Moments of vicious walkers and Möbius graph expansions

Makoto Katori* and Naoaki Komatsuda†

Department of Physics, Faculty of Science and Engineering, Chuo University, Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan

(Received 17 December 2002; published 27 May 2003)

A system of Brownian motions in one dimension all started from the origin and conditioned never to collide with each other in a given finite time interval $(0,T]$ is studied. The spatial distribution of such vicious walkers can be described by using the repulsive eigenvalue statistics of random Hermitian matrices and it was shown that the present vicious walker model exhibits a transition from the Gaussian unitary ensemble (GUE) statistics to the Gaussian orthogonal ensemble (GOE) statistics as the time t goes on from 0 to T. In the present paper, we characterize this GUE-to-GOE transition by presenting the graphical expansion formula for the moments of positions of vicious walkers. In the GUE limit $t\to 0$, only the ribbon graphs contribute and the problem is reduced to the classification of orientable surfaces by genus. Following the time evolution of the vicious walkers, however, the graphs with twisted ribbons, called Möbius graphs, increase their contribution to our expansion formula, and we have to deal with the topology of nonorientable surfaces. Application of the recent exact result of dynamical correlation functions yields closed expressions for the coefficients in the Möbius expansion using the Stirling numbers of the first kind.

DOI: 10.1103/PhysRevE.67.051110 PACS number(s): 05.40.−a, 02.50.Ey, 02.10.Ox

I. INTRODUCTION

The statistics of a set of one-dimensional random walks *conditioned never to collide* in a given time interval, say T, has its own importance in statistical physics, since, if we set $T\to\infty$, it realizes the one-dimensional Fermi statistics [1], and with $T<\infty$ it is used to analyze the models for wetting and melting phenomena [2]. We will refer to such noncolliding random walks and the continuum counterpart, noncolliding Brownian motions, simply as *vicious walks* following the terminology used by Fisher [2]. For the pioneering work on vicious walker models, see Refs. [3–7]. A generic setting of vicious walk problems is discussed in Ref. [8]. Recently the interest on the vicious walks in mathematical physics is renewed and growing very rapidly, for close relationships of the vicious walk problem with the study of ensembles of Young tableaux and the symmetric functions [9–11], the theories of orthogonal polynomials and random matrices [12,13], and some topics of representation theory and probability theory [14–16] have been clarified.

In an earlier paper [17], the continuum limit of noncolliding random walks on a lattice was taken by letting the temporal and spatial units $\Delta t,\Delta x$ go to zero with the relation $\Delta t\propto(\Delta x)^2$ and a system of noncolliding Brownian motions was derived. Since each random walk tends to be a Brownian motion in this *diffusion scaling limit*, such a construction of noncolliding Brownian motion is plausible, and indeed mathematical rigor can be established as a functional central limit theorem of vicious walks [16]. The important fact is that the repulsive interaction among the obtained Brownian particles is no longer contact interaction as in the original vicious random walks on a lattice but is long ranged. The origin of this long-ranged interaction is the restriction of allowed configurations by the noncolliding condition, so that we can say it is an example of entropy-origin effective force. Moreover, the following setting was considered in our vicious walks: (i) even after taking the continuum limit, still we let the time interval, in which the noncolliding condition is imposed, be finite $T<\infty$, and (ii) all the Brownian particles are assumed to start from an origin and the time interval of noncolliding is $(0,T]$.

In this setting, the process is temporally inhomogeneous. At the very early stage $t\ll T$ the repulsive interaction should be strong, since the Brownian motions will be restricted so that they will not collide for a long time period up to time T in the future. As the time t goes on, the strength of the repulsion is decreasing as is the remaining time until T, and attains its minimum at the final time $t=T$, at which there is no more restriction of motion in the future $t>T$. It was Dyson's idea that such systems of Brownian motions with long-ranged repulsion could have the equilibrium states, which can be described by the distribution functions of the eigenvalues of random matrices in the ensembles appropriately specified by the symmetry of the system [18]. The previous paper [17] showed that the spatial distribution of vicious walkers at $t\ll T$ realizes the eigenvalue statistics of the Gaussian unitary ensemble (GUE), one at $t=T$ does that of the Gaussian orthogonal ensemble (GOE), and the GUE-to-GOE transition is observed in the intermediate time, which is equivalent to the transition studied in the two-matrix model by Pandey and Mehta [19,20].

In the present paper, we will characterize this transition in distribution by calculating the moments of the positions of N particles as functions of time t,

$$M_{N,T}(t,k)=\left\langle \sum_{j=1}^{N} x_j^{2k}\right\rangle_t, \quad k=1,2,3,\ldots, \tag{1}$$

where x_j denotes the position of jth vicious walker and $\langle\cdot\rangle_t$ the average at time $t\in(0,T]$. In addition to such an interest of statistical physics, we can put emphasis that our

*Email address: katori@phys.chuo-u.ac.jp

†Email address: komatuda@phys.chuo-u.ac.jp

present study possesses another importance as an interesting application of the graphical expansion theory. In high energy physics, graphical expansions for the matrix models of SU(N) gauge theory were studied using *ribbon graphs* representing propagators and it was shown that the dominant graphs for large N are the planar ones and the leading corrections come from the graphs embedded in a torus, which are depressed by a factor $1/N^2$ with respect to the planar graphs [21,22]. On the other hand, it was clarified that in the graphical expansion of SO(N) gauge theory the leading corrections only are depressed by a factor $1/N$ for large N with respect to the dominant planar graphs, in which propagators can be represented by *twisted ribbons* [23,24]. In the gauge theory, the presence of non-Gaussian interaction terms of cubic or higher power is crucial and it leads to *triangulations of random surfaces*. Although the Gaussian matrix models associated with the present vicious walker model are not related to the triangulation problem of random surfaces and only give purely enumerative problems of surfaces, the proper structures of large N expansions in SU(N) and SO(N) gauge theories are also found in the GUE and GOE models, respectively. We can show that, if $T\to\infty$ in our vicious walker model, the moments of the walker positions (1) can be calculated by the graphical expansion method of the GUE using the ribbon graphs and the results are given in the form of power series in the inverse of the *square* of matrix size N,

$$M_{\mathrm{GUE}}(k)\propto N^{k+1}\sum_{g\geqslant 0}\varepsilon_g(k)\left(\frac{1}{N^2}\right)^g.$$

Here the coefficients $\varepsilon_g(k)$ are the numbers of *orientable surfaces of genus g* made from $2k$-gon by some specified procedure [25] (see also Ref. [26] and Sec. 5.5 in Ref. [20]). On the other hand, for the GOE we have to take into account the *nonorientable surfaces* as well as the orientable ones and the expansion is in the form of power series in $1/N$. In other words, in order to generate necessary surfaces, we need to use twisted ribbons, whose type of graphs is now called of *Möbius graphs* [27,28]. For moments (1) of our vicious walkers, we will perform the Möbius graph expansion in the present paper. Now the weights of the graphs are depending on the time t; in the limit $t\to 0$ all the weights on the twisted ribbons are zero, but they are growing as time t goes on, and at $t=T$ twisted ribbons are equally weighted as untwisted ones. This gives another characterization of the temporally inhomogeneity of the process and the GUE-to-GOE transition.

Quite recently Nagao, Tanemura, and one of the present authors applied the method of skew orthogonal polynomials and quaternion determinants developed for the multimatrix models [29–31] to the vicious walk problem and derived the quaternion determinantal expressions for dynamical correlation functions [32]. Using this result, we will present an expression of the coefficients in our expansion of moments using the Stirling number of the first kind.

The paper is organized as follows. In Sec. II, we briefly review the previous results reported in Ref. [17] and give the precise definition of the moments which we will study. Graphical representations are demonstrated in Sec. III. An application of the result of Ref. [32] is given in Sec. IV to give the expression for the coefficients of expansion and some concluding remarks are given in Sec. V. Appendixes A and B are prepared to derive the expression of the density function used in Sec. IV and the $1/N$ expansions of the one-point Green function discussed in Sec. V, respectively.

II. VICIOUS WALKS AND THE MATRIX MODEL

We study a continuum model of vicious walks, the noncolliding Brownian motions in the time interval $(0,T]$, constructed in Refs. [16,17] as the diffusion scaling limit of vicious random walks on a lattice. First we briefly review our previous results. The configuration space of the present N vicious walkers is $\mathbf{R}^N_<=\{\mathbf{x}=(x_1,x_2,\dots,x_N)\in\mathbf{R}^N; x_1<x_2<\cdots<x_N\}$, where $\mathbf{R}$ is a set of all real numbers. The probability density of vicious walkers at time $t\in(0,T]$ with the initial condition that all walkers start from the origin $\mathbf{0}$ is denoted by $\rho_{N,T}(t,\mathbf{x})$. It was given as

$$\rho_{N,T}(t,\mathbf{x})=Ce^{-|\mathbf{x}|^2/2t}h_N(\mathbf{x})\mathcal{N}_N(T-t,\mathbf{x}),\tag{2}$$

with $C=2^{-N/2}T^{N(N-1)/4}t^{-N^2/2}/\prod_{j=1}^N\Gamma(j/2)$ for $\mathbf{x}\in\mathbf{R}^N_<$, where $\Gamma(z)$ is the gamma function, $h_N(\mathbf{x})=\prod_{1\leqslant j<\ell\leqslant N}(x_\ell-x_j)$ and

$$\mathcal{N}_N(s,\mathbf{x})=\int_{\mathbf{R}^N_<}d\mathbf{y}\det_{1\leqslant j,\ell\leqslant N}\left(\frac{1}{\sqrt{2\pi s}}e^{-(x_j-y_\ell)/2s}\right).$$

By using the Harish-Chandra (Itzykson-Zuber) integral formula [33–35], we will see that $\rho_{N,T}(t,\mathbf{x})$ is proportional to the integral

$$h_N(\mathbf{x})^2\int dU\int dA\,\exp[-\mathrm{tr}\mathcal{H}(U^\dagger XU,A)],\tag{3}$$

with

$$\mathcal{H}(H,A)=\frac{T}{2t(T-t)}H^2-\frac{T}{t(T-t)}HA+\frac{T^2}{2t^2(T-t)}A^2,$$

where X is the $N\times N$ diagonal matrix with $X_{j\ell}=x_j\delta_{j\ell}$, and the integrals $\int dU$ and $\int dA$ are taken over the groups of $N\times N$ unitary matrices $\{U\}$ and real symmetric matrices $\{A\}$, respectively. The proportional coefficient is determined so that the probability density is normalized. On the other hand, the integral over A in Eq. (3) can be regarded as the convolution of the Gaussian distribution of complex Hermitian matrices H with variance $t(1-t/T)$ and that of real symmetric matrices A with variance t^2/T, and thus we have the expression

$$\rho_{N,T}(t,\mathbf{x}) \propto h_N(\mathbf{x})^2 \int dU \mu_{N,T}(t,U^\dagger XU), \tag{4}$$

with

$$\mu_{N,T}(t,H) \propto \exp\left(-\sum_{j,\ell} \left\{ \frac{(H^{\mathrm{R}}_{j\ell})^2}{2t} + \frac{(H^{\mathrm{I}}_{j\ell})^2}{2t(1-t/T)} \right\} \right), \tag{5}$$

where and in the following we use the abbreviations z^{R} and z^{I} for the real and the imaginary part of the complex number z, respectively, i.e., $z = z^{\mathrm{R}} + iz^{\mathrm{I}}$ for $z \in \mathbf{C}$ with $i = \sqrt{-1}$. Remark that, if we set

$$c = \sqrt{\frac{t(2T-t)}{T}} \tag{6}$$

and $\alpha^2 = 1 - t/T$, $c^N \mu_{N,T}(t,cH)$ is equal to the probability density of the two-matrix model of Pandey and Mehta with the parameter α [19,20]. Corresponding to changing the parameter α from 1 to 0 in the Pandey-Mehta two-matrix model, a GUE-to-GOE transition occurs in the time development of particle distribution in our vicious walks.

Now we define the quantity, which we will study in the present paper; the moment of particle positions in the vicious walks. Since distribution (2) of $\mathbf{x}$ is symmetric about the origin $\mathbf{0}$, all of the odd moments vanish. The even moments are defined and denoted as follows:

$$M_{N,T}(t,k) = \left\langle \sum_{j=1}^{N} x_j^{2k} \right\rangle_t = \int_{\mathbf{R}^N_<} d\mathbf{x} \sum_{j=1}^{N} x_j^{2k} \rho_{N,T}(t,\mathbf{x}) \tag{7}$$

for $k = 1,2, \ldots$.

III. GRAPHICAL EXPANSIONS

A. The Wick formula

First we notice that Eq. (4) is invariant under any permutation of $x_1, x_2, \ldots, x_N$. Then Eq. (7) is written as

$$M_{N,T}(t,k) \propto \frac{1}{N!} \int_{\mathbf{R}^N} d\mathbf{x} h_N(\mathbf{x})^2 \int dU \sum_{j=1}^{N} x_j^{2k} \mu_{N,T}(t,U^\dagger XU). \tag{8}$$

Next we introduce the integration measure for the $N\times N$ complex Hermitian matrices,

$$dH = \prod_{1\le j\le \ell \le N} dH^{\mathrm{R}}_{j\ell} \prod_{1\le j<\ell\le N} dH^{\mathrm{I}}_{j\ell}.$$

Since $dH \propto dU \times h_N(\mathbf{x})^2 d\mathbf{x}$, if $\mathbf{x} = (x_1, \ldots, x_N)$ are the eigenvalues of H and $d\mathbf{x} = \prod_{j=1}^{N} dx_j$ (e.g., see Ref. [20]), and $\sum_{j=1}^{N} x_j^{2k} = \mathrm{tr} H^{2k}$ for $H = U^\dagger XU$ with any unitary matrix U, Eq. (8) with Eq. (5) becomes

$$M_{N,T}(t,k) = \langle \mathrm{tr} H^{2k} \rangle, \tag{9}$$

where $\langle f \rangle = \int dH f \mu_{N,T}(t,H)$ for functions f of the elements of H with

$$\mu_{N,T}(t,H) = \prod_{j=1}^{N} \frac{e^{-(H^{\mathrm{R}}_{jj})^2/2t}}{\sqrt{2\pi t}} \prod_{1\le j<\ell\le N} \frac{e^{-(H^{\mathrm{R}}_{j\ell})^2/t}}{\sqrt{\pi t}} \times \prod_{1\le j<\ell\le N} \frac{e^{-(H^{\mathrm{I}}_{j\ell})^2/t(1-t/T)}}{\sqrt{\pi t(1-t/T)}}. \tag{10}$$

Note that

$$\mathrm{tr}(H^{2k}) = \sum_{j_1,j_2,\ldots,j_{2k}} H_{j_1 j_2} H_{j_2 j_3} \cdots H_{j_{2k-1} j_{2k}} H_{j_{2k} j_1},$$

where the sum is taken over all N^{2k} combinations of indices $j_1, j_2, \ldots, j_{2k}$, and that $H_{j\ell} = H^{\mathrm{R}}_{j\ell} + iH^{\mathrm{I}}_{j\ell}$ with the Hermitian condition $H^{\mathrm{R}}_{\ell j} = H^{\mathrm{R}}_{j\ell}, H^{\mathrm{I}}_{\ell j} = -H^{\mathrm{I}}_{j\ell}$, the integrand $\mathrm{tr} H^{2k}$ in Eq. (9) is a polynomial of the N^2-independent random variables $\{H^{\mathrm{R}}_{j\ell}; 1\le j\le \ell\le N\} \cup \{H^{\mathrm{I}}_{j\ell}; 1\le j<\ell\le N\}$. Since probability density (10) is a product of independent Gaussian integration kernels, we can apply the Wick formula with the variances

$$\langle (H^{\mathrm{R}}_{j\ell})^2 \rangle = \frac{t}{2}(1+\delta_{j\ell}), \quad \langle H^{\mathrm{R}}_{j\ell} H^{\mathrm{I}}_{j\ell} \rangle = 0,$$

$$\langle (H^{\mathrm{I}}_{j\ell})^2 \rangle = \frac{t(1-t/T)}{2}(1-\delta_{j\ell}), \tag{11}$$

for $1\le j\le \ell\le N$, where $\delta_{j\ell}$ is Kronecker's delta.

We can readily prove that Eq. (11) is equivalent to

$$\langle H_{j\ell} H_{mn} \rangle = \frac{c^2}{2}(\delta_{jn}\delta_{\ell m} + \gamma \delta_{jm}\delta_{\ell n}), \tag{12}$$

where c is given by Eq. (6) and

$$\gamma = \frac{t}{2T-t}. \tag{13}$$

The Wick formula for Eq. (9) is thus

$$M_{N,T}(t,k) = \sum_{j_1,j_2,\ldots,j_{2k}} \sum_{\pi\in S_{2k}:\mathrm{R}} \langle H_{j_{\pi(1)} j_{\pi(1)+1}} H_{j_{\pi(2)} j_{\pi(2)+1}} \rangle \langle H_{j_{\pi(3)} j_{\pi(3)+1}} H_{j_{\pi(4)} j_{\pi(4)+1}} \rangle \cdots \langle H_{j_{\pi(2k-1)} j_{\pi(2k-1)+1}} H_{j_{\pi(2k)} j_{\pi(2k)+1}} \rangle, \tag{14}$$

with the identification $j_{2k+1}=j_1$, where the first sum is taken over all N^{2k} combinations of indices $j_1,j_2,\ldots,j_{2k}$, and the second one over the set of permutations S_{2k} of $\{1,2,\ldots,2k\}$ with the restriction

$$\text{R:}\quad \pi(1)<\pi(3)<\cdots<\pi(2k-1),$$

$$\pi(2j-1)<\pi(2j),\quad 1\leq j\leq k.$$

The total number of the terms in the second summation is $(2k-1)!!$.

B. An example: The fourth moment

In this section, by performing calculation of the fourth moment $M_{N,T}(t,2)=\langle \mathrm{tr}H^4\rangle$, we will demonstrate how to obtain graphical expansions from the Wick formula (14) with variance (12). We start from the Wick formula for $M_{N,T}(t,2)$,

$$M_{N,T}(t,2)=\sum_{j_1,j_2,j_3,j_4}\{\langle H_{j_1j_2}H_{j_2j_3}\rangle\langle H_{j_3j_4}H_{j_4j_1}\rangle + \langle H_{j_1j_2}H_{j_3j_4}\rangle\langle H_{j_2j_3}H_{j_4j_1}\rangle + \langle H_{j_1j_2}H_{j_4j_1}\rangle \times\langle H_{j_2j_3}H_{j_3j_4}\rangle\},$$

which has $(2\times 2-1)!!=3$ terms in the summand of $j_1,\ldots,j_4$. Substitution of Eq. (12) and binomial expansion give the $3\times 2^2=12$ terms in the form

$$M_{N,T}(t,2)=\left(\frac{c^2}{2}\right)^2\sum_{\ell=1}^{12}L_\ell,$$

with

$$L_1=\sum_{j_1,j_2,j_3,j_4}\delta_{j_1j_3}\delta_{j_2j_2}\delta_{j_3j_1}\delta_{j_4j_4},\quad L_2=\sum_{j_1,j_2,j_3,j_4}\delta_{j_1j_3}\delta_{j_2j_2}\delta_{j_3j_4}\delta_{j_4j_1}\gamma,$$

$$L_3=\sum_{j_1,j_2,j_3,j_4}\delta_{j_1j_2}\delta_{j_2j_3}\delta_{j_3j_1}\delta_{j_4j_4}\gamma,\quad L_4=\sum_{j_1,j_2,j_3,j_4}\delta_{j_1j_2}\delta_{j_2j_3}\delta_{j_3j_4}\delta_{j_4j_1}\gamma^2,$$

$$L_5=\sum_{j_1,j_2,j_3,j_4}\delta_{j_1j_4}\delta_{j_2j_3}\delta_{j_2j_1}\delta_{j_3j_4},\quad L_6=\sum_{j_1,j_2,j_3,j_4}\delta_{j_1j_4}\delta_{j_2j_3}\delta_{j_2j_4}\delta_{j_3j_1}\gamma,$$

$$L_7=\sum_{j_1,j_2,j_3,j_4}\delta_{j_1j_3}\delta_{j_2j_4}\delta_{j_2j_1}\delta_{j_3j_4}\gamma,\quad L_8=\sum_{j_1,j_2,j_3,j_4}\delta_{j_1j_3}\delta_{j_2j_4}\delta_{j_2j_4}\delta_{j_3j_1}\gamma^2,$$

$$L_9=\sum_{j_1,j_2,j_3,j_4}\delta_{j_1j_1}\delta_{j_2j_4}\delta_{j_2j_4}\delta_{j_3j_3},\quad L_{10}=\sum_{j_1,j_2,j_3,j_4}\delta_{j_1j_1}\delta_{j_2j_4}\delta_{j_2j_3}\delta_{j_3j_4}\gamma,$$

$$L_{11}=\sum_{j_1,j_2,j_3,j_4}\delta_{j_1j_4}\delta_{j_2j_1}\delta_{j_2j_4}\delta_{j_3j_3}\gamma,\quad L_{12}=\sum_{j_1,j_2,j_3,j_4}\delta_{j_1j_4}\delta_{j_2j_1}\delta_{j_2j_3}\delta_{j_3j_4}\gamma^2.$$

Graphically, we prepare a square with four vertices labeled j_1,j_2,j_3,j_4 in a cyclic order for each term as shown in Fig. 1 and connect the vertices j_a and j_b by a line for each Kronecker's delta $\delta_{j_aj_b}$. We then regard these lines connecting vertices as the hems of ribbons connecting the two edges of the square. For example, the two lines connecting $j_1\leftrightarrow j_3$ and $j_2\leftrightarrow j_2$ in the term L_2 are considered as the two hems of a ribbon, say r_1, connecting the edges $\overline{j_1j_2}$ and $\overline{j_2j_3}$ of the square, while the lines $j_3\leftrightarrow j_4$ and $j_4\leftrightarrow j_1$ are as those of a ribbon, say r_2, connecting $\overline{j_3j_4}$ and $\overline{j_4j_1}$. There are two ways to connect two distinct edges by a ribbon, by untwisting as r_1 and by twisting as r_2 in the above example, respectively. For each twisted ribbon, we put a factor γ.

Next we take the summation over $j_1,\ldots,j_4$ in each term. Again consider the term L_2 for example. Under the restrictions on indices specified by the Kronecker deltas, $j_1=j_3=j_4$, only two indices, say j_1 and j_2, can be chosen to be arbitrary from $\{1,2,\ldots,N\}$. Then the summation over all possible choices of indices gives N^2 for this term. The contribution of L_2 is thus $N^2\gamma$. We list up the contributions of all terms in Fig. 1, and the sum of them gives

$$M_{N,T}(t,2)/(c^2/2)^2 =(N^3+N^2\gamma\times 2+N\gamma^2)\times 2+N+N\gamma\times 2+N^2\gamma^2 =N^3\left\{2+(4\gamma+\gamma^2)\frac{1}{N}+(1+2\gamma+2\gamma^2)\frac{1}{N^2}\right\}. \quad (15)$$

This shows that the 12 terms are classified into six equivalence classes $\{L_1,L_9\}$, $\{L_2,L_3,L_{10},L_{11}\}$, $\{L_4,L_{12}\}$, $\{L_5\}$,

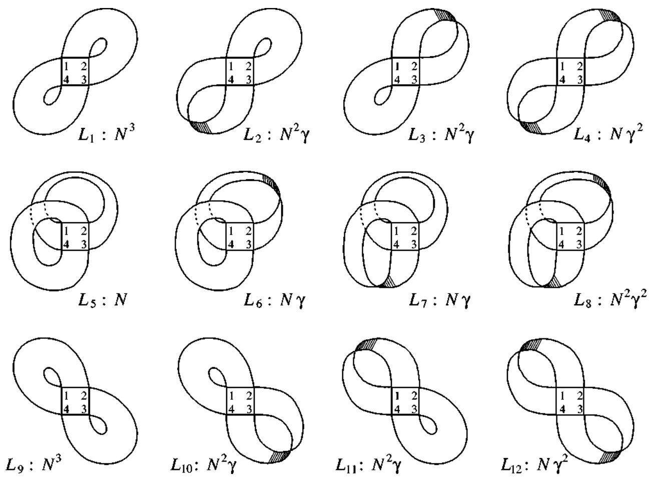

FIG. 1. Möbius graphs for the fourth moment.

$\{L_6, L_7\}$, and $\{L_8\}$ with respect to the contribution to the moment, and Fig. 1 implies that all graphs for the terms in an equivalence class are topologically equivalent.

C. General formula

For the general $2k$th moment, $M_{N,T}(t,k)$, $k \geqslant 1$, we have $(2k-1)!!$ Wick couplings in formula (14), each term of which is the k products of the variances $\langle H_{j\ell}H_{mn}\rangle$. By applying Eq. (12) and expanding in γ, we will have the $K = (2k-1)!! \times 2^k$ terms, $M_{N,T}(t,k)/(c^2/2)^k = \Sigma_{\ell=1}^{K} L_\ell$. As demonstrated in the above section, a one-to-one correspondence is established between terms $\{L_\ell\}$ and graphs, each of which consists of a $2k$-gon with its edges connected by k ribbons to each other. For each graph corresponding to L_ℓ, let φ_ℓ be the number of twisted ribbons in the k ribbons and V_ℓ be the "free indices" remaining after the identification of indices under the Kronecker delta conditions. Then the contribution from the term L_ℓ is given by $N^{V_\ell}\gamma^{\varphi_\ell}$. As mentioned at the end of the preceding section, we consider the equivalence classes of the terms having the same contribution to the $2k$th moment, and let each equivalence class be represented by a graph Γ. We let $V(\Gamma)$ and $\varphi(\Gamma)$ be the numbers of free indices and of twisted ribbons. Moreover, we denote the number of elements in the equivalence class Γ by $|\Gamma|$. In other words, $|\Gamma|$ is the number of ways to generate graphs, which are topologically equivalent with Γ, using a $2k$-gon and k ribbons by gluing edges of $2k$-gons by ribbons. Define $\mathcal{G}(k)$ as the collection of all graphs $\{\Gamma\}$ generated by the present procedure. Then we have

$$M_{N,T}(t,k) = \left(\frac{c^2}{2}\right)^k \sum_{\Gamma \in \mathcal{G}(k)} |\Gamma| N^{V(\Gamma)} \gamma^{\varphi(\Gamma)}. \tag{16}$$

Each graph Γ having no twisted ribbons, $\varphi(\Gamma) = 0$, defines a way of drawing a graph on an orientable surface S_Γ, called a *map*, and each map specifies the surface S_Γ on which the graph is drawn (see, for example, Refs. [26,36]). In general, the specified orientable surface has "holes" or "handles" and their number is called the *genus* $g(S_\Gamma)$. The genus is related to $V(\Gamma)$, $E(\Gamma) = k$ (the number of distinct edges; the original $2k$ sides of polygon were glued together in pairs by ribbons), and $F(\Gamma) = 1$ (the number of faces) through the Euler characteristic

$$\chi(S_\Gamma) \equiv V(\Gamma) - k + 1 = 2 - 2g(S_\Gamma). \tag{17}$$

Then the contribution from all graphs having no twisted ribbons is expressed as

$$\begin{aligned} M^0_{N,T}(t,k) &= \left(\frac{c^2}{2}\right)^k \sum_{\Gamma \in \mathcal{G}(k)} \mathbf{1}_{\{\varphi(\Gamma)=0\}} |\Gamma| N^{k+1-2g(S_\Gamma)} \\ &= \left(\frac{c^2}{2}\right)^k N^{k+1} \sum_{g=0}^{[k/2]} \varepsilon_g(k) \left(\frac{1}{N^2}\right)^g, \end{aligned} \tag{18}$$

where $\mathbf{1}_{\{\omega\}}$ is the indicator; $\mathbf{1}_{\{\omega\}} = 1$ if the condition ω is satisfied and $\mathbf{1}_{\{\omega\}} = 0$ otherwise, and $\varepsilon_g(k) = \Sigma_{\Gamma \in \mathcal{G}(k)} \mathbf{1}_{\{\varphi(\Gamma)=0, V(\Gamma)=k+1-2g\}} |\Gamma|$.

In the similar way, the graphs Γ having twisted ribbons, $\varphi(\Gamma) \geqslant 1$, are considered to define nonorientable surfaces S_Γ. The genus g for nonorientable surface may be defined by the Euler characteristics as [37]

$$\chi(S_\Gamma) = 2 - g(S_\Gamma)$$

instead of Eq. (17). Then all the contribution to the moment from such nonorientable surface graphs is

$$\begin{aligned} M^1_{N,T}(t,k) &= \left(\frac{c^2}{2}\right)^k \sum_{\Gamma \in \mathcal{G}(k)} \mathbf{1}_{\{\varphi(\Gamma)\geqslant 1\}} |\Gamma| N^{k+1-g(S_\Gamma)} \gamma^{\varphi(\Gamma)} \\ &= \left(\frac{c^2}{2}\right)^k N^{k+1} \sum_{g=1}^{k} \left(\frac{1}{N}\right)^g \sum_{m=1}^{k} \tilde{\varepsilon}_{g,m}(k) \gamma^m, \end{aligned} \tag{19}$$

where $\tilde{\varepsilon}_{g,m}(k) = \Sigma_{\Gamma \in \mathcal{G}(k)} \mathbf{1}_{\{\varphi(\Gamma)=m, V(\Gamma)=k+1-g\}} |\Gamma|$. Moment (16) is then given by the summation

$$M_{N,T}(t,k) = M^0_{N,T}(t,k) + M^1_{N,T}(t,k). \tag{20}$$

It should be noted that γ defined by Eq. (13) is a monotonically increasing function of t and changes its value from $\gamma = 0$ to 1 as the time passes from $t = 0$ to T. The above formula (19) shows the fact that the contribution from Möbius graphs with twisted ribbons is growing in time and at $t = T$ twisted ribbons contribute with the same weights as untwisted ribbons (the GOE case).

IV. CALCULATION BY DENSITY FUNCTION

Set

$$\rho_{N,T}(t,\mathbf{x};T,\mathbf{y}) = \frac{C}{(N!)^2} e^{-|\mathbf{x}|^2/2t} h_N(\mathbf{x}) \operatorname{sgn}[h_N(\mathbf{y})] \times \det_{1 \leqslant j,k \leqslant N} \left(\frac{e^{-(x_j - y_k)^2/2(T-t)}}{\sqrt{2\pi(T-t)}} \right)$$

for $\mathbf{x},\mathbf{y}\in\mathbf{R}^N$. It is easy to confirm that $\rho_{N,T}(t,\mathbf{x};T,\mathbf{y})$ is invariant under any permutation of $x_1,\ldots,x_N$ and that of $y_1,\ldots,y_N$, and Eq. (2) is equal to $N!\int d\mathbf{y}\rho_{N,T}(t,\mathbf{x};T,\mathbf{y})$, if $\mathbf{x}\in\mathbf{R}^N_<$. Then the density function at time t is defined as

$$\rho(t,x)=N\int\prod_{j=2}^{N}dx_j\int d\mathbf{y}\rho_{N,T}(t,\mathbf{x};T,\mathbf{y}), \tag{21}$$

and the $2k$th moment is given by

$$M_{N,T}(t,k)=\int x^{2k}\rho(t,x)dx. \tag{22}$$

Let $H_j(x)$ be the jth Hermitian polynomial, satisfying the orthogonality $\int e^{-x^2}H_j(x)H_\ell(x)dx=h_j\delta_{j\ell}$ with $h_j=2^j j!\sqrt{\pi}$. As shown in Appendix A, the general formula for the dynamical correlation functions of vicious walks reported in Refs. [32,38] gives the expression

$$\rho(t,x)=\frac{1}{c}e^{-(x/c)^2}\sum_{j=0}^{N-1}\frac{1}{h_j}\{H_j(x/c)\}^2+\frac{\gamma}{2^{N-1}c\sqrt{\pi}}e^{-(x/c)^2}H_{N-1}(x/c)\frac{1}{(N/2-1)!}\times\sum_{j=0}^{\infty}\frac{(N/2+j)!}{(N+2j+1)!}\gamma^jH_{N+2j+1}(x/c). \tag{23}$$

Substituting Eq. (23) into Eq. (22) and replacing the integral variable x by $y=x/c$ give

$$\begin{aligned}M_{N,T}(t,k)/c^{2k}&=\sum_{j=1}^{N-1}\frac{1}{h_j}\int y^{2k}\{H_j(y)\}^2e^{-y^2}dy+\frac{\gamma}{2^{N-1}\sqrt{\pi}(N/2-1)!}\sum_{j=0}^{\infty}\frac{(N/2+j)!}{(N+2j+1)!}\gamma^j\int y^{2k}H_{N-1}(y)H_{N+2j+1}(y)e^{-y^2}dy\\&=\frac{1}{2^{2k+N}\sqrt{\pi}(N-1)!}\sum_{j=0}^{k}\frac{(2k)!}{j!(2k-2j)!}\left[\int H_{2k-2j}(y)\{H_N(y)\}^2e^{-y^2}dy\right.\\&\quad\left.-\int H_{2k-2j}(y)H_{N-1}(y)H_{N+1}(y)e^{-y^2}dy\right]+\frac{\gamma}{2^{2k+N-1}\sqrt{\pi}(N/2-1)!}\sum_{j=0}^{\infty}\frac{(N/2+j)!}{(N+2j+1)!}\gamma^j\\&\quad\times\sum_{\ell=0}^{k}\frac{(2k)!}{\ell!(2k-2\ell)!}\int H_{2k-2\ell}(y)H_{N-1}(y)H_{N+2j+1}(y)e^{-y^2}dy.\end{aligned}$$

In the second equality, we used the Christoffel-Darboux formula (see p. 193 in Ref. [39])

$$\sum_{j=0}^{N-1}\frac{1}{h_j}\{H_j(y)\}^2=\frac{1}{2^N\sqrt{\pi}(N-1)!}[\{H_N(y)\}^2-H_{N-1}(y)H_{N+1}(y)],$$

and the relation

$$(2y)^{2k}=\sum_{j=0}^{k}\frac{(2k)!}{j!(2k-2j)!}H_{2k-2j}(y).$$

Now we apply the integration formula for the triple of Hermitian polynomials (see p. 290 in Ref. [40]):

$$\int H_j(y)H_\ell(y)H_m(y)e^{-y^2}dy=\begin{cases}\dfrac{j!\ell!m!}{(s-j)!(s-\ell)!(s-m)!}2^s\sqrt{\pi} & \text{if } j+\ell+m=2s \text{ is even}\\ & \text{and } s-j\geqslant 0,\ s-\ell\geqslant 0,\ s-m\geqslant 0,\\ 0 & \text{otherwise.}\end{cases}$$

Then we arrive at $M_{N,T}(t,k)=M^0_{N,T}(t,k)+M^1_{N,T}(t,k)$, with

$$M^0_{N,T}(t,k)=\left(\frac{c^2}{2}\right)^k\frac{(2k)!}{2^kk!}\sum_{j=0}^{k}\binom{k}{j}\binom{N}{k-j+1}2^{k-j}, \tag{24}$$

$$M^1_{N,T}(t,k)=\left(\frac{c^2}{2}\right)^k\sum_{j=0}^{k-1}\sum_{\ell=0}^{k-j-1}\frac{2^{\ell-j}(N/2+\ell)!N!(2k)!}{(N/2)!(N+\ell-k+j)!(k-j-\ell-1)!(k-j+\ell+1)!j!}\gamma^{\ell+1}. \tag{25}$$

Here we do not repeat the explanation how to characterize the quantity $\varepsilon_g(k)$ in Eq. (18) by using formula (24), and only mention that it is obtained as the solution of the recurrence relation

$$(k+1)\varepsilon_g(k)=(4k-2)\varepsilon_g(k-1)+(k-1)(2k-1)(2k-3)\varepsilon_{g-1}(k-2), \tag{26}$$

with the boundary conditions

$$\varepsilon_g(0)=\begin{cases}1 & \text{if } g=0,\\ 0 & \text{otherwise.}\end{cases}$$

See Ref. [25], Sec. 5.5 in Ref. [20], and Ref. [26] for details.

In order to express the expansion in $1/N$ for Eq. (25), here we introduce the number $s(n,k)$ defined as the coefficients of the expansion

$$x(x-1)\cdots(x-n+1)=\sum_{\ell=1}^{n} s(n,\ell)x^{\ell}$$

for $n\geqslant 1$. It is known that $s(n,\ell)(-1)^{n-\ell}$ is the number of elements in the set S_n of all permutations of $\{1,2,\ldots,n\}$, which are products of ℓ disjoint cycles. These numbers $s(n,\ell)$ are called *the Stirling numbers of the first kind* [41]. For example, there are $3!=6$ distinct permutations of $\{1,2,3\}$. We denote a permutation $1\to a, 2\to b, 3\to c$ simply by $[abc]$ $(a,b,c\in\{1,2,3\}, a\neq b\neq c)$. We regard $[231]$ as a cycle $1\to 2\to 3\to 1$, $[213]$ as a product of two cycles $1\to 2\to 1$ and $3\to 3$, and the identity transformation $[123]$ as that of three cycles $1\to 1, 2\to 2$, and $3\to 3$. In this example, we see $s(3,1)=2$ (for there are two elements $[231]$ and $[312]$ with $\ell=1$ in S_3), $s(3,3)=1$ ($[123]$) and $s(3,2)=-3$ (other three permutations). For convenience, we will assume here that $s(n,\ell)=0$ if $n\leqslant 0$, or $\ell\leqslant 0$ or $n<\ell$. Then we have expression (19) from Eq. (25) with the coefficients

$$\tilde{\varepsilon}_{g,m}(k)=\frac{(2k)!}{2^k}\sum_{j=k-g+1}^{k}\sum_{\ell}\frac{(-1)^{m-\ell}2^{m+j-\ell}}{(k-j)!(j-m)!(j+m)!}\times s(m,\ell)s(j-m+1,k-g-\ell+2). \tag{27}$$

It is easy to confirm that $s(n,1)=(-1)^{n-1}(n-1)!$, $s(n,n-1)=-n(n-1)/2$, and $s(n,n)=1$ for any $n=1,2,3,\ldots$, and numerical tables of $s(n,\ell)$ are found in Ref. [41]. For example, if we set $k=2$, Eq. (27) gives $\tilde{\varepsilon}_{1,1}(2)=4s(1,1)s(2,2)=4$, $\tilde{\varepsilon}_{1,2}(2)=s(2,2)s(1,1)=1$, $\tilde{\varepsilon}_{2,1}(2)=6\{s(1,1)s(1,1)+2s(1,1)s(2,1)/3\}=2$, and $\tilde{\varepsilon}_{2,2}(2)=-2s(2,1)s(1,1)=2$, and having $\varepsilon_0(2)=2, \varepsilon_1(2)=1$, result (15) is again obtained through the general formula (20) with Eqs. (18) and (19).

V. CONCLUDING REMARKS

In the present paper, we performed the graphical expansions for the moments of the positions of vicious walkers. The obtained formula (20) with Eqs. (18) and (19) can be regarded as the power series of the number of walkers N. Here we consider the large N limit. Let $\hat{M}_{N,T}(t,k)$ be the dominant term of Eq. (20) in $N\gg 1$. Then

$$\hat{M}_{N,T}(t,k)=\left(\frac{c^2}{2}\right)^k N^{k+1}\varepsilon_0(k), \tag{28}$$

that is, only the contribution from the genus-zero orientable surfaces will survive in the limit $N\to\infty$. It is well known that $\varepsilon_0(k)$'s are given by the Catalan numbers C_k (see, for example, Ref. [42]),

$$\varepsilon_0(k)=C_k=\frac{1}{k+1}\binom{2k}{k}, \tag{29}$$

and their generating function is

$$a(\zeta)=\frac{1}{2\zeta}\{1-2\zeta-\sqrt{1-4\zeta}\}=\sum_{k=1}^{\infty}\zeta^k C_k. \tag{30}$$

Corresponding to Eq. (28), define the density function $\hat{\rho}(t,x)$ as

$$\hat{M}_{N,T}(t,k)=\int x^{2k}\hat{\rho}(t,x)dx.$$

Multiplying both sides by z^k and taking the summation over k from 0 to ∞, we will have

$$\frac{1}{c^2 z}[1-\sqrt{1-2c^2Nz}]=\int\frac{\hat{\rho}(t,x)}{1-zx^2}dx,$$

where Eqs. (28)–(30) were used. If we set $z=1/2c^2N$, it becomes

$$\int\frac{\hat{\rho}(t,x)}{2N-(x/c)^2}dx=1.$$

This integral equation can be solved as [20]

$$\hat{\rho}(t,x)=\begin{cases}\dfrac{1}{\pi c}\sqrt{2N-(x/c)^2} & \text{if } |x|\leqslant\sqrt{2N}c,\\ 0 & \text{otherwise.}\end{cases}$$

In the large N limit, the density function will keep a semicircle shape independently of the time evolution, in which only the width of the semicircle depends on time and simply scaled by c [32]. In other words, *Wigner's semicircle law* is universal in $N\to\infty$.

The universal property of random matrix theory at the large N limit and its finite-N corrections have been studied by calculating the Green functions in the form of $1/N$ expansions in the field theory [21–23,43]. In order to compare our present results with the previous ones, here we consider the one-point Green function defined by

$$G_{N,T}(t,z)=\left\langle\frac{1}{N}\sum_{j=1}^{N}\frac{1}{z-\xi_j}\right\rangle_t, \tag{31}$$

with $\xi_j = x_j/(\sqrt{N}c)$ for $z \in \mathbf{C}$. It is nothing but the generating function of moments (1),

$$G_{N,T}(t,z) = \frac{1}{Nz}\sum_{k=0}^{\infty}\left(\frac{1}{Nc^2z^2}\right)^k M_{N,T}(t,k).$$

We define the coefficients $\overline{M}_T^{(n)}(t,k)$ in the expansion of the moment as

$$M_{N,T}(t,k) = \left(\frac{c^2}{2}\right)^k N^{k+1}\sum_{n=0}^{\infty}\frac{1}{N^n}\overline{M}_T^{(n)}(t,k). \tag{32}$$

Then we have the $1/N$ expansion of the Green function as

$$G_{N,T}(t,z) = \sum_{n=0}^{\infty}\frac{1}{N^n}G_T^{(n)}(t,z), \tag{33}$$

where

$$G_T^{(n)}(t,z) = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{2z^2}\right)^k \overline{M}_T^{(n)}(t,k). \tag{34}$$

Since we have obtained the closed expressions for any moments, Eq. (20) with Eqs. (24) and (25), the Green function (31) is completely determined, and the coefficients $G_T^{(n)}(t,z)$ in its $1/N$ expansion (33) can be derived for any order n through formulas (18) and (19) with the known result of $\varepsilon_g(k)$ [25] and Eq. (27). For example, as shown in Appendix B, the present results give the following expressions for the first three terms:

$$G_T^{(0)}(t,z) = \frac{1+a(\zeta)}{z}, \tag{35}$$

$$G_T^{(1)}(t,z) = 2z\frac{a(\zeta)}{1-a(\zeta)^2}\frac{\gamma a(\zeta)}{1-\gamma a(\zeta)}, \tag{36}$$

and

$$\begin{aligned} G_T^{(2)}(t,z) = {} & \frac{\zeta^2}{3z}\frac{\partial^2}{\partial\zeta^2}\frac{1}{1-a(\zeta)^2} + \frac{\zeta}{6z}\frac{\partial}{\partial\zeta}\frac{1}{1-a(\zeta)^2} \\ & + \frac{1}{2z}\frac{\partial}{\partial\zeta}\left[\frac{a(\zeta)}{1-a(\zeta)^2}\left\{2\gamma\frac{\partial}{\partial\gamma}-1\right\}\frac{\gamma a(\zeta)}{1-\gamma a(\zeta)}\right] \\ & - \frac{1}{2z\zeta}\frac{a(\zeta)}{1-a(\zeta)^2}\left\{3\gamma\frac{\partial}{\partial\gamma}-1\right\}\frac{\gamma a(\zeta)}{1-\gamma a(\zeta)}, \end{aligned} \tag{37}$$

with $\zeta = 1/(2z^2)$. By using Eq. (30), we can show that setting $\gamma=1$ reduces them to Eqs. (24), (26), and (28) with $\beta=1$ (the GOE case) given by Itoi, respectively, who derived them by solving the loop equations [43]. On the other hand, we can confirm that, if we set $\gamma=0$, $G_T^{(n)}(t,z/\sqrt{2})/\sqrt{2}$ gives the $\beta=2$ (GUE) results of Itoi. Some details are given in Appendix B. Our result is general, and it will reproduce all the previous results of $1/N$ expansions for the GUE and GOE as the special cases with $\gamma=0$ and 1, respectively.

Our formula also gives a power series in γ with N-dependent coefficients. Let $\widetilde{M}(k;\gamma,N) = M_{N,T}(t,k)/(c^2/2)^k$. Then

$$\widetilde{M}(k;\gamma,N) = \sum_{m=0}^{k}\gamma^m f_{k,m}(N),$$

where $f_{k,m}(N)$ are the polynomials of degree $k+1$ in N for $m=0$ and of degree k in N for $1 \leqslant m \leqslant k$. Explicit expression of $f_{k,m}(N)$ is immediately obtained from Eq. (25). We remark that the coefficients of the highest order in γ, $f_{k,k}(N)$, are equal to the *zonal polynomials* $Z_{(k)}(s_1,s_2,\ldots,s_k)$, if we set all the variables $s_1=s_2=\cdots=s_k=N$ [44].

As shown by Eq. (4) with Eq. (5), the present system of vicious walkers can be regarded as a Gaussian matrix model, and thus exactly solvable as demonstrated in this paper. We would like to state, however, that the probability density (2) of the positions of walkers $\mathbf{x}$ is not in the simple Gaussian form multiplied by $h_N(\mathbf{x})$, when $0<t<T$, due to the factor $\mathcal{N}_N(T-t,\mathbf{x})$. Combination of the Pfaffian representation of this factor given in Refs. [16,17] and the facts that $\mathrm{Pf}(A) = (\det A)^{1/2}$ for any even-dimensional antisymmetric matrix A and $\det A = e^{\mathrm{tr}\ln A}$, Eq. (2) is written for even N as

$$\rho_{N,T}(t,\mathbf{x}) \propto h_N(\mathbf{x})\exp[-V(\mathbf{x})],$$

with the nonharmonic potential

$$V(\mathbf{x}) = -\frac{|\mathbf{x}|^2}{2t} + \frac{1}{2}\mathrm{tr}[\ln F(T-t,\mathbf{x})],$$

where $F(s,\mathbf{x})$ is the $N\times N$ antisymmetric matrix with the element $F_{jk}(s,\mathbf{x}) = (2/\sqrt{\pi})\mathrm{erf}[(x_k-x_j)/2\sqrt{s}]$ with $\mathrm{erf}(u) = \int_0^x du e^{-u^2}$. Then the present results for $0<t<T$ are nontrivial.

In summary, we demonstrated the GUE-to-GOE transition in time for the vicious walk model by presenting the graphical expansion formula of the moments of walker's positions. The weights of contributing graphs are time developing and our formula interpolates the genus expansion of orientable graphs for the GUE and that of nonorientable graphs for the GOE by using a time parameter $\gamma = t/(2T-t)$. The formula provides a power series in the number of walkers N and it was shown that the exact expression of dynamical correlation functions recently reported by Nagao *et al.* [32] is very useful in order to evaluate the coefficients in the series. By comparing with the previous results of $1/N$ expansion of the one-point Green function, we showed that our results are general and valid. Further applications of the quaternion determinantal expressions of dynamical correlations of vicious walkers to the graphical expansions of more general types of moments (e.g., correlators of moments at different times [45]) will be interesting future problems.

ACKNOWLEDGMENTS

The authors thank S. Kakei and T. Sasamoto for letting them know of the papers by M. Mulase. They also thank T. Nagao for useful comments on the present work.

APPENDIX A: DENSITY FUNCTION

Let $H_j(x)$ be the jth Hermite polynomial defined in Sec. IV. We can read the density function (21) from Refs. [32,38] by setting $M=1$ as

$$\rho(t,x)=\sum_{\ell=0}^{N/2-1}\frac{1}{r_\ell}[\Phi_{2\ell}(x)R_{2\ell+1}(x)-\Phi_{2\ell+1}(x)R_{2\ell}(x)], \tag{A1}$$

where

$$R_\ell(x)=\gamma^{\ell/2}\sum_{j=0}^{\ell}\alpha_{\ell j}H_j(x/c)\gamma^{-j/2}, \tag{A2}$$

$$\Phi_\ell(x)=\int dyR_\ell(y)\int_{-\infty<z<z'<\infty}dzdz'\begin{vmatrix} p(y,z) & p(x,z)\\ p(y,z') & p(x,z')\end{vmatrix},$$

$$r_\ell=\frac{2}{\pi}\gamma^{2\ell+1/2}\left(\frac{c}{2}\right)^{4\ell+1}h_{2\ell}, \tag{A3}$$

with

$$\alpha_{2\ell j}=(c/2)^{2\ell}\delta_{2\ell j},$$

$$\alpha_{2\ell+1 j}=(c/2)^{2\ell+1}(\delta_{2\ell+1 j}-4\ell\,\delta_{2\ell-1 j}),$$

and

$$p(x,y)=\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\frac{e^{-(x-y)^2/2(T-t)}}{\sqrt{2\pi(T-t)}}.$$

The function $p(x,y)$ can be expanded using the Hermite polynomials as

$$p(x,y)=\frac{e^{-(x/c)^2}e^{y^2/2T}}{\sqrt{2\pi t}\sqrt{2\pi(T-t)}}\exp\left\{-\frac{(y/\sqrt{T}-\sqrt{\gamma}x/c)^2}{1-\gamma}\right\}$$
$$=\frac{e^{-(x/c)^2}e^{-y^2/2T}}{\sqrt{2\pi t}\sqrt{2T-t}}\sum_{j=0}^{\infty}\frac{\gamma^{j/2}}{h_j}H_j(x/c)H_j(y/\sqrt{T}).$$

Then, for $\ell=0,1,2,\ldots,$ we will have

$$\Phi_{2\ell}(x)=\frac{r_\ell}{c^2}e^{-(x/c)^2}\sum_{j\geqslant 2\ell+1}\frac{\gamma^{\{j-(2\ell+1)\}/2}}{h_j}\beta_{j2\ell+1}H_j(x/c),$$

$$\Phi_{2\ell+1}(x)=-\frac{r_\ell}{c^2}e^{-(x/c)^2}\sum_{j\geqslant 2\ell}\frac{\gamma^{(j-2\ell)/2}}{h_j}\beta_{j2\ell}H_j(x/c), \tag{A4}$$

where $\beta_{j\ell}$'s satisfy the relation

$$\sum_{\ell=s}^{j}\beta_{j\ell}\alpha_{\ell s}=\delta_{js},\quad 0\leqslant s\leqslant j.$$

Substituting Eqs. (A2)–(A4) into Eq. (A1) gives Eq. (23).

It should be noted that, though the $M=1$ case of Ref. [32] is equivalent to the Pandey-Mehta two-matrix model [19], the present expression (23) is more useful for the moment calculation as shown in Sec. IV.

APPENDIX B: COEFFICIENTS IN 1/*N* EXPANSION OF THE GREEN FUNCTION

By definition of $\bar{M}_T^{(n)}(t,k)$ in Eqs. (32), (18), and (19), we have

$$\bar{M}_T^{(0)}(t,k)=\varepsilon_0(k)=C_k, \tag{B1}$$

$$\bar{M}_T^{(1)}(t,k)=\sum_{m=1}^{k}\tilde{\varepsilon}_{1,m}(k)\gamma^m=\sum_{m\geqslant 1}\binom{2k}{k+m}\gamma^m, \tag{B2}$$

$$\bar{M}_T^{(2)}(t,k)=\varepsilon_1(k)+\sum_{m=1}^{k}\tilde{\varepsilon}_{2,m}(k)\gamma^m=\frac{1}{12}\frac{(2k)!}{k!(k-2)!}$$
$$+\frac{1}{2}\sum_{m\geqslant 1}\left\{(2m-1)\binom{2k}{k+m}(k+1)\right.$$
$$\left.-(3m-1)\binom{2k}{k+m}\right\}, \tag{B3}$$

where we have used Eqs. (27) and (29), and the fact $\varepsilon_1(k)=(2k)!/[12k!(k-2)!]$, which is obtained from Eq. (26) with Eq. (29). Through Eqs. (30) and (34), Eq. (B1) immediately gives Eq. (35). In order to derive Eqs. (36) and (37) from Eqs. (B2) and (B3), respectively, we can use the identity given as Eq. (C5) in Ref. [46],

$$\sum_{n=1}^{\infty}\binom{2n}{n+m}\zeta^{n+1}=\frac{a(\zeta)^{m+1}}{1-a(\zeta)^2}\quad\text{for } m\geqslant 0,$$

and its derivatives with respect to ζ. It is easy to see that

$$G_T^{(0)}(t,z)=z-\sqrt{z^2-2},$$

and $G_T^{(1)}(t,z)$ is zero at $\gamma=0$. Moreover, we have obtained the following expressions for $0\leqslant 1-\gamma \leqslant 1$:

$$G_T^{(1)}(t,z)=-\frac{1}{2}\left[\frac{1}{\sqrt{z^2-2}}-\frac{z}{z^2-2}\right]-\frac{1}{2(z^2-2)^{3/2}}(1-\gamma)+O((1-\gamma)^2),$$

$$G_T^{(2)}(t,z)=\frac{2z^2+3-2z\sqrt{z^2-2}}{4(z^2-2)^{5/2}}-\frac{z(z^2+6)-(z^2+1)\sqrt{z^2-2}}{4(z^2-2)^3}\times(1-\gamma)+O((1-\gamma)^2),$$

and

$$G_T^{(2)}(t,z)=\frac{1}{4(z^2-2)^{5/2}} \quad \text{at} \quad \gamma=0.$$

[1] P.-G. de Gennes, J. Chem. Phys. **48**, 2257 (1968).
[2] M.E. Fisher, J. Stat. Phys. **34**, 667 (1984).
[3] D.A. Huse and M.E. Fisher, Phys. Rev. B **29**, 239 (1984).
[4] P.J. Forrester, J. Phys. A **22**, L609 (1989); **23**, 1259 (1990); **24**, 203 (1991).
[5] D.K. Arrowsmith, P. Mason, and J.W. Essam, Physica A **177**, 267 (1991).
[6] S. Mukherji and S.M. Bhattacharjee, J. Phys. A **26**, L1139 (1993); Phys. Rev. E **48**, 3427 (1993); **52**, 3301(E) (1995).
[7] J.W. Essam and A.J. Guttmann, Phys. Rev. E **52**, 5849 (1995).
[8] J. Cardy and M. Katori, J. Phys. A **36**, 609 (2003).
[9] A.J. Guttmann, A.L. Owczarek, and X.G. Viennot, J. Phys. A **31**, 8123 (1998).
[10] C. Krattenthaler, A.J. Guttmann, and X.G. Viennot, J. Phys. A **33**, 8835 (2000).
[11] C. Krattenthaler, A.J. Guttmann, and X.G. Viennot, J. Stat. Phys. **110**, 1069 (2003).
[12] J. Baik, Commun. Pure Appl. Math. **53**, 1385 (2000).
[13] T. Nagao and P.J. Forrester, Nucl. Phys. B **620[FS]**, 551 (2002).
[14] D.J. Grabiner, Ann. Inst. Henri Poincaré, Probab. Statist. **35**, 177 (1999).
[15] N. O'Connell and M. Yor, Elect. Comm. Prob. **7**, 1 (2002).
[16] M. Katori and H. Tanemura, e-print math-PR/0203286.
[17] M. Katori and H. Tanemura, Phys. Rev. E **66**, 011105 (2002).
[18] F.J. Dyson, J. Math. Phys. **3**, 1191 (1962).
[19] A. Pandey and M.L. Mehta, Commun. Math. Phys. **87**, 449 (1983); M.L. Mehta and A. Pandey, J. Phys. A **16**, 2655 (1983).
[20] M.L. Mehta, *Random Matrices*, 2nd ed. (Academic, New York, 1991).
[21] G. 't Hooft, Nucl. Phys. B **72**, 461 (1974).
[22] E. Brézin, C. Itzykson, G. Parisi, and J.B. Zuber, Commun. Math. Phys. **59**, 35 (1978).
[23] G.M. Cicuta, Lett. Nuovo Cimento **35**, 87 (1982).
[24] E. Brezin and H. Neuberger, Nucl. Phys. B **350**, 513 (1991).
[25] J. Harer and D. Zagier, Invent. Math. **85**, 457 (1986).
[26] A. Zvonkin, Math. Comput. Modell. **26**, 281 (1997); available from http://dept-info.labri.u-bordeaux.fr/zvonkin/
[27] P.G. Silvestrov, Phys. Rev. E **55**, 6419 (1997).
[28] M. Mulase and A. Waldron, e-print math-ph/0206011.
[29] T. Nagao and P. Forrester, Nucl. Phys. B **563[PM]**, 547 (1999).
[30] P.J. Forrester, T. Nagao, and G. Honner, Nucl. Phys. B **553**, 601 (1999).
[31] T. Nagao, Nucl. Phys. B **602**, 622 (2001).
[32] T. Nagao, M. Katori, and H. Tanemura, Phys. Lett. A **307**, 29 (2003).
[33] Harish-Chandra, Am. J. Math. **79**, 87 (1957).
[34] C. Itzykson and J.-B. Zuber, J. Math. Phys. **21**, 411 (1980).
[35] M.L. Mehta, Commun. Math. Phys. **79**, 327 (1981).
[36] A. Okounkov, Int. Math. Res. Notices **20**, 1043 (2000).
[37] W.P. Thurston, *Three-Dimensional Geometry and Topology* (Princeton University Press, Princeton, NJ, 1997).
[38] M. Katori, T. Nagao, and H. Tanemura, e-print math-PR/0301143.
[39] H. Bateman, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 2.
[40] H. Bateman, *Tables of Integral Transforms*, edited by A. Erdélyi (McGraw-Hill, New York, 1954), Vol. 2.
[41] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun (Dover, New York, 1965).
[42] N.J.A. Sloane, *Handbook of Integer Sequences* (Academic, New York, 1973).
[43] C. Itoi, Nucl. Phys. B **493[FS]**, 651 (1997).
[44] A.M. Mathai, S. B. Provost, and T. Hayakawa, *Bilinear Forms and Zonal Polynomials*, Lecture Notes in Statistics Vol. 102 (Springer, New York, 1995).
[45] C. Itoi and Y. Sakamoto, e-print cond-mat/9702156.
[46] M. Katori and N. Inui, J. Phys. A **30**, 2975 (1997).