mu(\boldsymbol{x})s_\mu(\boldsymbol{y})$$

multivariate Taylor expansion

Schur func. expansion

$$= h_N(\boldsymbol{x})h_N(\boldsymbol{y}) \times \left\{\prod_{j=1}^{N}\frac{1}{(N-j)!} \times 1 \times 1 + \text{higher order terms}\right\}$$

anti-symmetric in $\boldsymbol{x}$ and $\boldsymbol{y}$ (Fermion)

$$f_N(t,\boldsymbol{y}|\boldsymbol{x}) = h_N(\boldsymbol{x})h_N(\boldsymbol{y}) \times \frac{1}{(2\pi t)^{N/2}} e^{-(|\boldsymbol{x}|^2+|\boldsymbol{y}|^2)/2t} \sum_{\mu:\ell(\mu)\le N} \prod_{j=1}^{N} \frac{1}{(\mu_j+N-j)!} s_\mu\left(\frac{\boldsymbol{x}}{\sqrt{t}}\right) s_\mu\left(\frac{\boldsymbol{y}}{\sqrt{t}}\right)$$

symmetric in $\boldsymbol{x}$ and $\boldsymbol{y}$ (Boson)

Boson-Fermion correspondence in $d=1$ system

in $\frac{|\boldsymbol{x}|}{\sqrt{t}} \to 0$

$$\left(\frac{1}{\sqrt{t}}\right)^{N(N-1)/2} h_N(\boldsymbol{y})$$

$$f_N(t,\boldsymbol{y}|\boldsymbol{x}) \simeq \frac{1}{(2\pi t)^{N/2}} e^{-|\boldsymbol{y}|^2/2t} h_N\left(\frac{\boldsymbol{x}}{\sqrt{t}}\right) h_N\left(\frac{\boldsymbol{y}}{\sqrt{t}}\right) \times \prod_{j=1}^{N} \frac{1}{(N-j)!}$$

$$= \frac{t^{-N(N+1)/4}}{C_2(N)} e^{-|\boldsymbol{y}|^2/2t} h_N\left(\frac{\boldsymbol{x}}{\sqrt{t}}\right) h_N(\boldsymbol{y})$$

$$= \frac{t^{-N^2/2}}{C_2(N)} e^{-|\boldsymbol{y}|^2/2t} h_N(\boldsymbol{x}) h_N(\boldsymbol{y})$$

$$C_2(N) = (2\pi)^{N/2} \prod_{j=1}^{N} \Gamma(j)$$

$$\frac{1}{\prod_{j=1}^{N}(j-1)!} = \frac{1}{\prod_{j=1}^{N}\Gamma(j)}$$

Note that

$$\mathcal{H}(N) \simeq \mathbb{R}^{N^2} \implies \dim \mathcal{H}(N) = N^2$$

$$\mathcal{N}_N(t,\boldsymbol{x}) \equiv \int_{\mathbb{W}_N^{\mathrm{A}}} d\boldsymbol{y}\, f_N(t,\boldsymbol{y}|\boldsymbol{x})$$

$\frac{|\boldsymbol{x}|}{\sqrt{t}} \to 0$

$$\simeq \int_{\mathbb{W}_N^{\mathrm{A}}} d\boldsymbol{y}\, \frac{1}{C_2(N) t^{N/2}} e^{-|\boldsymbol{y}|^2/2t} h_N\left(\frac{\boldsymbol{x}}{\sqrt{t}}\right) h_N\left(\frac{\boldsymbol{y}}{\sqrt{t}}\right)$$

$$= \frac{1}{C_2(N)} h_N\left(\frac{\boldsymbol{x}}{\sqrt{t}}\right) \int_{\mathbb{W}_N^{\mathrm{A}}} d\frac{\boldsymbol{y}}{\sqrt{t}} e^{-|\boldsymbol{y}|^2/2t} h_N\left(\frac{\boldsymbol{y}}{\sqrt{t}}\right)$$

a version of the Selberg integral

$$2^{N/2} \prod_{j=1}^{N} \Gamma(j/2) \equiv C_1(N)$$

$$\mathcal{N}_N(t,\boldsymbol{x}) \simeq \frac{C_1(N)}{C_2(N)} h_N\left(\frac{\boldsymbol{x}}{\sqrt{t}}\right) \quad \text{in} \quad \frac{|\boldsymbol{x}|}{\sqrt{t}} \to 0$$

$$= \frac{C_1(N)}{C_2(N)} t^{-N(N-1)/4} h_N(\boldsymbol{x})$$

$$\sim t^{-N(N-1)/4} \quad \text{for a fixed } \boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}}$$

survival probab. exponent
intersection exponent

power-law decay of survival probability (nonequilibrium critical phenomenon)

Remark

- Pfaffian
For an even integer n and an antisymmetric $n \times n$ matrix $A = (a_{ij})$ we put

$$\underset{1 \le i,j \le n}{\mathrm{Pf}} \left[a_{ij} \right] = \frac{1}{(n/2)!} \sum_{\sigma: \sigma(2k-1) < \sigma(2k), 1 \le k \le n/2} \mathrm{sgn}(\sigma) a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(n-1)\sigma(n)},$$

where the summation is extended over all permutations σ of $(1, 2, \ldots, n)$ with restriction $\sigma(2k-1) < \sigma(2k)$, $k = 1, 2, \ldots, n/2$.

- de Bruin's identity

Lemma [de Bruin]
Let $z_i(x), 1 \le i \le \widehat{N}$ be an integrable piecewise continuous function on a region $\Lambda \subset \mathbb{R}$. Let $\mathbb{W}_N^{\mathrm{A}}(\Lambda) = \left\{ \boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}} : x_i \in \Lambda, 1 \le i \le N \right\}$. Then

$$\int_{\mathbb{W}_N^{\mathrm{A}}(\Lambda)} d\boldsymbol{x} \det_{1 \le i,j \le N} \left[z_i(x_j) \right] = \underset{1 \le i,j \le \widehat{N}}{\mathrm{Pf}} \left[Z_{ij} \right],$$

where where

$$\widehat{N} = \begin{cases} N, & \text{if } N = \text{even}, \\ N+1, & \text{if } N = \text{odd}, \end{cases}$$

$$I_i = \int_\Lambda z_i(x) dx,$$
$$I_{ij} = \int_{(x_1,x_2) \in \Lambda^2 : x_1 < x_2} \det \begin{bmatrix} z_i(x_1) & z_i(x_2) \\ z_j(x_1) & z_j(x_2) \end{bmatrix} dx_1 dx_2,$$

and

$$Z_{ij} = \begin{cases} I_{ij}, & \text{if } \ 1 \le i,j \le N, \\ I_i, & \text{if } \ 1 \le i \le N, j = N+1, \\ -I_j, & \text{if } \ i = N+1, 1 \le j \le N, \\ 0, & \text{if } \ i = j = N+1. \end{cases}$$

- Then the survival probability is given by a pfaffian.

$$\mathcal{N}(s, \boldsymbol{x}) = \mathop{\mathrm{Pf}}_{1 \le i,j \le \widehat{N}} \left[F_{ij}^{\mathrm{A}} \left(\frac{\boldsymbol{x}}{\sqrt{2s}} \right) \right],$$

$\boldsymbol{x}/\sqrt{2s} = (x_1/\sqrt{2s}, \dots, x_N/\sqrt{2s})$, and

$$F_{ij}^{\mathrm{A}}(\boldsymbol{y}) = \begin{cases} \Psi(y_j - y_i), & \text{if } 1 \le i, j \le N, \\ 1, & \text{if } 1 \le i \le N, j = N+1, \\ -1, & \text{if } i = N+1, 1 \le j \le N, \\ 0, & \text{if } i = j = N+1, \end{cases}$$

with

$$\Psi(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-v^2} dv.$$

3.3 $|x| \to 0$ limit and $T \to \infty$ limit

A: $|\boldsymbol{x}| \to 0$ **limit**

$$
\begin{aligned}
g_N^T(0,\mathbf{0};t,\boldsymbol{y}) &\equiv \lim_{|\boldsymbol{x}|\to 0} g_N^T(0,\boldsymbol{x};t,\boldsymbol{y}) \\
&= \lim_{|\boldsymbol{x}|\to 0} \frac{f_N(t,\boldsymbol{y}|\boldsymbol{x})\mathcal{N}_N(T-t,\boldsymbol{y})}{\mathcal{N}_N(T,\boldsymbol{x})} \\
&= \mathcal{N}_N(T-t,\boldsymbol{y}) \lim_{|\boldsymbol{x}|\to 0} \frac{f_N(t,\boldsymbol{y}|\boldsymbol{x})}{\mathcal{N}_N(T,\boldsymbol{x})} \\
&= \mathcal{N}_N(T-t,\boldsymbol{y}) \lim_{|\boldsymbol{x}|\to 0} \frac{\dfrac{t^{-N^2/2}}{C_2(N)} e^{-|\boldsymbol{y}|^2/2t} h_N(\boldsymbol{x}) h_N(\boldsymbol{y})}{\dfrac{C_1(N)}{C_2(N)} T^{-N(N-1)/4} h_N(\boldsymbol{x})} \\
&= \frac{T^{N(N-1)/4} t^{-N^2/2}}{C_1(N)} h_N(\boldsymbol{y}) e^{-|\boldsymbol{y}|^2/2t} \mathcal{N}_N(T-t,\boldsymbol{y})
\end{aligned}
$$

Remark. If we set

$$t = T,$$

then

$$\mathcal{N}(T-T, \boldsymbol{y}) = \mathcal{N}(0, \boldsymbol{y}) = 1 \quad \text{for} \quad \boldsymbol{y} \in \mathbb{W}_N^{\mathrm{A}}.$$

Therefore

$$g_N^T(0, \mathbf{0}; T, \boldsymbol{y}) = \frac{T^{-N(N+1)/4}}{C_1(N)} h_N(\boldsymbol{y}) e^{-|\boldsymbol{y}|^2/2t}.$$

GOE eigenvalue distribution with variance $\sigma^2 = T$

$$\frac{N(N+1)}{2} = N + \frac{N(N-1)}{2}$$

$\mathcal{S}(N) \equiv$ space of $N \times N$ real symmetric matrices

$$\dim \mathcal{S}(N) = \frac{N(N+1)}{2}$$

B: $T \to \infty$ limit

$$\begin{aligned} p_N(t, \boldsymbol{y}|\boldsymbol{x}) &\equiv \lim_{T\to\infty} g_N^T(0, \boldsymbol{x}; t, \boldsymbol{y}) \\ &= f_N(t, \boldsymbol{y}|\boldsymbol{x}) \lim_{T\to\infty} \frac{\mathcal{N}_N(T-t, \boldsymbol{y})}{\mathcal{N}_N(T, \boldsymbol{x})} \\ &= f_N(t, \boldsymbol{y}|\boldsymbol{x}) \lim_{T\to\infty} \frac{\frac{C_1(N)}{C_2(N)}(T-t)^{-N(N-1)/4} h_N(\boldsymbol{y})}{\frac{C_1(N)}{C_2(N)} T^{-N(N-1)/4} h_N(\boldsymbol{x})} \\ &= f_N(t, \boldsymbol{y}|\boldsymbol{x}) \frac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})}. \end{aligned}$$

$$\begin{aligned} p_N(t, \boldsymbol{y}|\mathbf{0}) &\equiv \lim_{T\to\infty} g_N^T(0, \mathbf{0}; t, \boldsymbol{y}) \\ &= \lim_{T\to\infty} \frac{T^{N(N-1)/4} t^{-N^2/2}}{C_1(N)} h_N(\boldsymbol{y}) e^{-|\boldsymbol{y}|^2/2t} \mathcal{N}_N(T-t, \boldsymbol{y}) \\ &= \frac{t^{-N^2/2}}{C_1(N)} h_N(\boldsymbol{y}) e^{-|\boldsymbol{y}|^2/2t} \lim_{T\to\infty} T^{N(N-1)/4} \frac{C_1(N)}{C_2(N)} (T-t)^{-N(N-1)/4} h_N(\boldsymbol{y}) \\ &= \frac{t^{-N^2/2}}{C_2(N)} h_N(\boldsymbol{y})^2 e^{-|\boldsymbol{y}|^2/2t}. \end{aligned}$$

GUE eigenvalue distribution with variance $\sigma^2 = t$

3.4 Transition from GUE to GOE in Noncolliding BM with duration T starting from $\mathbf{0}$

- For

$$g_N^T(0,\mathbf{0};t,\boldsymbol{x}) = \frac{T^{N(N-1)/4}t^{-N^2/2}}{C_1(N)} h_N(\boldsymbol{x}) e^{-|\boldsymbol{x}|^2/2t} \mathcal{N}_N(T-t,\boldsymbol{x}),$$

let

$$\boldsymbol{X}^T(t) = (X_1^T(t), X_2^T(t), \dots, X_N^T(t)), \quad t \in [0,T]$$

be the process "noncolliding BM with duration T starting from $\mathbf{0}$", whose transition probability density is given by g_N^T.

(i) When $0 < t \ll T$ for finite $|\boldsymbol{x}| < \infty$

$$\mathcal{N}_N(T-t,\boldsymbol{x}) \simeq \frac{C_1(N)}{C_2(N)} T^{-N(N-1)/4} h_N(\boldsymbol{x}).$$

Then

$$g_N^T(0,\mathbf{0};t,\boldsymbol{x}) \simeq \frac{t^{-N^2/2}}{C_2(N)} h_N^2(\boldsymbol{x}) e^{-|\boldsymbol{x}|^2/2t},$$

that is, well approximated by GUE.

(ii) When $t \to T$, that is, $t \simeq T$,

$$\begin{aligned} g_N^T(0,\mathbf{0};t,\boldsymbol{x}) &\simeq g_N^T(0,\mathbf{0};T,\boldsymbol{x}) \\ &= \frac{t^{-N(N+1)/4}}{C_1(N)} h_N(\boldsymbol{x}) e^{-|\boldsymbol{x}|^2/2t}. \end{aligned}$$

That is, well approximated by GUE.

Transition from GUE to GOE
temporally inhomogeneous system
Two-matrix model of Pandey and Mehta

3.5 Generalized Imhof relation

- $T \to \infty$ limit:

$$p_N(t, \boldsymbol{y}|\boldsymbol{x}) = \begin{cases} f_N(t, \boldsymbol{y}|\boldsymbol{x}) \dfrac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})} & \boldsymbol{x} \neq \boldsymbol{0} \\ \dfrac{t^{-N^2/2}}{C_2(N)} h_N(\boldsymbol{y})^2 e^{-|\boldsymbol{y}|^2/2t} & \boldsymbol{x} = \boldsymbol{0} \end{cases}$$

"noncolliding BM" = temporally homogeneous system

$$\boldsymbol{X}(t) = (X_1(t), X_2(t), \dots, X_N(t)), \quad t \in [0, \infty)$$

- For finite $T < \infty$,

$$\boldsymbol{X}^T(t) = (X_1^T(t), X_2^T(t), \dots, X_N^T(t)), \quad t \in [0, T]$$

with the t.p.d.

$$g_N^T(0, \boldsymbol{0}; t, \boldsymbol{y}) = \frac{T^{N(N-1)/4} t^{-N^2/2}}{C_1(N)} h_N(\boldsymbol{y}) e^{-|\boldsymbol{y}|^2/2t} \mathcal{N}_N(T-t, \boldsymbol{y}).$$

- generalized Imhof relation (originally BES$^{(3)}$ vs. meander)

$$
\begin{aligned}
g_N^T(0, \mathbf{0}; T, \boldsymbol{y}) &= \frac{T^{N(N-1)/4} T^{-N^2/2}}{C_1(N)} h_N(\boldsymbol{y}) e^{-|\boldsymbol{y}|^2/2T} \mathcal{N}_N(T-T, \boldsymbol{y}) \\
&= \frac{C_2(N)}{C_1(N)} T^{N(N-1)/4} \frac{1}{h_N(\boldsymbol{y})} p_N(T, \boldsymbol{y}|\mathbf{0})
\end{aligned}
$$

$$\Downarrow$$

$$
\mathbb{P}(\boldsymbol{X}^T(\cdot) \in d\omega) = \frac{C_2(N)}{C_1(N)} T^{N(N-1)/4} \frac{1}{h_N(\omega(T))} \mathbb{P}(\boldsymbol{X}(\cdot) \in d\omega)
$$

absolute continuity

3.6 Stochastic-calculus derivation of Harish-Chandra-Itzykson-Zuber integral

- 2nd Day: GUE-process

$$\Xi(t) = (s_{jk}(t) + i a_{jk}(t))_{1 \le j,k \le N}, \quad t \in [0, \infty).$$

eigenvalue process = the Dyson model

$$\boldsymbol{X}(t) = (X_1(t), \dots, X_N(t)), \quad t \in [0, \infty),$$
$$dX_j(t) = dB_j(t) + \sum_{1 \le k \le N: k \ne j} \frac{dt}{X_j(t) - X_k(t)}, \quad 1 \le j \le N, \quad t \in [0, \infty).$$

- Let $\beta_{jk}(t), 1 \le j < k \le N$ be indep. one-dim. Brownian bridges with duration T,

$$\beta_{jk}(t) = \widetilde{B}_{jk}(t) - \int_0^t \frac{\beta_{jk}(s)}{T-s} ds, \quad 0 \le t \le T,$$

and set

$$\alpha_{jk}(t) = \begin{cases} \frac{1}{\sqrt{2}} \beta_{jk}(t), & j < k \\ 0, & j = k \\ -\frac{1}{\sqrt{2}} \beta_{kj}(t), & j > k. \end{cases}$$

- Consider the $\mathcal{H}(N)$-valued (*i.e.* $N \times N$-hermitian-matrix-valued) diffusion process

$$\Xi^T(t) = (s_{jk}(t) + i \alpha_{jk}(t))_{1 \le j,k \le N}, \quad t \in [0, \infty).$$

Let

$$\widetilde{\boldsymbol{X}}^T(t) = (\widetilde{X}_1^T(t), \dots, \widetilde{X}_N^T(t)), \quad t \in [0, \infty)$$

be the eigenvalue process of $\Xi^T(t)$.

- We can prove that

  $\widetilde{\boldsymbol{X}}^T(t)$ and $\boldsymbol{X}(t)$ satisfy the generalized Imhof relation

$$\widetilde{\boldsymbol{X}}^T(t) = \boldsymbol{X}^T(t), \quad t \in [0, T] \ : \quad \text{equivalent}$$

eigenvalue process of $\Xi^T(t)$

temporally inhomogeneous noncolliding BM with duration T starting from $\mathbf{0}$

- For $\boldsymbol{X}^T(t)$: the t.p.d. is given by

$$
\begin{aligned}
g_N^T(0, \mathbf{0}; t, \boldsymbol{y}) &= \frac{T^{N(N-1)/4} t^{-N^2/2}}{C_1(N)} h_N(\boldsymbol{y}) e^{-|\boldsymbol{y}|^2/2t} \mathcal{N}_N(T-t, \boldsymbol{y}) \\
&= \frac{T^{N(N-1)/4} t^{-N^2/2}}{C_1(N)} h_N(\boldsymbol{y}) e^{-|\boldsymbol{y}|^2/2t} \int_{\mathbb{W}_N^{\mathrm{A}}} d\boldsymbol{z} \det_{1 \leq j,k \leq N} \left[\frac{1}{\sqrt{2\pi(T-t)}} \exp\left\{ -\frac{(y_j - z_k)^2}{2(T-t)} \right\} \right] \\
&= \frac{(2\pi)^{-N/2}}{C_1(N)} \sigma^{-N} \alpha^{N(N+1)/4} h_N(\boldsymbol{y}) \int_{\mathbb{W}_N^{\mathrm{A}}} d\boldsymbol{a}\, e^{-\alpha|\boldsymbol{a}|^2/2} \det_{1 \leq j,k \leq N} \left[\exp\left\{ -\frac{1}{2\sigma^2} (y_j - a_k)^2 \right\} \right],
\end{aligned}
$$

where

$$
\frac{t}{T} z_j = a_j, \quad \frac{t(T-t)}{T} = \sigma^2, \quad \frac{T}{t^2} = \alpha.
$$

- For $\widetilde{\boldsymbol{X}}^T(t)$

$$\widetilde{g}_N^T(0, \mathbf{0}; t, \boldsymbol{y}) = \frac{\widetilde{C}_2(N)}{C_2(N)} h_N(\boldsymbol{y})^2 \int_{\mathrm{U}(N)} dU \, q_N^T(0, O; t, U^\dagger \Lambda_{\boldsymbol{y}} U),$$

where dU is the Haar measure of the space $\mathrm{U}(N)$ normalized as $\int_{\mathrm{U}(N)} dU$
$\Lambda_{\boldsymbol{y}} = \mathrm{diag}\{y_1, \dots, y_N\}$,
$\widetilde{C}_2(N) = 2^{N/2}\pi^{N^2/2}$, O=$N \times N$ zero matrix.
$q_N^T(0, O; t, H) =$ t.p.d. for $\Xi^T(t)$ from O to $H \in \mathcal{H}(N)$.

- Introduce
$\mathcal{H}(N)$-valued process $\Theta^{(1)}(t) = (\theta_{jk}^{(1)}(t))_{1 \le j,k \le N}$ with

$$\theta_{jk}^{(1)}(t) = \begin{cases} \frac{1}{\sqrt{2}} \left\{ B_{jk}(t) - \frac{t}{T} B_{jk}(T) \right\} + \frac{i}{\sqrt{2}} \beta_{jk}(t), & j < k \\ B_{jj}(t) - \frac{t}{T} B_{jj}(T), & j = k \end{cases}$$

$\mathcal{S}(N)$-valued process (*i.e.* $N \times N$- real symmetric matrix-valued process
$\Theta^{(2)}(t) = (\theta_{jk}^{(2)}(t))_{1 \le j,k \le N}$ with

$$\theta_{jk}^{(2)}(t) = \begin{cases} \frac{t}{\sqrt{2}T} B_{jk}(T), & j < k \\ \\ \frac{t}{T} B_{jj}(T), & j = k \end{cases}$$

Then

$$\Xi(t) = \Theta^{(1)}(t) + \Theta^{(2)}(t).$$

- $\Theta^{(1)}(t)$: the GUE process with $\mathrm{E}[\theta_{jj}^{(1)}(t)^2] = \sigma^2 = \dfrac{t(T-t)}{T}$

  $\Theta^{(2)}(t)$: the GOE process with $\mathrm{E}[\theta_{jj}^{(2)}(t)^2] = \dfrac{1}{\alpha} = \dfrac{t^2}{T}$

$$\Theta^{(1)}(t) \quad \perp \quad \Theta^{(2)}(t)$$

$$\begin{aligned} q_N^T(0, O; t, H) &= \int_{\mathcal{S}(N)} \nu(dA)\, \mu^{\mathrm{GOE}}\left(A, \frac{1}{\alpha}\right) \mu^{\mathrm{GUE}}(H - A, \sigma^2) \\ &= \frac{\sigma^{-N^2} \alpha^{N(N+1)/4}}{\widetilde{C}_1(N)\widetilde{C}_2(N)} \frac{\widetilde{C}_1(N)}{C_1(N)} \int_{\mathbb{W}_N^{\mathrm{A}}} d\boldsymbol{a}\, h_N(\boldsymbol{a}) \exp\left\{ -\frac{\alpha}{2}|\boldsymbol{a}|^2 - \frac{1}{2\sigma^2} \mathrm{Tr}(H - \Lambda_\alpha)^2 \right\}, \end{aligned}$$

where

$$\begin{aligned} &\widetilde{C}_1(N) = 2^{N/2} \pi^{N(N+1)/4}, \\ &\nu(dA) = \frac{\widetilde{C}_1(N)}{C_1(N)} h_N(\boldsymbol{a}) dV\, d\boldsymbol{a}, \end{aligned}$$

dV = the Haar measure of the space $\mathrm{O}(N)$ normalized as $\int_{\mathrm{O}(N)} dV = 1$.

- $\boldsymbol{X}^T(t) = \widetilde{\boldsymbol{X}}^T(t), \quad t \in [0, T]$

$$\frac{C_1(N)\sigma^{N^2-N}}{(2\pi)^{N/2} h_N(\boldsymbol{y})} \int_{\mathbb{W}_N^{\mathrm{A}}} d\boldsymbol{a}\, e^{-\alpha|\boldsymbol{a}|/2} \det_{1 \leq j,k \leq N} \left[\exp\left\{ -\frac{1}{2\sigma^2}(y_j - a_k)^2 \right\} \right]$$
$$= \int_{\mathbb{W}_N^{\mathrm{A}}} d\boldsymbol{a}\, h_N(\boldsymbol{a}) e^{-\alpha|\boldsymbol{a}|^2/2} \int_{\mathrm{U}(N)} dU \, \exp\left\{ -\frac{1}{2\sigma^2} \mathrm{Tr}(U^\dagger \Lambda_{\boldsymbol{y}} U - \Lambda_{\boldsymbol{a}})^2 \right\}$$

for every $\alpha > 0$.

- Harish-Chandra-Itzykson-Zuber integral

$$\frac{C_1(N)\sigma^{N^2}}{h_N(\boldsymbol{y}) h_N(\boldsymbol{a})} \det_{1 \leq j,k \leq N} \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2}(y_j - a_k)^2 \right\} \right]$$
$$= \int_{\mathrm{U}(N)} dU \, \exp\left\{ -\frac{1}{2\sigma^2} \mathrm{Tr}(U^\dagger \Lambda_{\boldsymbol{y}} U - \Lambda_{\boldsymbol{a}})^2 \right\}.$$