Probab. Theory Relat. Fields (2007) 138:113–156
DOI 10.1007/s00440-006-0015-4

Infinite systems of noncolliding generalized meanders and Riemann–Liouville differintegrals

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Received: 3 November 2005 / Revised: 2 May 2006 /
Published online: 6 September 2006

Abstract Yor's generalized meander is a temporally inhomogeneous modification of the $2(\nu+1)$-dimensional Bessel process with $\nu > -1$, in which the inhomogeneity is indexed by $\kappa \in [0, 2(\nu+1))$. We introduce the noncolliding particle systems of the generalized meanders and prove that they are Pfaffian processes, in the sense that any multitime correlation function is given by a Pfaffian. In the infinite particle limit, we show that the elements of matrix kernels of the obtained infinite Pfaffian processes are generally expressed by the Riemann–Liouville differintegrals of functions comprising the Bessel functions J_ν used in the fractional calculus, where orders of differintegration are determined by $\nu - \kappa$. As special cases of the two parameters (ν, κ), the present infinite systems include the quaternion determinantal processes studied by Forrester, Nagao and Honner and by Nagao, which exhibit the temporal transitions between the universality classes of random matrix theory.

Keywords Noncolliding generalized meanders · Bessel processes · Random matrix theory · Fredholm Pfaffian and determinant · Riemann–Liouville differintegrals

Mathematics Subject Classification (2000) 60J60 · 15A52 · 26A33 · 60G55

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1 Introduction

The random matrix (RM) theory was introduced originally as an approximation theory of *statistics* of nuclear energy levels [30]. It should be noted that at the same time as the standard theory was established for three ensembles called the Gaussian unitary, orthogonal, and symplectic ensembles (GUE, GOE, GSE) [11], Dyson proposed to study *stochastic processes* of interacting particles such that the eigenvalue statistics of RMs are realized in distribution of particle positions on $\mathbb{R}$ [10]. Dyson's Brownian motion model is a one-parameter family of N-particle systems, $\mathbf{Z}^{(\beta)}(t) = (Z_1^{(\beta)}(t), \dots, Z_N^{(\beta)}(t))$, described by the stochastic differential equations

$$\mathrm{d}Z_i^{(\beta)}(t) = \mathrm{d}B_i(t) + \frac{\beta}{2} \sum_{1 \le j \le N, j \neq i} \frac{1}{Z_i^{(\beta)}(t) - Z_j^{(\beta)}(t)} \mathrm{d}t,$$
$$t \in [0, \infty), \quad 1 \le i \le N, \tag{1}$$

where $B_i(t), i = 1, 2, \dots, N$ are independent standard Brownian motions and the parameter β equals 2, 1, and 4 for GUE, GOE, and GSE, respectively. Due to the strong repulsive forces, which are long-ranged and act between any pair of particles, intersections of particle trajectories are prohibited for $\beta \ge 1$ [43] (see also [7]). In this one-parameter family, the $\beta = 2$ case (i.e. the GUE case) is the simplest and the most understood, since its equivalence with the N particle systems of Brownian motions *conditioned never to collide with each other* can be proved [16].

The standard (Wigner–Dyson) theory has been extended by adding three chiral versions of RM ensembles in the particle physics of QCD [19,46,51,52], and by introducing the four additional ensembles so-called the Bogoliubov–de Gennes classes in the mesoscopic physics [1,2]. Here we note that the chiral ensembles have a parameter $\nu \in \{0, 1, 2, \dots\}$ in addition to β. In these totally ten ensembles [1,2,54], chiral GUE (chGUE), class C and class D can be regarded as natural extensions of the GUE, in the sense that these eigenvalue statistics are also realized in appropriate noncolliding systems of stochastic particle systems: König and O'Connell showed that the chGUE with the parameter $\nu \in \{0, 1, 2, \dots\}$ corresponds to the noncolliding systems of $2(\nu + 1)$-dimensional squared Bessel processes [28]. The present authors clarified that the eigenvalue statistics in the classes C and D are realized by the noncolliding systems of the Brownian motions *with an absorbing wall at the origin* and of the Brownian motions *reflecting at the origin* [26,27]. Since the absorbing and reflecting Brownian motions are directly related with the three-dimensional and one-dimensional Bessel processes, respectively (see, for example [41]), the stochastic differential equations of these noncolliding particle systems are generally given by

$$\mathrm{d}\widetilde{Z}_i^{(\nu)}(t) = \mathrm{d}B_i(t) + \sum_{1 \le j \le N, j \ne i} \left\{ \frac{1}{\widetilde{Z}_i^{(\nu)}(t) - \widetilde{Z}_j^{(\nu)}(t)} + \frac{1}{\widetilde{Z}_i^{(\nu)}(t) + \widetilde{Z}_j^{(\nu)}(t)} \right\} \mathrm{d}t + \frac{2\nu+1}{2} \frac{1}{\widetilde{Z}_i^{(\nu)}(t)} \mathrm{d}t, \qquad t \in [0,\infty), \quad 1 \le i \le N, \tag{2}$$

with reflecting barrier condition at the origin in case $\nu = -1/2$. Therefore, the difference of (nonstandard) RM ensembles can be attributed to the difference of dimensionality of the Bessel processes, whose noncolliding sets realize the statistics of the RM ensembles [26]. Here we remind that the d-dimensional Bessel process is defined as the process of the radial coordinate (the modulus) of a Brownian motion in $\mathbb{R}^d$. To realize other $10 - 4 = 6$ RM ensembles by conditioned stochastic processes may be much more difficult (see [49]), but we demonstrated that, if we consider appropriate noncolliding systems of *temporally inhomogeneous* processes defined only in a finite time-interval $[0, T]$, we can observe the transitions of distributions into the six distributions as the time t approaches the final time T [25,26]. The interesting fact is that the processes that can be used instead of the Bessel processes (2) should have one more parameter κ in addition to ν. This two-parameter family of temporally inhomogeneous processes indexed by $(\nu, \kappa), \nu > -1, \kappa \in [0, 2(\nu + 1))$ is equivalent with the family of processes already studied by Yor. He called them the *generalized meanders* [53].

From the viewpoint of RM theory, studying time-development of stochastic systems by calculating, for example, the multitime correlation functions corresponds to considering multimatrix models. In particular, the temporally inhomogeneous processes will be identified with such matrix models that matrices with different symmetries are coupled in a chain [22–24,34]. Determination of all multitime correlation functions of systems, which allows us to determine scaling limits associated with the infinity limit of matrix sizes (i.e. the infinite-particle limit) is one of the main topics of the modern theory of RM [30]. The finite and infinite particle systems showing the orthogonal–unitary and symplectic–unitary transitions, and transitions between class C to class CI were studied and multitime correlation functions were determined by Forrester, Nagao and Honner (FNH) [15], and by Nagao [32], respectively. The system in the Laguerre ensemble with $\beta = 1$ initial condition reported in the former paper can be regarded as the $\nu = \kappa \in \{0, 1, 2, \dots\}$ case of the noncolliding system of the generalized meanders and the system reported in the latter paper as the $(\nu, \kappa) = (1/2, 1)$ case.

If we think about the system of generalized meanders apart from the RM theory, however, we can consider the parameters ν and κ as real numbers, and not necessarily integers nor half-integers. In the present paper, we calculate the multitime correlation functions of noncolliding systems of (squared) generalized meanders for arbitrary values of parameters, provided they satisfy the condition $\nu > -1, \kappa \in [0, 2(\nu + 1))$ so that the systems do not collapse. We first define the N particle systems in a finite time-interval $[0, T]$

and take the $N = T \to \infty$ limit to construct the two-parameter family of infinite particle systems. We prove that the multitime characteristic function is given by a Fredholm Pfaffian [40] and thus any multitime correlation function is given by a Pfaffian. Similarly to the results by FNH [15] and Nagao [32] and their temporally-homogeneous version (the determinantal process with the extended Bessel kernel [50]), the elements of the matrix kernels of Pfaffians are expressed using the Bessel functions, but we clarify the fact that they are generally given by the *Riemann–Liouville differintegrals* of the functions comprising the Bessel functions, which are used in *fractional calculus* (see, for example, [36,38,44]). This structure will explain the origin of the multiple integral expressions for the elements of the matrix kernels reported by FNH [15] and Nagao [32].

The paper is organized as follows. In Sect. 2, the definitions of the generalized meanders of Yor and their noncolliding systems are given and the Riemann–Liouville differintegrals of the Bessel functions with appropriate factors are introduced. The main theorem for the infinite particle limit (Theorem 2.1) is then given. It is demonstrated that, if we take a further limit in the system of Theorem 2.1, we will obtain the temporally homogeneous system of infinite number of particles, which is a determinantal process with the extended Bessel kernel studied in [50] (see also [37]). Using the properties of the Riemann–Liouville differintegrals, we show that Theorem 2.1 includes the results by FNH [15] and Nagao [32] as special cases. Section 3 is devoted to prove that for any finite number of particles N, the present system is a *Pfaffian process* (Theorem 3.1), in the sense that any multitime correlation function is given by a Pfaffian [40]. These Pfaffian processes may be regarded as the continuous space–time version of the Pfaffian point processes and Pfaffian Schur processes studied by Borodin and Rains [4]. Soshnikov used the term *Pfaffian ensembles* in [6,47,48]. See also [13,17,20,39,45] in the context of study of nonequilibrium phenomena in the polynuclear growth models, and [14,35] in that of shape fluctuations of crystal facets. The processes studied in [15,32] are also Pfaffian processes, since the ‘quaternion determinantal expressions’ of correlation functions, introduced and developed by Dyson, Mehta, Forrester, and Nagao [12,29–31,33], are readily transformed to Pfaffian expressions. The method of skew-orthogonal functions associated with the Laguerre polynomials is used in Sect. 4 in order to perform matrix inversion and give explicit expressions for the elements of matrix kernels of Pfaffians. Asymptotics in $T = N \to \infty$ are studied in Sect. 5. Appendices are given to show proofs of formulae and lemmas used in the text.

At the end of this introduction, we would like to refer to the papers [8, 18], which reported the further extensions of RM theory in physics and the representation theory. We hope that the present paper will demonstrate the fruitfulness of developing the probability theory of interacting infinite particle systems in connection with the extensive study of (multi-)matrix models in the RM theory.

2 Definition of processes and results

2.1 Noncolliding systems of generalized meanders

Let $\mathbb{Z}$ and $\mathbb{R}$ be the sets of integers and real numbers, respectively, and set $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}_0$, and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Let $\Gamma(c), c \in \mathbb{R} \setminus (\mathbb{Z}_- \cup \{0\})$, be the Gamma function: $\Gamma(c) = \int_0^\infty dy\, e^{-y} y^{c-1}$ for $c > 0$, and $\Gamma(c) = \Gamma(c + [-c] + 1)/\{c(c+1)\cdots(c + [-c])\}$ for $c \in (-\infty, 0) \setminus \mathbb{Z}_-$, where $[c]$ is the largest integer that is less than or equal to the real number c. For $t > 0, x, y \in \mathbb{R}_+$ and $\nu > -1$ we denote by $G_t^{(\nu)}(t; y|x)$ the transition probability density of a $2(\nu+1)$-dimensional *Bessel process* [5,41],

$$G^{(\nu)}(t; y|x) = \frac{y^{\nu+1}}{x^\nu} \frac{1}{t} e^{-(x^2+y^2)/2t} I_\nu\left(\frac{xy}{t}\right), \quad x > 0, \quad y \in \mathbb{R}_+,$$
$$G^{(\nu)}(t; y|0) = \frac{y^{2\nu+1}}{2^\nu \Gamma(\nu+1) t^{\nu+1}} e^{-y^2/2t}, \quad y \in \mathbb{R}_+,$$

where $I_\nu(z)$ is the modified Bessel function : $I_\nu(z) = \sum_{n=0}^\infty (z/2)^{2n+\nu}/\{\Gamma(n+1)\Gamma(\nu+n+1)\}$. For $T > 0$, $\kappa \in [0, 2(\nu+1))$, we put

$$h_T^{(\nu,\kappa)}(t, x) = \int_0^\infty dy\, G^{(\nu)}(T-t; y|x) y^{-\kappa}, \quad x \in \mathbb{R}_+, \quad t \in [0, T],$$

and

$$G_T^{(\nu,\kappa)}(s, x; t, y) = \frac{1}{h_T^{(\nu,\kappa)}(s, x)} G^{(\nu)}(t-s; y|x) h_T^{(\nu,\kappa)}(t, y), \tag{3}$$
$$G_T^{(\nu,\kappa)}(0, 0; t, y) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\kappa/2)} (2T)^{\kappa/2} G^{(\nu)}(t; y|0) h_T^{(\nu,\kappa)}(t, y), \tag{4}$$

for $x > 0$, $y \in \mathbb{R}_+$, $0 \leq s \leq t \leq T$. This transition probability density $G_T^{(\nu,\kappa)}(s, x; t, y)$ defines the temporally inhomogeneous process in a finite time-interval $[0, T]$, which is called a *generalized meander*. In particular, when $\nu = 1/2$ and $\kappa = 1$, it is identified with the process called a *Brownian meander* (see Chap. 3 in [53]).

Now we consider the N-particle system of generalized meanders conditioned that they never collide in a finite time-interval $[0, T]$. Let

$$\mathbb{R}_{+<}^N = \left\{\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}_+^N : 0 \leq x_1 < x_2 < \cdots < x_N\right\}.$$

According to the determinantal formula of Karlin and McGregor [21], the transition probability density is given as

$$g_{N,T}^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = \frac{f_{N,T}^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y})\mathcal{N}_{N,T}^{(\nu,\kappa)}(T-t,\mathbf{y})}{\mathcal{N}_{N,T}^{(\nu,\kappa)}(T-s,\mathbf{x})}, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}_{+<}^N, \tag{5}$$

for $0 \le s \le t \le T$, where $f_{N,T}^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = \det_{1\le j,k\le N}\left[G_T^{(\nu,\kappa)}(s,x_j,t,y_k)\right]$ and

$$\mathcal{N}_{N,T}^{(\nu,\kappa)}(t,\mathbf{x}) = \int_{\mathbb{R}_{+<}^N} d\mathbf{y} f_{N,T}^{(\nu,\kappa)}(T-t,\mathbf{x};T,\mathbf{y}).$$

Since $h_T^{(\nu,0)}(t,x) = 1$, $G_T^{(\nu,0)}(s,x;t,y) = G^{(\nu)}(t-s;y|x)$ and thus $f_{N,T}^{(\nu,0)}$ is temporally homogeneous and independent of T, we will write $f_N^{(\nu)}(t-s;\mathbf{y}|\mathbf{x})$ for $f_{N,T}^{(\nu,0)}(s,\mathbf{x};t,\mathbf{y})$. Moreover, note that

$$f_{N,T}^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = \frac{1}{h_T^{(\nu,\kappa)}(s,\mathbf{x})} f_N^{(\nu)}(t-s;\mathbf{y}|\mathbf{x}) h_T^{(\nu,\kappa)}(t,\mathbf{y}),$$

where $h_T^{(\nu,\kappa)}(t,\mathbf{x}) \equiv \prod_{j=1}^N h_T^{(\nu,\kappa)}(t,x_j)$ and $h_T^{(\nu,\kappa)}(T,\mathbf{x}) = \prod_{j=1}^N x_j^{-\kappa}$. Then Eq. (5) can be written as

$$g_{N,T}^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = \frac{1}{\widetilde{\mathcal{N}}_N^{(\nu,\kappa)}(T-s,\mathbf{x})} f_N^{(\nu)}(t-s;\mathbf{y}|\mathbf{x})\widetilde{\mathcal{N}}_N^{(\nu,\kappa)}(T-t,\mathbf{y}), \tag{6}$$

where

$$\widetilde{\mathcal{N}}_N^{(\nu,\kappa)}(t,\mathbf{x}) = \int_{\mathbb{R}_{+<}^N} d\mathbf{y}\, f_N^{(\nu)}(t;\mathbf{y}|\mathbf{x}) \prod_{j=1}^N y_j^{-\kappa}. \tag{7}$$

In our previous paper [26] it was shown that, taking the limit $\mathbf{x} \to \mathbf{0} \equiv (0,0,\dots,0)$ at the initial time $s=0$, Eq. (6) becomes

$$g_{N,T}^{(\nu,\kappa)}(0,\mathbf{0};t,\mathbf{y}) = C_{N,T}^{\nu,\kappa}(t)\prod_{j=1}^N G^{(\nu)}(t,y_j|0) \prod_{1\le j<k\le N}(y_k^2-y_j^2)\,\widetilde{\mathcal{N}}_N^{(\nu,\kappa)}(T-t,\mathbf{y}), \tag{8}$$

for $\nu > -1$ and $\kappa \in [0, 2(\nu+1))$, where

$$C_{N,T}^{\nu,\kappa}(t) = \frac{T^{(N+\kappa-1)N/2} t^{-(N-1)N}}{2^{N(N-\kappa-1)/2}} \prod_{j=1}^N \frac{\Gamma(\nu+1)\Gamma(1/2)}{\Gamma(j/2)\,\Gamma((j+1+2\nu-\kappa)/2)}.$$

The N-particle system of *noncolliding generalized meanders all starting from the origin* $\mathbf{0}$ *at time* 0 is defined by the transition probability density $g_{N,T}^{(\nu,\kappa)}$ given above and it will be denoted by $\mathbf{X}(t), t \in [0,T]$ in the present paper. It makes a two-parameter family of temporally inhomogeneous processes parameterized by $\nu > -1$ and $\kappa \in [0, 2(\nu+1))$.

We denote by $\mathfrak{X}$ the space of countable subsets ξ of $\mathbb{R}$ satisfying $\sharp(\xi \cap K) < \infty$ for any compact subset K. For $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \bigcup_{\ell=1}^{\infty} \mathbb{R}^\ell$, we denote $\{x_i\}_{i=1}^n \in \mathfrak{X}$ simply by $\{\mathbf{x}\}$. Then $\Xi_N^{\mathbf{X}}(t) = \{\mathbf{X}(t)\}, t \in [0,T]$, is the diffusion process on the set $\mathfrak{X}$ with transition density function $\mathfrak{g}_{N,T}^{(\nu,\kappa)}(s,\xi;t,\eta)$, $0 \le s \le t \le T$:

$$\mathfrak{g}_{N,T}^{(\nu,\kappa)}(s,\xi;t,\eta) = \begin{cases} g_{N,T}^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}), & \text{if } s > 0,\ \sharp\xi = \sharp\eta = N, \\ g_{N,T}^{(\nu,\kappa)}(0,\mathbf{0};t,\mathbf{y}), & \text{if } s = 0,\ \xi = \{0\},\ \sharp\eta = N, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the elements of $\mathbb{R}_{+<}^N$ with $\xi = \{\mathbf{x}\}$, $\eta = \{\mathbf{y}\}$.

For the given time interval $[0,T]$, we consider the M intermediate times $0 < t_1 < t_2 < \cdots < t_M < T$. For convenience, we set $t_0 = 0, t_{M+1} = T$. For $\mathbf{x}^{(m)} \in \mathbb{R}^N$, $1 \le m \le M+1$, and $N' = 1, 2, \dots, N$, we put $\mathbf{x}_{N'}^{(m)} = \left(x_1^{(m)}, x_2^{(m)}, \dots, x_{N'}^{(m)}\right)$ and $\xi_m^{N'} = \{\mathbf{x}_{N'}^{(m)}\}$. Then the multitime transition density function of the process $\Xi_N^{\mathbf{X}}(t)$ is given by

$$\mathfrak{g}_{N,T}^{(\nu,\kappa)}\left(0,\{0\};t_1,\xi_1^N;\dots;t_{M+1},\xi_{M+1}^N\right) = \prod_{m=0}^{M} \mathfrak{g}_{N,T}^{(\nu,\kappa)}\left(t_m,\xi_m^N;t_{m+1},\xi_{m+1}^N\right), \tag{9}$$

where we assume $\xi_0^N = \{0\}$. For a sequence $\{N_m\}_{m=1}^{M+1}$ of positive integers less than or equal to N, we define the $(N_1, N_2, \dots, N_{M+1})$-multitime correlation function by

$$\rho_{N,T}^{\mathbf{X}}\left(t_1,\xi_1^{N_1};t_2,\xi_2^{N_2};\dots;t_{M+1},\xi_{M+1}^{N_{M+1}}\right) = \prod_{m=1}^{M+1} \frac{1}{(N-N_m)!}$$
$$\times \int_{\prod_{m=1}^{M+1}\mathbb{R}_+^{N-N_m}} \prod_{m=1}^{M+1} \prod_{j=N_m+1}^{N} dx_j^{(m)}\, \mathfrak{g}_{N,T}^{(\nu,\kappa)}\left(0,\{0\};t_1,\xi_1^N;\dots;t_{M+1},\xi_{M+1}^N\right). \tag{10}$$

Associated with the generalized meander (3) and (4), we consider a temporally inhomogeneous diffusion process with transition probability density

$$p_T^{(\nu,\kappa)}(0,0;t,y) \equiv G_T^{(\nu,\kappa)}(0,0;t,\sqrt{y})\frac{1}{2}y^{-1/2}, \quad y \in \mathbb{R}_+,$$
$$p_T^{(\nu,\kappa)}(s,x;t,y) \equiv G_T^{(\nu,\kappa)}(s,\sqrt{x}:t,\sqrt{y})\frac{1}{2}y^{-1/2}, \quad x > 0, \quad y \in \mathbb{R}_+,$$

$0 \leq s < t \leq T$, and call it a *squared generalized meander*. The N-particle system of *noncolliding squared generalized meanders* $\mathbf{Y}(t), t \in [0, T]$, is then defined by

$$\mathbf{Y}(t) = \left(X_1(t)^2, X_2(t)^2, \dots, X_N(t)^2\right), \quad t \in [0, T].$$

The correlation function $\rho^{\mathbf{Y}}_{N,T}$ of $\Xi^{\mathbf{Y}}_N(t) = \{\mathbf{Y}(t)\}$ is obtained from Eq. (10) through the relation

$$\begin{aligned} &\rho^{\mathbf{Y}}_{N,T}\left(t_1, \zeta_1^{N_1}; t_2, \zeta_2^{N_2}; \dots; t_{M+1}, \zeta_{M+1}^{N_{M+1}}\right) \\ &\quad = \rho^{\mathbf{X}}_{N,T}\left(t_1, \xi_1^{N_1}; t_2, \xi_2^{N_2}; \dots; t_{M+1}, \xi_{M+1}^{N_{M+1}}\right) \prod_{m=1}^{M+1} \prod_{j=1}^{N_m} \frac{1}{2x_j^{(m)}}, \end{aligned} \tag{11}$$

where $\xi_m^{N_m} = \{\mathbf{x}^{(m)}_{N_m}\}, \zeta_m^{N_m} = \{\mathbf{y}^{(m)}_{N_m}\}$ with $x_j^{(m)} = \sqrt{y_j^{(m)}}$, $1 \leq j \leq N_m$, $1 \leq m \leq M+1$.

2.2 Riemann–Liouville differintegrals of Bessel functions

We consider the following left and right Riemann–Liouville differintegrals for integrable functions f on $\mathbb{R}_+$,

$${}_0\mathbf{D}^c_x f(x) = \frac{1}{\Gamma(n-c)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_0^x (x-y)^{n-c-1} f(y) \mathrm{d}y, \tag{12}$$

$${}_x\mathbf{D}^c_\infty f(x) = \frac{1}{\Gamma(n-c)} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_x^\infty (y-x)^{n-c-1} f(y) \mathrm{d}y, \tag{13}$$

where $c \in \mathbb{R}$ and $n = [c+1]_+$ with the notation $x_+ = \max\{x, 0\}$. It is easy to confirm that, if $c \in \mathbb{N}_0$, both of them are reduced to the ordinary multiple derivative,

$${}_0\mathbf{D}^c_x f(x) = (-1)^c \; {}_x\mathbf{D}^c_\infty f(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^c f(x),$$

and, if $c \in \mathbb{Z}_-$, they are equal to the multiple integrals,

$$\begin{aligned} {}_0\mathbf{D}^c_x f(x) &= \int_0^x \mathrm{d}y_{|c|-1} \int_0^{y_{|c|-1}} \mathrm{d}y_{|c|-2} \cdots \int_0^{y_2} \mathrm{d}y_1 \int_0^{y_1} \mathrm{d}y_0 f(y_0), \\ {}_x\mathbf{D}^c_\infty f(x) &= \int_x^\infty \mathrm{d}y_{|c|-1} \int_{y_{|c|-1}}^\infty \mathrm{d}y_{|c|-2} \cdots \int_{y_2}^\infty \mathrm{d}y_1 \int_{y_1}^\infty \mathrm{d}y_0 f(y_0). \end{aligned}$$

For $c \in (-\infty, 0) \setminus \mathbb{Z}_-$ Eqs. (12) and (13) define *fractional integrals*, and for $c \in \mathbb{R}_+ \setminus \mathbb{N}_0$ *fractional differentials*. The Riemann–Liouville differintegrals are most often used in the fractional calculus (see, for example, [36,38,44]).

Let $J_\nu(z)$ be the Bessel functions: $J_\nu(z) = \sum_{\ell=0}^{\infty}(-1)^\ell (z/2)^{2\ell+\nu}/\{\Gamma(\nu+\ell+1)\ell!\}$. We define functions $\widetilde{J}_\nu$ and $\widehat{J}_\nu$ as

$$\widetilde{J}_\nu(\theta,\eta,x,s) = (\theta\eta x)^{\nu/2} J_\nu(2\sqrt{\theta\eta x}) e^{2s\theta\eta}, \tag{14}$$

$$\widehat{J}_\nu(\theta,\eta,x,s) = (\theta\eta x)^{-\nu/2} J_\nu(2\sqrt{\theta\eta x}) e^{2s\theta\eta}. \tag{15}$$

We will use the following abbreviations for the Riemann–Liouville differintegrals of order $c \in \mathbb{R}$ of $\widetilde{J}_\nu$ and $\widehat{J}_\nu$,

$$\widetilde{J}_\nu^{(c)}(\theta,\eta,x,s) = {}_0\mathbf{D}_\eta^c \widetilde{J}_\nu(\theta,\eta,x,s), \quad \theta,\eta > 0, \quad s \in \mathbb{R}, \tag{16}$$

$$\widehat{J}_\nu^{(c)}(\theta,\eta,x,s) = {}_\eta\mathbf{D}_\infty^c \widehat{J}_\nu(\theta,\eta,x,s), \quad \theta,\eta > 0, \quad s < 0. \tag{17}$$

We note that, if $c \in \mathbb{R} \setminus \mathbb{N}_0$, $\widetilde{J}_\nu^{(c)}$ can be expanded as

$$\widetilde{J}_\nu^{(c)}(\theta,\eta,x,s) = \frac{1}{\Gamma(-c)} \sum_{n=0}^{\infty} \frac{(-1)^n \eta^{n-c}}{n!(n-c)} \widetilde{J}_\nu^{(n)}(\theta,\eta,x,s), \tag{18}$$

for $\theta,\eta > 0$, $s \in \mathbb{R}$. It is also noted that, since $\widehat{J}_\nu(\theta,\eta,x,s) \to 0$ exponentially fast as $\eta \to \infty$, if $s\theta < 0$,

$$\widehat{J}_\nu^{(c)}(\theta,\eta,x,s) = \frac{1}{\Gamma(n-c)} \int_\eta^\infty d\xi \; (\xi-\eta)^{n-c-1} \widehat{J}_\nu^{(n)}(\theta,\xi,x,s), \tag{19}$$

for $\theta,\eta > 0$, $s < 0$, where $n = [c+1]_+$.

2.3 Results

We put

$$\mathfrak{a} = \mathfrak{a}(\nu,\kappa) = \nu - \frac{\kappa}{2}, \qquad \mathfrak{b} = \mathfrak{b}(\nu,\kappa) = \nu - \kappa, \tag{20}$$

and introduce functions $\mathcal{D}(s,x;t,y)$, $\tilde{\mathcal{I}}(s,x;t,y)$, $\mathcal{S}(s,x;t,y)$, and $\widetilde{\mathcal{S}}(s,x;t,y)$, $x,y \in \mathbb{R}_+$, $s,t < 0$,

$$\mathcal{D}(s,x;t,y) = \frac{1}{4(xy)^{\kappa/2}} \int_0^1 d\theta\, \theta^{1-\kappa} \Big[\widetilde{J}_\nu^{(-\mathfrak{b})}(\theta,1,x,-s) \widetilde{J}_\nu^{(-\mathfrak{b}-1)}(\theta,1,y,-t) - \widetilde{J}_\nu^{(-\mathfrak{b}-1)}(\theta,1,x,-s) \widetilde{J}_\nu^{(-\mathfrak{b})}(\theta,1,y,-t) \Big],$$

$$\tilde{\mathcal{I}}(s,x;t,y) = (xy)^{\kappa/2} \int_1^\infty d\theta\, \theta^{\kappa-1} \Big[\widehat{J}_\nu^{(\mathfrak{b}+1)}(\theta,1,x,s) \int_1^\infty d\xi\, \xi^{\mathfrak{a}} \widehat{J}_\nu^{(\mathfrak{b}+1)}(\theta,\xi,y,t) - \widehat{J}_\nu^{(\mathfrak{b}+1)}(\theta,1,y,t) \int_1^\infty d\xi\, \xi^{\mathfrak{a}} \widehat{J}_\nu^{(\mathfrak{b}+1)}(\theta,\xi,x,s) \Big],$$

$$\mathcal{S}(s,x;t,y) = \frac{1}{2} \left(\frac{x}{y}\right)^{\kappa/2} \int_0^1 d\theta \Big[\widehat{J}_\nu^{(\mathfrak{b}+1)}(\theta,1,x,s) \widetilde{J}_\nu^{(-\mathfrak{b}-1)}(\theta,1,y,-t) - \left\{ \mathfrak{a}\widetilde{J}_\nu^{(-\mathfrak{b}-1)}(\theta,1,y,-t) - \widetilde{J}_\nu^{(-\mathfrak{b})}(\theta,1,y,-t) \right\} \times \int_1^\infty d\xi\, \xi^{\mathfrak{a}} \widehat{J}_\nu^{(\mathfrak{b}+1)}(\theta,\xi,x,s) \Big], \tag{21}$$

and

$$\widetilde{\mathcal{S}}(s,x;t,y) = \mathcal{S}(s,x;t,y) - \mathbf{1}_{(s<t)} \left(\frac{y}{x}\right)^{\mathfrak{b}/2} \mathcal{G}(s,x;t,y), \tag{22}$$

where $\mathbf{1}_{(\omega)}$ is the indicator function: $\mathbf{1}_{(\omega)} = 1$ if ω is satisfied and $\mathbf{1}_{(\omega)} = 0$ otherwise, and

$$\mathcal{G}(s,x;t,y) = \int_0^\infty d\theta\, J_\nu(2\sqrt{\theta x}) J_\nu(2\sqrt{\theta y}) e^{2(s-t)\theta}. \tag{23}$$

For an integer N and a skew-symmetric $2N \times 2N$ matrix $A = (a_{ij})$, the Pfaffian is defined as

$$\mathrm{Pf}(A) = \mathrm{Pf}_{1\le i<j\le 2N}(a_{ij}) = \frac{1}{N!} \sum_\sigma \mathrm{sgn}(\sigma) a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2N-1)\sigma(2N)}, \tag{24}$$

where the summation is extended over all permutations σ of $(1,2,\dots,2N)$ with restriction $\sigma(2k-1)<\sigma(2k), k=1,2,\dots,N$. We put

$$\widehat{\Xi}_N^{\mathbf{Y}}(s)=\{Y_1(T_N+s),Y_2(T_N+s),\dots,Y_N(T_N+s)\},\quad s\in[-T_N,0),$$

and $\widehat{\Xi}_N^{\mathbf{Y}}(s)=\{0\}, s\in(-\infty,-T_N)$. Then we can state the main theorem in the present paper.

Theorem 2.1. *Let $T_N=N$. Then the process $\widehat{\Xi}_N^{\mathbf{Y}}(s), s\in(-\infty,0)$ converges to the process $\widehat{\Xi}_\infty^{\mathbf{Y}}(s), s\in(-\infty,0)$, as $N\to\infty$, in the sense of finite dimensional distributions, whose correlation functions $\rho^{\mathbf{Y}}$ are given by*

$$\rho^{\mathbf{Y}}\left(s_1,\{\mathbf{y}_{N_1}^{(1)}\};s_2,\{\mathbf{y}_{N_2}^{(2)}\};\dots;s_M,\{\mathbf{y}_{N_M}^{(M)}\}\right)=\mathrm{Pf}\left[\mathcal{A}\left(\mathbf{y}_{N_1}^{(1)},\mathbf{y}_{N_2}^{(2)},\dots,\mathbf{y}_{N_M}^{(M)}\right)\right],$$

for any $M\geq 1$, any sequence $\{N_m\}_{m=1}^M$ of positive integers, and any strictly increasing sequence $\{s_m\}_{m=1}^{M+1}$ of nonpositive numbers with $s_{M+1}=0$, where $\mathcal{A}\left(\mathbf{y}_{N_1}^{(1)},\mathbf{y}_{N_2}^{(2)},\dots,\mathbf{y}_{N_M}^{(M)}\right)$ is the $2\sum_{m=1}^M N_m\times 2\sum_{m=1}^M N_m$ skew-symmetric matrix defined by

$$\mathcal{A}\left(\mathbf{y}_{N_1}^{(1)},\mathbf{y}_{N_2}^{(2)},\dots,\mathbf{y}_{N_M}^{(M)}\right)=\left(\mathcal{A}^{m,n}(y_i^{(m)},y_j^{(n)})\right)_{1\leq i\leq N_m,1\leq j\leq N_n,1\leq m,n\leq M}$$

with 2×2 matrices $\mathcal{A}^{m,n}(x,y)$;

$$\mathcal{A}^{m,n}(x,y)=\begin{pmatrix}\mathcal{D}(s_m,x;s_n,y) & \widetilde{\mathcal{S}}(s_n,y;s_m,x)\\ -\widetilde{\mathcal{S}}(s_m,x;s_n,y) & -\tilde{\mathcal{I}}(s_m,x;s_n,y)\end{pmatrix}.$$

In the infinite-particle system defined by Theorem 2.1, we can take the further limit:

$$s_m\to-\infty\quad\text{with the time differences } s_n-s_m \text{ fixed},\quad 1\leq m,n\leq M.$$

In this limit, $\mathcal{D}(s_m,x;s_n,y)\tilde{\mathcal{I}}(s_m,x;s_n,y)\to 0$, $1\leq m,n\leq M$, as we show in Appendix C.. Therefore, we can replace $\mathcal{D}$ and $\tilde{\mathcal{I}}$ by zeros in the matrices. Then the Pfaffian is reduced to an ordinary determinant of the $\sum_{m=1}^{M+1}N_m\times\sum_{m=1}^{M+1}N_m$ matrix,

$$\mathbb{A}\left(\mathbf{y}_{N_1}^{(1)},\mathbf{y}_{N_2}^{(2)},\dots,\mathbf{y}_{N_M}^{(M)}\right)=\left(\mathbf{a}^{m,n}(y_i^{(m)},y_j^{(n)})\right)_{1\leq i\leq N_m,1\leq j\leq N_n,1\leq m,n\leq M}$$

with the elements $\mathbf{a}^{m,n}\left(y_i^{(m)}, y_j^{(n)}\right) = \widetilde{\mathbb{S}}\left(s_m, y_i^{(m)}; s_n, y_j^{(n)}\right)$, where

$$\widetilde{\mathbb{S}}(s,x;t,y) = \begin{cases} \int_0^1 \mathrm{d}\theta\, J_\nu(2\sqrt{\theta x})J_\nu(2\sqrt{\theta y})e^{2(s-t)\theta}, & \text{if } s > t, \\ \dfrac{J_\nu(2\sqrt{x})\sqrt{y}J'_\nu(2\sqrt{y}) - J_\nu(2\sqrt{y})\sqrt{x}J'_\nu(2\sqrt{x})}{x-y}, & \text{if } s = t, \\ -\int_1^\infty \mathrm{d}\theta\, J_\nu(2\sqrt{\theta x})J_\nu(2\sqrt{\theta y})e^{2(s-t)\theta}, & \text{if } s < t, \end{cases}$$

with $J'_\nu(z) = \mathrm{d}J_\nu(z)/\mathrm{d}z$. Hence, in this limit we obtain a temporally homogeneous system of infinite number of particles, whose correlation functions are given by

$$\widetilde{\rho}^{\mathbf{Y}}\left(s_1, \{\mathbf{y}_{N_1}^{(1)}\}; \ldots; s_M, \{\mathbf{y}_{N_M}^{(M)}\}\right) = \det \mathbb{A}\left(\mathbf{y}_{N_1}^{(1)}, \ldots, \mathbf{y}_{N_M}^{(M)}\right). \tag{25}$$

Remark 1. Forrester et al. [15] studied the orthogonal–unitary and symplectic–unitary universality transitions in RM theory by giving the quaternion determinantal expressions of (two-time) correlation functions for parametric RM models. One of their results for the *Laguerre ensemble with* $\beta = 1$ *initial condition*, which shows the orthogonal–unitary transition, can be reproduced from Theorem 2.1 by setting

$$\text{(i)} \qquad \kappa = \nu \qquad \Longleftrightarrow \qquad \mathfrak{a} = \frac{\nu}{2}, \quad \mathfrak{b} = 0, \qquad \text{where} \quad \nu \in \mathbb{N}_0.$$

This fact may be readily seen, if we notice that by definition

$$\begin{aligned} \widetilde{\mathcal{J}}_\nu^{(-1)}(\theta, 1, x, s) &= \int_0^1 \mathrm{d}\eta\, (\theta\eta x)^{\nu/2} J_\nu(2\sqrt{\theta\eta x})e^{2s\theta\eta} \\ &= \theta^{-1}x^{\nu/2}\int_0^\theta \mathrm{d}u\, u^{\nu/2}J_\nu(2\sqrt{ux})e^{2su}. \end{aligned}$$

Remark 2. Nagao's result on the multitime correlation functions for *vicious random walk with a wall* [32] can be regarded as the special case of Theorem 2.1, in which

$$\text{(ii)} \qquad \nu = \frac{1}{2}, \quad \kappa = 1 \qquad \Longleftrightarrow \qquad \mathfrak{a} = 0, \quad \mathfrak{b} = -\frac{1}{2}.$$

This fact can be confirmed by noting that, by definition (16) with $\widetilde{J}_{1/2}(\theta,0;x,s) = 0$,

$$\widetilde{J}^{(-1/2)}_{1/2}(\theta,1,x,s) = \frac{(\theta x)^{1/4}}{\sqrt{\pi}} \int_0^1 d\eta\, (1-\eta)^{-1/2} \eta^{1/4} J_{1/2}(2\sqrt{\theta\eta x}) e^{2s\theta\eta},$$

$$\widetilde{J}^{(1/2)}_{1/2}(\theta,1,x,s) = \frac{(\theta x)^{1/4}}{\sqrt{\pi}} \int_0^1 d\eta\, (1-\eta)^{-1/2} \frac{d}{d\eta} \left\{ \eta^{1/4} J_{1/2}(2\sqrt{\theta\eta x}) e^{2s\theta\eta} \right\},$$

by Eq. (19),

$$\widehat{J}^{(1/2)}_{1/2}(\theta,\eta,x,s) = -\frac{(\theta x)^{-1/4}}{\sqrt{\pi}} \int_\eta^\infty d\xi\, (\xi-\eta)^{-1/2} \frac{d}{d\xi} \left\{ \xi^{-1/4} J_{1/2}(2\sqrt{\theta\xi x}) e^{2s\theta\xi} \right\},$$

for $s < 0$ and, by definition (17),

$$\int_1^\infty d\xi \widehat{J}^{(1/2)}_{1/2}(\theta,\xi,x,s) = \frac{(\theta x)^{-1/4}}{\sqrt{\pi}} \int_1^\infty d\eta\, (\eta-1)^{-1/2} \eta^{-1/4} J_{1/2}(2\sqrt{\theta\eta x}) e^{2s\theta\eta},$$

for $s < 0$. In this case, the system shows the transition between the class C and class CI of the Bogoliubov–de Gennes universality classes of nonstandard RM theory [26,27,32].

Remark 3. From the results for finite noncolliding processes [26], we expect that, when

$$\text{(iii)} \qquad \kappa = \nu + 1 \qquad \Longleftrightarrow \qquad \mathfrak{a} = \frac{\nu-1}{2}, \quad \mathfrak{b} = -1, \qquad \text{where} \quad \nu \in \mathbb{N}_0,$$

the present infinite particle system will show the transition from the chiral GUE to the chiral GOE of the universality classes and when

$$\text{(iv)} \qquad \nu = -\frac{1}{2}, \quad \kappa = 0 \qquad \Longleftrightarrow \qquad \mathfrak{a} = \mathfrak{b} = -\frac{1}{2},$$

that from the class D to the 'real-component version' of class D of the Bogoliubov–de Gennes universality classes [26].

Remark 4. Following the argument given in [20,39], tightness in time can be proved and transition phenomena observed in the limit $s_M \to 0$ may be generally discussed, which will be reported elsewhere.

Remark 5. The homogeneous system (25) was studied in [37,50].

3 Correlation functions given by Pfaffians

3.1 The multitime transition density

If we put

$$\widetilde{G}^{(\nu,\kappa)}(t,y|x) = G^{(\nu)}(t,y|x)\left(\frac{y}{x}\right)^{-\kappa}, \quad x>0, \quad y \in \mathbb{R}_+,$$
$$\widetilde{G}^{(\nu,\kappa)}(t,y|0) = G^{(\nu)}(t,y|0)y^{-\kappa}, \quad y \in \mathbb{R}_+,$$

the multitime transition densition (9) with $t_0 = 0, t_{M+1} = T$, and $\xi_0 = \{0\}$ is written as

$$\begin{aligned}
&\mathfrak{g}^{(\nu,\kappa)}_{N,T}\left(0,\{0\};t_1,\left\{\mathbf{x}^{(1)}_N\right\};\ldots;t_{M+1},\left\{\mathbf{x}^{(M+1)}_N\right\}\right) \\
&= C^{\nu,\kappa}_{N,T}(t_1)\prod_{1\le j<k\le N}\left\{\left(x^{(1)}_k\right)^2-\left(x^{(1)}_j\right)^2\right\}\prod_{1\le j<k\le N}\mathrm{sgn}\left(x^{(M+1)}_k - x^{(M+1)}_j\right) \\
&\quad\times\prod_{j=1}^N \widetilde{G}^{(\nu,\kappa)}\left(t_1,x^{(1)}_j|0\right)\prod_{m=1}^M \det_{1\le j,k\le N}\left[\widetilde{G}^{(\nu,\kappa)}\left(t_{m+1}-t_m,x^{(m+1)}_j|x^{(m)}_k\right)\right],
\end{aligned}$$

where Eqs. (6) and (8) with Eq. (7) are used.

Through the relation (11), the multitime transition density for the process $\{\mathbf{Y}(t)\}, t \in [0,T]$, denoted by $\mathfrak{p}^{(\nu,\kappa)}_{N,T}$ is then written as

$$\begin{aligned}
&\mathfrak{p}^{(\nu,\kappa)}_{N,T}\left(0,\{0\};t_1,\left\{\mathbf{y}^{(1)}\right\};\ldots;t_{M+1},\left\{\mathbf{y}^{(M+1)}\right\}\right) \\
&= C^{\nu,\kappa}_{N,T}(t_1)h_N(\mathbf{y}^{(1)})\mathrm{sgn}\left(h_N(\mathbf{y}^{(M+1)})\right)\prod_{k=1}^N \widetilde{p}^{(\nu,\kappa)}(t_1,y^{(1)}_k|0) \\
&\quad\times\prod_{m=1}^M \det_{1\le j,k\le N}\left[\widetilde{p}^{(\nu,\kappa)}(t_{m+1}-t_m,y^{(m+1)}_j|y^{(m)}_k)\right],
\end{aligned} \tag{26}$$

where

$$h_N(\mathbf{y}) \equiv \prod_{1\le i<j\le N}(y_j-y_i), \quad \mathbf{y}\in\mathbb{R}^N,$$
$$\begin{aligned}
\widetilde{p}^{(\nu,\kappa)}(t,y|0) &\equiv \widetilde{G}^{(\nu,\kappa)}(t,\sqrt{y}|0)\frac{1}{2}y^{-1/2} \\
&= \frac{y^{\mathfrak{a}}}{2^{\nu+1}\Gamma(\nu+1)t^{\nu+1}}e^{-y/2t}, \quad y\in\mathbb{R}_+,
\end{aligned}$$

$$\widetilde{p}^{(\nu,\kappa)}(t-s,y|x) \equiv \widetilde{G}^{(\nu,\kappa)}(t-s,\sqrt{y}|\sqrt{x})\frac{1}{2}y^{-1/2}$$
$$= \frac{e^{-(x+y)/\{2(t-s)\}}}{2(t-s)}\left(\frac{y}{x}\right)^{\mathfrak{b}/2} I_\nu\left(\frac{\sqrt{xy}}{t-s}\right), \quad x>0, \quad y \in \mathbb{R}_+. \tag{27}$$

Expectations related to the process $\{\mathbf{Y}(t_1)\}, \{\mathbf{Y}(t_2)\}, \dots, \{\mathbf{Y}(t_{M+1})\}$ are denoted by $\mathbb{E}^{\mathbf{Y}}_{N,T}$:

$$\mathbb{E}^{\mathbf{Y}}_{N,T}\Big[f(\{\mathbf{Y}(t_1)\},\dots,\{\mathbf{Y}(t_{M+1})\})\Big]$$
$$= \left(\frac{1}{N!}\right)^{M+1} \int_{\mathbb{R}_+^{N(M+1)}} \prod_{m=1}^{M+1} d\mathbf{y}^{(m)} f(\left\{\mathbf{y}^{(1)}\right\},\dots,\left\{\mathbf{y}^{(M+1)}\right\})$$
$$\times \mathfrak{p}^{(\nu,\kappa)}_{N,T}\Big(0,\{0\};t_1,\{\mathbf{y}^{(1)}\};\dots;t_{M+1},\{\mathbf{y}^{(M+1)}\}\Big). \tag{28}$$

3.2 Fredholm Pfaffian representation of characteristic function and Pfaffian process

For simplicity of expressions, we assume from now on that the number of particles N is even. The references [32,33] will be useful to give necessary modifications to the following expressions in the case that N is odd. Let $C_0(\mathbb{R})$ be the set of all continuous real functions with compact supports. For $\mathbf{f} = (f_1, f_2, \dots, f_{M+1}) \in C_0(\mathbb{R})^{M+1}$, and $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_{M+1}) \in \mathbb{R}^{M+1}$, the multitime characteristic function is defined for the process $\{\mathbf{Y}(t)\}, t \in [0,T]$ as

$$\Psi^{\mathbf{Y}}_{N,T}(\mathbf{f};\boldsymbol{\theta}) = \mathbb{E}^{\mathbf{Y}}_{N,T}\left[\exp\left\{\sqrt{-1}\sum_{m=1}^{M+1}\theta_m \sum_{i_m=1}^{N} f_m(Y_{i_m}(t_m))\right\}\right] \tag{29}$$

Let $\chi_m(x) = e^{\sqrt{-1}\theta_m f_m(x)} - 1$, $1 \le m \le M+1$. Then by the definition of multitime correlation function (11) with (10), we have

$$\Psi^{\mathbf{Y}}_{N,T}(\mathbf{f};\boldsymbol{\theta}) = \sum_{N_1=0}^{N} \cdots \sum_{N_{M+1}=0}^{N} \prod_{m=1}^{M+1} \frac{1}{N_m!}$$
$$\times \int_{\mathbb{R}_+^{N_1}} d\mathbf{y}^{(1)}_{N_1} \cdots \int_{\mathbb{R}_+^{N_{M+1}}} d\mathbf{y}^{(M+1)}_{N_{M+1}} \prod_{m=1}^{M+1}\prod_{i^{(m)}=1}^{N_m} \chi_m\Big(y^{(m)}_{i^{(m)}}\Big)$$
$$\times \rho^{\mathbf{Y}}_{N,T}\Big(t_1,\{\mathbf{y}^{(1)}_{N_1}\};\dots;t_{M+1},\{\mathbf{y}^{(M+1)}_{N_{M+1}}\}\Big), \tag{30}$$

that is, the multitime characteristic function is a generating function of multitime correlation functions $\rho^{\mathbf{Y}}_{N,T}$.

We consider a vector space $\mathcal{V}$ with the orthonormal basis $\Big\{|m,x\rangle\Big\}_{1\le m\le M+1, x\in\mathbb{R}_+}$, which satisfies

$$\langle m,x|n,y\rangle = \delta_{mn}\delta(x-y), \quad m,n=1,2,\dots,M+1, x,y\in\mathbb{R}_+, \tag{31}$$

where δ_{mn} and $\delta(x-y)$ denote Kronecker's delta and Dirac's δ-measure, respectively. We introduce the operators $\hat{J},\hat{p},\hat{p}_+,\hat{p}_-$ and $\hat{\chi}$ acting on $\mathcal{V}$ as follows

$$\langle m,x|\hat{J}|n,y\rangle = \mathbf{1}_{(m=n=M+1)}\mathrm{sgn}(y-x), \tag{32}$$

$$\begin{aligned}\langle m,x|\hat{p}|n,y\rangle &= \mathbf{1}_{(m<n)}\widetilde{p}^{(\nu,\kappa)}(t_n-t_m,y|x) + \mathbf{1}_{(m>n)}\widetilde{p}^{(\nu,\kappa)}(t_m-t_n,x|y) \\ &\quad +\mathbf{1}_{(m=n)}\delta(x-y),\end{aligned} \tag{33}$$

$$\langle m,x|\hat{p}_+|n,y\rangle = \mathbf{1}_{(m<n)}\widetilde{p}^{(\nu,\kappa)}(t_n-t_m,y|x) = \langle n,y|\hat{p}_-|m,x\rangle, \tag{34}$$

$$\langle m,x|\hat{\chi}|n,y\rangle = \chi_m(x)\delta_{mn}\delta(x-y), \tag{35}$$

and we will use the convention

$$\begin{aligned}\langle m,x|\hat{A}|n,y\rangle\langle n,y|\hat{B}|\ell,z\rangle &= \sum_{n=1}^{M+1}\int_{\mathbb{R}_+} \mathrm{d}y\, A(m,x;n,y)B(n,y;\ell,z) \\ &= \langle m,x|\hat{A}\hat{B}|\ell,z\rangle,\end{aligned}$$

for operators $\hat{A}$ and $\hat{B}$ with $\langle m,x|\hat{A}|n,y\rangle = A(m,x;n,y)$ and $\langle m,x|\hat{B}|n,y\rangle = B(m,x;n,y)$.

Let $M_i(x)$ be an arbitrary polynomial of x with degree i in the form $M_i(x) = b_i x^i + \cdots$ with a constant $b_i \neq 0$ for $i\in\mathbb{N}_0$. Since the product of differences $h_N(\mathbf{x})$ is equal to the Vandermonde determinant, we have

$$h_N(\mathbf{x}) = \left\{\prod_{k=1}^{N} b_{k-1}\right\}^{-1} \det_{1\le i,j\le N}\Big[M_{i-1}(x_j)\Big]. \tag{36}$$

Then we consider the set of linearly independent vectors $\Big\{|i\rangle\,;\, i\in\mathbb{N}\Big\}$ in $\mathcal{V}$ defined by

$$|i\rangle = |m,x\rangle\langle m,x|i\rangle,$$

where

$$\langle m,x|i\rangle = \langle i|m,x\rangle = \int_{\mathbb{R}_+} \mathrm{d}y M_{i-1}(y)\widetilde{p}^{(\nu,\kappa)}(t_1,y|0)\widetilde{p}^{(\nu,\kappa)}(t_m-t_1,x|y), \tag{37}$$

$i \in \mathbb{N}, m = 1, 2, \dots, M+1, x \in \mathbb{R}_+$. We will use the convention

$$\langle i|\hat{A}|j\rangle\langle j|\hat{B}|m,x\rangle = \sum_{j=1}^{\infty} A_{ij} B_j^{(m)}(x) = \langle i|\hat{A} \circ \hat{B}|m,x\rangle,$$

for $A_{ij} = \langle i|\hat{A}|j\rangle$ and $B_j^{(m)}(x) = \langle j|\hat{B}|m,x\rangle$. It should be noted that the vecors $\Big\{|i\rangle\,;\, i \in \mathbb{N}\Big\}$ are not assumed to be mutually orthogonal. By these vectors, however, any operator $\hat{A}$ on $\mathcal{V}$ may have a semi-infinite matrix representation $A = \Big(\langle i|\hat{A}|j\rangle\Big)_{i,j \in \mathbb{N}}$. If the matrix A representing an operator $\hat{A}$ is invertible, we define the operator $\hat{A}^{\triangle}$ so that its matrix representation is the inverse of A

$$\Big(\langle i|\hat{A}^{\triangle}|j\rangle\Big)_{i,j \in \mathbb{N}} = A^{-1}, \tag{38}$$

that is, $\langle i|\hat{A}|j\rangle\langle j|\hat{A}^{\triangle}|k\rangle = \langle i|\hat{A} \circ \hat{A}^{\triangle}|k\rangle = \delta_{ik}$, $i, k \in \mathbb{N}$.

Let $\mathcal{P}_N$ be a linear operator projecting $\text{Span}\Big\{|i\rangle\,;\, i \in \mathbb{N}\Big\}$ to its N-dimensional subspace $\text{Span}\Big\{|i\rangle\,;\, i = 1, 2, \dots, N\Big\}$ such that

$$\langle i|\mathcal{P}_N|m,x\rangle = \langle m,x|\mathcal{P}_N|i\rangle = \begin{cases} \langle i|m,x\rangle, & \text{if } 1 \le i \le N, \\ 0, & \text{otherwise.} \end{cases}$$

We will use the abbreviation $\hat{A}_N = \mathcal{P}_N \hat{A} \mathcal{P}_N$ for an operator $\hat{A}$. If the $N \times N$ matrix defined by $A_N = (\langle i|\hat{A}_N|j\rangle)_{1 \le i,j \le N}$ is invertible, then $(\hat{A}_N)^{\triangle}$ is defined so that $\Big(\langle i|(\hat{A}_N)^{\triangle}|j\rangle\Big)_{1 \le i,j \le N} = (A_N)^{-1}$, and $\langle i|(\hat{A}_N)^{\triangle}|j\rangle = 0$, if $i \ge N+1$ or $j \ge N+1$.

As shown in Appendix A., we can prove that

$$\Big\{\Psi_{N,T}^{\mathbf{Y}}(\mathbf{f};\boldsymbol{\theta})\Big\}^2 = \text{Det}\left(I_2 \delta_{mn} \delta(x-y) + \begin{pmatrix} \tilde{S}^{m,n}(x,y) & \tilde{I}^{m,n}(x,y) \\ D^{m,n}(x,y) & \tilde{S}^{n,m}(y,x) \end{pmatrix} \chi_n(y)\right), \tag{39}$$

where Det denotes the Fredholm determinant. Here I_2 is the unit matrix with size 2,

$$\begin{aligned} D^{m,n}(x,y) &= -\langle m,x| \circ (\hat{J}_N)^{\triangle} \circ |n,y\rangle, \\ S^{m,n}(x,y) &= \langle m,x|\hat{p}\hat{J} \circ (\hat{J}_N)^{\triangle} \circ |n,y\rangle, \\ I^{m,n}(x,y) &= -\langle m,x|\hat{p}\hat{J} \circ (\hat{J}_N)^{\triangle} \circ \hat{J}\hat{p}|n,y\rangle, \end{aligned} \tag{40}$$

and

$$
\begin{aligned}
\tilde{S}^{m,n}(x,y) &= S^{m,n}(x,y) - \langle m,x|\hat{p}_+|n,y\rangle \\
\tilde{I}^{m,n}(x,y) &= I^{m,n}(x,y) + \langle m,x|\hat{p}\hat{J}\hat{p}|n,y\rangle.
\end{aligned}
\tag{41}
$$

It implies that the multitime characteristic function is given by the *Fredholm Pfaffian* [40],

$$
\Psi^{\mathbf{Y}}_{N,T}(\mathbf{f};\boldsymbol{\theta}) = \mathrm{PF}\Big(J_2\delta_{mn}\delta(x-y) + \sqrt{\chi_m(x)}A^{m,n}(x,y)\sqrt{\chi_n(y)}\Big), \tag{42}
$$

where $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$
\begin{aligned}
A^{m,n}(x,y) &= J_2 \begin{pmatrix} \tilde{S}^{m,n}(x,y) & \tilde{I}^{m,n}(x,y) \\ D^{m,n}(x,y) & \tilde{S}^{n,m}(y,x) \end{pmatrix} \\
&= \begin{pmatrix} D^{m,n}(x,y) & \tilde{S}^{n,m}(y,x) \\ -\tilde{S}^{m,n}(x,y) & -\tilde{I}^{m,n}(x,y) \end{pmatrix}.
\end{aligned}
\tag{43}
$$

It is defined by

$$
\begin{aligned}
&\mathrm{PF}\Big(J_2\delta_{mn}\delta(x-y) + \sqrt{\chi_m(x)}A^{m,n}(x,y)\sqrt{\chi_n(y)}\Big) \\
&= \sum_{N_1=0}^{N} \cdots \sum_{N_{M+1}=0}^{N} \prod_{m=1}^{M+1} \frac{1}{N_m!} \\
&\quad \times \int_{\mathbb{R}_+^{N_1}} d\mathbf{y}^{(1)}_{N_1} \cdots \int_{\mathbb{R}_+^{N_{M+1}}} d\mathbf{y}^{(M+1)}_{N_{M+1}} \prod_{m=1}^{M+1} \prod_{i^{(m)}=1}^{N_m} \chi_m\Big(y^{(m)}_{i^{(m)}}\Big) \\
&\quad \times \mathrm{Pf}\Big(A\Big(\mathbf{y}^{(1)}_{N_1}, \mathbf{y}^{(2)}_{N_2}, \ldots, \mathbf{y}^{(M+1)}_{N_{M+1}}\Big)\Big),
\end{aligned}
\tag{44}
$$

where $A\Big(\mathbf{y}^{(1)}_{N_1}, \mathbf{y}^{(2)}_{N_2}, \ldots, \mathbf{y}^{(M+1)}_{N_{M+1}}\Big)$ denotes the $2\sum_{m=1}^{M+1} N_m \times 2\sum_{m=1}^{M+1} N_m$ skew-symmetric matrices constructed from Eq. (43) as

$$
A\Big(\mathbf{y}^{(1)}_{N_1}, \mathbf{y}^{(2)}_{N_2}, \ldots, \mathbf{y}^{(M+1)}_{N_{M+1}}\Big) = \Big(A^{m,n}(y^{(m)}_i, y^{(n)}_j)\Big)_{1\le i\le N_m, 1\le j\le N_n, 1\le m,n\le M+1}
$$

for $N_m = 1, 2, \ldots, N, 1 \le m \le M+1$. Comparison of Eqs. (30) and (42) with Eq. (44) immediately gives the following statement.

Theorem 3.1. *The N-particle non-colliding system of squared generalized meanders $\mathbf{Y}(t), t \in [0,T]$ is a Pfaffian process, in the sense that any multitime correlation function is given by a Pfaffian*

$$
\rho^{\mathbf{Y}}_{N,T}\left(t_1, \{\mathbf{y}^{(1)}_{N_1}\}; \ldots; t_{M+1}, \{\mathbf{y}^{(M+1)}_{N_{M+1}}\}\right) = \mathrm{Pf}\left(A\Big(\mathbf{y}^{(1)}_{N_1}, \ldots, \mathbf{y}^{(M+1)}_{N_{M+1}}\Big)\right).
$$

4 Skew-orthogonal functions and matrix inversion

4.1 Skew-symmetric inner products

Consider the $N \times N$ skew-symmetric matrix $A_0 = ((A_0)_{ij})_{1 \le i,j \le N}$ with

$$(A_0)_{ij} = \langle i|\hat{J}_N|j\rangle = \langle i|m,x\rangle\langle m,x|\hat{J}|n,y\rangle\langle n,y|j\rangle, \quad i,j = 1,2,\ldots,N. \tag{45}$$

In order to clarify the fact that each element $(A_0)_{ij}$ is a functional of the polynomials $M_{i-1}(x)$ and $M_{j-1}(x)$ through Eq. (37), we introduce the skew-symmetric inner product

$$\langle f,g\rangle \equiv \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, F(x,y)\widetilde{p}^{(\nu,\kappa)}(t_1,x|0)\widetilde{p}^{(\nu,\kappa)}(t_1,y|0)f(x)g(y), \tag{46}$$

where

$$F(x,y) = \int_0^\infty \mathrm{d}w \int_0^w \mathrm{d}z \begin{vmatrix} \widetilde{p}^{(\nu,\kappa)}(T-t_1,z|x) & \widetilde{p}^{(\nu,\kappa)}(T-t_1,w|x) \\ \widetilde{p}^{(\nu,\kappa)}(T-t_1,z|y) & \widetilde{p}^{(\nu,\kappa)}(T-t_1,w|y) \end{vmatrix}, \tag{47}$$

for $x,y \in \mathbb{R}_+$. Then we have the expression

$$(A_0)_{ij} = \langle M_{i-1}, M_{j-1}\rangle, \quad i,j = 1,2,\ldots,N. \tag{48}$$

We now rewrite the skew-symmetric inner product (46) by using the simpler one

$$\begin{aligned}\langle f,g\rangle_* &= -\langle g,f\rangle_* \\ &\equiv \int_0^\infty \mathrm{d}w\, e^{-w/2}w^{\mathfrak{a}} \int_0^w \mathrm{d}z\, e^{-z/2}z^{\mathfrak{a}}\Big\{f(z)g(w) - f(w)g(z)\Big\},\end{aligned} \tag{49}$$

which we call the *elementary skew-symmetric inner product.* Remind that $\widetilde{p}^{(\nu,\kappa)}$ is given by Eq. (27) using the modified Bessel function. We will expand it in terms of the Laguerre polynomials, $L_j^\alpha(x) = (x^{-\alpha}e^x/j!)(\mathrm{d}/\mathrm{d}x)^j(e^{-x}x^{j+\alpha})$, $\alpha \in \mathbb{R}$, $j \in \mathbb{N}_0$, using the formula

$$\sum_{j=0}^\infty \frac{\Gamma(j+1)L_j^\nu(x)L_j^\nu(y)r^j}{\Gamma(j+1+\nu)} = \frac{(xyr)^{-\nu/2}}{1-r}e^{-\frac{(x+y)r}{1-r}}I_\nu\left(\frac{2\sqrt{xyr}}{1-r}\right), \tag{50}$$

for $|r| < 1, \nu > -1$. (See the corresponding calculation for the noncolliding Brownian particles in [22], where the heat kernel was expanded in terms of the

Hermite polynomials.) For this purpose, it is useful to introduce the variables

$$c_n = \frac{t_n(2T - t_n)}{T}, \qquad \chi_n = \frac{2T - t_n}{t_n}, \quad n = 1, 2, \ldots, M+1,$$

since we can see that

$$\widetilde{p}^{(\nu,\kappa)}(t_n - t_m, c_n\eta | c_m\xi) = \frac{1}{2(t_n - t_m)} I_\nu \left(\frac{2\sqrt{\xi\eta\chi_n/\chi_m}}{1 - \chi_n/\chi_m} \right) \left(\frac{c_n\eta}{c_m\xi} \right)^{\mathfrak{b}/2}$$
$$\times \exp\left[-\left(\frac{1}{1 - \chi_n/\chi_m} - 1 + \frac{t_m}{2T} \right) \xi - \left(\frac{1}{1 - \chi_n/\chi_m} - \frac{t_n}{2T} \right) \eta \right],$$

and, if we apply the formula (50) with $r = \chi_n/\chi_m, x = \xi$ and $y = \eta$, it is written as

$$\widetilde{p}^{(\nu,\kappa)}(t_n - t_m, c_n\eta | c_m\xi) = \left(\frac{t_m}{t_n} \right)^{\nu+1} c_m^{-\mathfrak{a}-1} \xi^{\kappa/2} (c_n\eta)^{\mathfrak{a}}$$
$$\times \exp\left[-\frac{t_m}{2T}\xi - \left(1 - \frac{t_n}{2T} \right) \eta \right] \sum_{j=0}^{\infty} \frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} \left(\frac{\chi_n}{\chi_m} \right)^j L_j^\nu(\xi) L_j^\nu(\eta). \tag{51}$$

That is, c_n and χ_n give the spatial scale of spread of N particles and the proper temporal factor at time t_n, respectively. (See Eq. (17) and explanation below it in [34], where the variable c_n was determined by showing that the one-particle density obeys Wigner's semicircle law scaled by c_n for the non-colliding Brownian particles.) In particular, for $n = M+1$ we have

$$\widetilde{p}^{(\nu,\kappa)}(T - t_m, T\eta | c_m\xi) = \frac{t_m^{\nu+1}}{T^{\kappa/2+1}} c_m^{-\mathfrak{a}-1} \xi^{\kappa/2} \eta^{\mathfrak{a}} \exp\left[\left(1 - \frac{t_m}{T} \right) \frac{\xi}{2} \right]$$
$$\times e^{-\xi/2} e^{-\eta/2} \sum_{j=0}^{\infty} \frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} \chi_m^{-j} L_j^\nu(\xi) L_j^\nu(\eta), \tag{52}$$

since $c_{M+1} = T$ and $\chi_{M+1} = 1$. Then we obtain the relation

$$\left\langle f\left(\frac{\cdot}{c_1} \right), g\left(\frac{\cdot}{c_1} \right) \right\rangle = \frac{2^{-2\nu-2} T^{-\kappa}}{\Gamma(\nu+1)^2} \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \, e^{-x} e^{-y} x^\nu y^\nu f(x) g(y)$$
$$\times \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \chi_1^{-j-k} L_j^\nu(x) L_k^\nu(y)$$
$$\times \left\langle \frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} L_j^\nu, \frac{\Gamma(k+1)}{\Gamma(k+1+\nu)} L_k^\nu \right\rangle_*. \tag{53}$$

4.2 Skew-orthogonal polynomials

For $n \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$ we define

$$\binom{n+\alpha}{n} = \begin{cases} \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}, & \text{if } n \in \mathbb{N}, \alpha \notin \mathbb{Z}_-, \\ \dfrac{(-1)^n \Gamma(-\alpha)}{\Gamma(n+1)\Gamma(-n-\alpha)}, & \text{if } n \in \mathbb{N}, n+\alpha \in \mathbb{Z}_-, \\ 0, & \text{if } n \in \mathbb{N}, \alpha \in \mathbb{Z}_-, n+\alpha \in \mathbb{N}_0, \\ 1, & \text{if } n = 0, \\ 0, & \text{if } n \in \mathbb{Z}_-. \end{cases} \tag{54}$$

Note that for $n \in \mathbb{N}, \alpha \in \mathbb{Z}_-$ with $n+\alpha \leq -1$,

$$\binom{n+\alpha}{n} = (-1)^n \binom{-\alpha-1}{n}.$$

By this definition, the equality

$$\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n x^{n+\alpha}\bigg|_{x=1} = \binom{n+\alpha}{n} \tag{55}$$

holds for $n \in \mathbb{N}_0,\ \alpha \in \mathbb{R}$. Then Laguerre polynomials can be expressed as

$$L_j^{\alpha}(x) = \sum_{\ell=0}^{j} \frac{(-1)^\ell}{\ell!} \binom{j+\alpha}{j-\ell} x^\ell \tag{56}$$

for any $\alpha \in \mathbb{R}$. Remark that applying Eq. (55) to the equation

$$\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n x^{n+\alpha} = \frac{1}{(n-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1} x^{(n-1)+\alpha} + x\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n x^{n+(\alpha-1)},$$

with $x = 1$ and putting $\beta = \alpha + n$, we have the identity

$$\binom{\beta}{n} = \binom{\beta-1}{n} + \binom{\beta-1}{n-1}, \quad n \in \mathbb{Z}, \quad \beta \in \mathbb{R}. \tag{57}$$

We use the following orthogonal relations and formulae on Laguerre polynomials, which hold for $\alpha, \beta > -1$;

$$\int_0^\infty L_j^\alpha(x) L_k^\alpha(x) x^\alpha e^{-x} \mathrm{d}x = \frac{\Gamma(\alpha+j+1)}{\Gamma(j+1)} \delta_{jk}, \quad j,k \in \mathbb{N}_0, \tag{58}$$

$$x \frac{\mathrm{d}}{\mathrm{d}x} L_j^\alpha(x) = j L_j^\alpha(x) - (j+\alpha) L_{j-1}^\alpha(x), \quad j \in \mathbb{N}, \tag{59}$$

$$L_j^\alpha(x) = -\frac{\mathrm{d}}{\mathrm{d}x} L_{j+1}^\alpha(x) + \frac{\mathrm{d}}{\mathrm{d}x} L_j^\alpha(x), \quad j \in \mathbb{N}_0, \tag{60}$$

$$L_j^\beta(x) = \sum_{k=0}^{j} \binom{j-k+\beta-\alpha-1}{j-k} L_k^\alpha(x), \quad j \in \mathbb{N}_0. \tag{61}$$

Remark 6. The identities (59) and (60) are given as Eqs. (6.2.6) and (6.2.7) in [3]. The relation (61) is proved in [3] as (6.2.37) only when $\beta \geq \alpha > -1$. The identity (see Eq. (54) and [42])

$$\sum_{\ell=0}^{k} \binom{\ell-\alpha-1}{\ell} \binom{k-\ell+\alpha-1}{k-\ell} = \delta_{k0},$$

can be used to invert the relation (61) to the form

$$L_\ell^\alpha(x) = \sum_{j=0}^{\ell} \binom{\ell-j+\alpha-\beta-1}{\ell-j} L_j^\beta(x), \quad \ell \in \mathbb{N}_0.$$

Therefore, the validity of Eq. (61) for $\beta \geq \alpha > -1$ implies that for $\alpha > \beta > -1$.

We introduce the polynomials

$$F_j(x) = -\frac{\mathrm{d}}{\mathrm{d}x} L_{j+1}^{2\mathfrak{a}}(x), \quad j \in \mathbb{N}_0, \tag{62}$$

$$G_j(x) = \frac{\mathrm{d}}{\mathrm{d}x} \left\{ L_{j+1}^{2\mathfrak{a}}(x) - \frac{j+2\mathfrak{a}}{j} L_{j-1}^{2\mathfrak{a}}(x) \right\}, \quad j \in \mathbb{N}. \tag{63}$$

For $k \in \mathbb{N}_0, j = 0, 1, 2, \ldots, k$, let

$$\alpha_{k,j} = \binom{k-j+\mathfrak{b}}{k-j}, \qquad \text{if } k \text{ is even},$$

$$\alpha_{k,j} = \frac{k+2\mathfrak{a}}{k} \binom{k-2-j+\mathfrak{b}}{k-2-j} - \binom{k-j+\mathfrak{b}}{k-j}, \quad \text{if } k \text{ is odd}, \tag{64}$$

The following lemmas are derived from the relations (58)–(61) with some calculations.

Lemma 4.1. *For* $\ell \in \mathbb{N}_0$

$$F_{2\ell}(x) = \sum_{j=0}^{2\ell} \alpha_{2\ell,j} L_j^{\nu}(x), \tag{65}$$

$$G_{2\ell+1}(x) = \sum_{j=0}^{2\ell+1} \alpha_{2\ell+1,j} L_j^{\nu}(x). \tag{66}$$

Lemma 4.2. *For* $q, \ell \in \mathbb{N}_0$

$$\langle F_{2q}, G_{2\ell+1}\rangle_* = -\langle G_{2\ell+1}, F_{2q}\rangle_* = r_q^* \delta_{q\ell}, \tag{67}$$

$$\langle F_{2q}, F_{2\ell}\rangle_* = 0, \tag{68}$$

$$\langle G_{2q+1}, G_{2\ell+1}\rangle_* = 0, \tag{69}$$

with

$$r_q^* \equiv \frac{4\Gamma(2q+2\mathfrak{a}+2)}{(2q+1)!} = 4\Gamma(2\mathfrak{a}+1)\binom{2q+2\mathfrak{a}+1}{2q+1}. \tag{70}$$

Then if we define the monic polynomials in x of degree k for $k \in \mathbb{N}_0$ as

$$R_k(x) = k!\left(\frac{c_1}{\chi_1}\right)^k \sum_{j=0}^{k} \alpha_{k,j} L_j^{\nu}\left(\frac{x}{c_1}\right)\chi_1^j, \tag{71}$$

Lemma 4.2 gives the following through the relation (53) and the orthogonality of the Laguerre polynomials (58).

Lemma 4.3. *For* $q, \ell \in \mathbb{N}_0$

$$\langle R_{2q}, R_{2\ell+1}\rangle = -\langle R_{2\ell+1}, R_{2q}\rangle = r_q \delta_{q\ell},$$
$$\langle R_{2q}, R_{2\ell}\rangle = 0, \quad \langle R_{2q+1}, R_{2\ell+1}\rangle = 0,$$

where

$$r_q = 2^{-2\nu} T^{-\kappa}\left(\frac{t_1^2}{T}\right)^{4q+1} \frac{(2q)!\Gamma(2q+2+2\mathfrak{a})}{\Gamma(\nu+1)^2}. \tag{72}$$

The choice of the polynomials $F_j(x)$ and $G_j(x)$ in Eqs. (62) and (63), and their explicit expansions in terms of the Laguerre polynomials (Lemma 4.1) are crucial, since they enable us to determine the appropriate skew-orthogonal polynomials (Lemma 4.3). As shown below, we are able to inverse the skew-symmetric matrix A_0 given by Eq. (45) readily for arbitrary (even) N, by using these skew-orthogonal polynomials.

4.3 Matrix inversion

Let $b_{2k} = b_{2k+1} = r_k^{-1/2}, k \in \mathbb{N}_0$, and determine the polynomials $\{M_i(x)\}_{0 \le i \le N-1}$ in Eq. (37) as

$$M_i(x) = b_i R_i(x), \quad i = 0, 1, \dots, N-1.$$

Then by Eqs. (45) and (48) and Lemma 4.3, we have the equality

$$\langle i|\hat{J}_N|j\rangle = (J_N)_{ij}, \quad i, j = 1, 2, \dots, N, \tag{73}$$

where $J_N = I_{N/2} \otimes J_2$. It is interesting to compare this result with Eq. (32). Since $J_N^2 = -I_N$, we can immediately obtain the inversion matrix appearing in Eq. (40) as

$$\langle i|(\hat{J}_N)^{\triangle}|j\rangle = -(J_N)_{ij}, \quad i, j = 1, 2, \dots, N. \tag{74}$$

If we consider a semi-infinite matrix

$$J \equiv \lim_{N\to\infty} J_N = \Big(\langle i|\hat{J}|j\rangle\Big)_{i,j\in\mathbb{N}},$$

its inverse matrix may be given by

$$J^{-1} = \Big(\langle i|\hat{J}^{\triangle}|j\rangle\Big)_{i,j\in\mathbb{N}} = -J.$$

Using expansions (86), (93) and Lemma 5.4 given below with Lemmas 4.1 and 4.2, we can show

$$\langle m, x|\hat{p}|n, y\rangle = \langle m, x|\hat{p}\hat{J}|i\rangle\langle i|\hat{J}^{\triangle}|j\rangle\langle j|n, y\rangle,$$

and so

$$\langle m, x|\hat{p}\hat{J}\hat{p}|n, y\rangle = \langle m, x|\hat{p}\hat{J}|i\rangle\langle i|\hat{J}^{\triangle}|j\rangle\langle j|\hat{J}\hat{p}|n, y\rangle.$$

Then Eq. (41) is written as

$$\tilde{S}^{m,n}(x, y) = \begin{cases} \langle m, x|\hat{p}\hat{J}|i\rangle\langle i|(\hat{J}_N)^{\triangle}|j\rangle\langle j|n, y\rangle, & \text{if } m \ge n, \\ -\langle m, x|\hat{p}\hat{J}|i\rangle\langle i|(\hat{J}^{\triangle} - (\hat{J}_N)^{\triangle})|j\rangle\langle j|n, y\rangle, & \text{if } m < n, \end{cases}$$

$$\tilde{I}^{m,n}(x, y) = \langle m, x|\hat{p}\hat{J}|i\rangle\langle i|(\hat{J}^{\triangle} - (\hat{J}_N)^{\triangle})|j\rangle\langle j|\hat{J}\hat{p}|n, y\rangle. \tag{75}$$

Now we introduce the notations, just following the previous papers for multimatrix models [15,31,33], as

$$R_i^{(m)}(x) \equiv \frac{1}{b_i}\langle m,x|i+1\rangle$$
$$= \int_0^\infty \mathrm{d}y\, R_i(y)\widetilde{p}^{(\nu,\kappa)}(t_1,y|0)\widetilde{p}^{(\nu,\kappa)}(t_m-t_1,x|y), \tag{76}$$
$$\Phi_i^{(m)}(x) \equiv -\frac{1}{b_i}\langle m,x|\hat{p}\hat{J}|i+1\rangle$$
$$= \int_0^\infty \mathrm{d}y\, R_i^{(m)}(y)F^{(m)}(y,x), \tag{77}$$

for $i=0,1,\dots,N-1, m=1,2,\dots,M+1$, where

$$F^{(m)}(x,y) = \int_0^\infty \mathrm{d}w \int_0^w \mathrm{d}z \begin{vmatrix} \widetilde{p}^{(\nu,\kappa)}(T-t_m,z|x) & \widetilde{p}^{(\nu,\kappa)}(T-t_m,w|x) \\ \widetilde{p}^{(\nu,\kappa)}(T-t_m,z|y) & \widetilde{p}^{(\nu,\kappa)}(T-t_m,w|y) \end{vmatrix}. \tag{78}$$

It should be noted that $R_i^{(1)}(x) = R_i(x)\widetilde{p}^{(\nu,\kappa)}(t_1,x|0), 0 \le i \le N-1$, and $F^{(1)}(x,y) = F(x,y)$, where $R_i(x)$ and $F(x,y)$ were defined by Eqs. (71) and (47), respectively. Then we arrive at the following explicit expressions for the elements of matrix kernel (43) of our Pfaffian processes,

$$D^{m,n}(x,y) = D_N^{m,n}(x,y) = \sum_{\ell=0}^{(N/2)-1} \frac{1}{r_\ell}\Big[R_{2\ell}^{(m)}(x)R_{2\ell+1}^{(n)}(y) - R_{2\ell+1}^{(m)}(x)R_{2\ell}^{(n)}(y)\Big],$$
$$\tilde{I}^{m,n}(x,y) = \tilde{I}_N^{m,n}(x,y) = -\sum_{\ell=N/2}^{\infty} \frac{1}{r_\ell}\Big[\Phi_{2\ell}^{(m)}(x)\Phi_{2\ell+1}^{(n)}(y) - \Phi_{2\ell+1}^{(m)}(x)\Phi_{2\ell}^{(n)}(y)\Big],$$
$$S^{m,n}(x,y) = S_N^{m,n}(x,y) = \sum_{\ell=0}^{(N/2)-1} \frac{1}{r_\ell}\Big[\Phi_{2\ell}^{(m)}(x)R_{2\ell+1}^{(n)}(y) - \Phi_{2\ell+1}^{(m)}(x)R_{2\ell}^{(n)}(y)\Big], \tag{79}$$

and

$$\tilde{S}^{m,n}(x,y) = \tilde{S}_N^{m,n}(x,y) = S^{m,n}(x,y) - \widetilde{p}^{(\nu,\kappa)}(t_n-t_m,y|x)\mathbf{1}_{(m<n)}. \tag{80}$$

5 Asymptotic behavior of correlation functions

In this section, we give the proof of our main theorem (Theorem 2.1), by estimating the $N \to \infty$ asymptotic of matrix kernel (43) of Theorem 3.1.

Elementary calculation needed for the estimation is summarized in Appendix B. Here $a_N \sim b_N, N \to \infty$ means $a_N/b_N \to 1, N \to \infty$. We assume that $T = N, t_m = T + s_m, 1 \le m \le M+1$ with $s_1 < s_2 < \cdots < s_M < s_{M+1} = 0$. We put

$$L_j^{\nu}(x, -s_m) = L_j^{\nu}(x)\, \chi_m^j, \quad \widehat{L}_j^{\nu}(x, s_m) = \frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} L_j^{\nu}(x)\, \chi_m^{-j}. \tag{81}$$

5.1 Asymptotics of $R_k(x)$ and $R_k^{(m)}(x)$

Let

$$\widehat{R}_k(x) = \frac{1}{k!}\left(\frac{t_1^2}{T}\right)^{-k} R_k(x) = \sum_{j=0}^{k} \alpha_{k,j} L_j^{\nu}\left(\frac{x}{c_1}, -s_1\right).$$

Since $c_1 \sim N = T$,

$$\begin{aligned} \widehat{R}_{2\ell}(x) &\sim I(2\ell, \mathfrak{b}), \\ \widehat{R}_{2\ell+1}(x) &\sim \frac{2\mathfrak{a}}{2\ell+1} I(2\ell-1, \mathfrak{b}) - I(2\ell+1, \mathfrak{b}-1) - I(2\ell, \mathfrak{b}-1) \\ &\sim \frac{\mathfrak{a}}{\ell} I(2\ell, \mathfrak{b}) - 2I(2\ell, \mathfrak{b}-1), \quad N \to \infty, \end{aligned} \tag{82}$$

where

$$I(q,c) \equiv \sum_{j=0}^{q} \binom{q-j+c}{q-j} L_j^{\nu}\left(\frac{x}{N}, -s_1\right) = \sum_{j=0}^{q} \binom{j+c}{j} L_{q-j}^{\nu}\left(\frac{x}{N}, -s_1\right),$$

for $q \in \mathbb{N}$ and $c \in \mathbb{R}$. We set

$$2\ell = N\theta,$$

and examine the asymptotic behavior of $I(2\ell, c)$ as $N \to \infty$ with some $\theta \in (0, \infty)$. When $c \in \mathbb{Z}_-$, $\binom{j+c}{j} = (-1)^j \binom{-c-1}{j}$. Then from Eq. (B.10) in Lemma B.2 with $j = 2\ell$ (*i.e.* $\eta = 1$ in Eq. (B.5)), we can easily see

$$I(2\ell, c) = \sum_{j=0}^{-c-1} \binom{j+c}{j} L_{2\ell-j}^{\nu}\left(\frac{x}{N}, -s_1\right) \sim \frac{(N\theta)^{c+\nu+1}}{(\theta x)^{\nu}} \widetilde{J}_{\nu}^{(-c-1)}(\theta, 1, x, -s_1),$$

$N \to \infty$. This result is generalized to the following lemma.

Lemma 5.1. *For any $c \in \mathbb{R}$, $\theta \in (0,\infty)$, we have*

$$I(2\ell,c) \sim \frac{(N\theta)^{c+\nu+1}}{(\theta x)^{\nu}} \widetilde{J}_{\nu}^{(-c-1)}(\theta,1,x,-s_1), \quad N \to \infty. \tag{83}$$

Proof.

$$\begin{aligned} I(2\ell,c) &= \sum_{p=0}^{2\ell} \binom{p+c}{p} L_{2\ell-p}^{\nu}\left(\frac{x}{N},-s_1\right) \\ &= \sum_{p=0}^{2\ell} \binom{p+c}{p} \sum_{k=0}^{p-1} \sum_{q=0}^{1} (-1)^{q+1} \binom{1}{q} L_{2\ell-k-q}^{\nu}\left(\frac{x}{N},-s_1\right) \\ &\quad + \sum_{p=0}^{2\ell} \binom{p+c}{p} L_{2\ell}^{\nu}\left(\frac{x}{N},-s_1\right). \end{aligned}$$

Repeating this procedure, we have

$$I(2\ell,c) = \sum_{k=0}^{\infty} (-1)^k a_k(2\ell,c) \sum_{q=0}^{k} (-1)^q \binom{k}{q} L_{2\ell-q}^{\nu}\left(\frac{x}{N},-s_1\right) \tag{84}$$

with

$$a_k(2\ell,c) = \sum_{p=0}^{2\ell} \binom{p+c}{p} \sum_{j_1=0}^{p-1} \sum_{j_2=0}^{j_1-1} \cdots \sum_{j_k=0}^{j_{k-1}-1} 1 = \sum_{p=0}^{2\ell} \binom{p+c}{p} \binom{p}{k}.$$

Using Eq. (57), we can rewrite $a_k(2\ell,c)$ as

$$\begin{aligned} a_k(2\ell,c) &= \sum_{p=0}^{2\ell} \left[\binom{p+c+1}{p} \binom{p}{k} - \binom{p+c}{p-1} \binom{p-1}{k} \right. \\ &\quad \left. + \binom{p+c}{p-1} \left\{ \binom{p-1}{k} - \binom{p}{k} \right\} \right] \\ &= \binom{2\ell+c+1}{2\ell} \binom{2\ell}{k} - a_{k-1}(2\ell-1,c+1). \end{aligned}$$

Using this equation recursively, we obtain

$$a_k(2\ell,c) = \sum_{r=0}^{k} (-1)^r \binom{2\ell+c+1}{2\ell-r} \binom{2\ell-r}{k-r} = \binom{2\ell+c+1}{2\ell-k} \binom{c+k}{k}. \tag{85}$$

Thus Eq. (84) with Eq. (85) gives

$$I(2\ell,c)=\sum_{k=0}^{\infty}(-1)^k\binom{2\ell+c+1}{2\ell-k}\binom{c+k}{k}\sum_{q=0}^{k}(-1)^q\binom{k}{q}L_{2\ell-q}^{\nu}\left(\frac{x}{N},-s_1\right).$$

By simple calculation with the estimate (B.2) and (B.10) of Lemma B.2, we obtain Eq. (83) through the expression (18). □

From above asymptotic of $I(2\ell,c)$ with Eq. (82) we have the following proposition.

Proposition 5.2. (1) *Suppose that $\ell\in\mathbb{N}$ and $2\ell\sim N\theta$, $N\to\infty$ for some $\theta\in(0,\infty)$. Then*

$$\widehat{R}_{2\ell}(x)\sim\frac{(N\theta)^{\mathfrak{b}+\nu}}{(\theta x)^{\nu}}\widetilde{J}_{\nu}^{(-\mathfrak{b}-1)}(\theta,1,x,-s_1),\quad N\to\infty.$$

(2) *Suppose that $\ell\in\mathbb{N}_0$ and $2\ell+1\sim N\theta$, $N\to\infty$ for some $\theta\in(0,\infty)$. Then*

$$\widehat{R}_{2\ell+1}(x)\sim\frac{2(N\theta)^{\mathfrak{b}+\nu}}{(\theta x)^{\nu}}\left[\mathfrak{a}\widetilde{J}_{\nu}^{(-\mathfrak{b}-1)}(\theta,1,x,-s_1)-\widetilde{J}_{\nu}^{(-\mathfrak{b})}(\theta,1,x,-s_1)\right],\\ N\to\infty.$$

We next examine asymptotic of $R_k^{(m)}(x)$. From the definition (76) and the expression (51)

$$\begin{aligned}R_k^{(m)}(x)&=c_1\int_0^{\infty}d\eta\,R_k(c_1\eta)\widetilde{p}^{(\nu,\kappa)}(t_1,c_1\eta|0)\widetilde{p}^{(\nu,\kappa)}(t_m-t_1,x|c_1\eta)\\&=\frac{k!}{2^{\nu+1}\Gamma(\nu+1)}\left(\frac{c_1}{\chi_1}\right)^k\left(\frac{1}{t_m}\right)^{\nu+1}x^{\mathfrak{a}}\exp\left[\left(-2+\frac{t_m}{T}\right)\frac{x}{2c_m}\right]\\&\quad\times\sum_{j=0}^{k}\alpha_{k,j}L_j^{\nu}\left(\frac{x}{c_m},-s_m\right).\end{aligned}\tag{86}$$

We put

$$\widehat{R}_k^{(m)}(x)=\frac{2^{\nu}T^{\nu}\Gamma(\nu+1)}{\Gamma(k+1+2\mathfrak{a})}\left(\frac{\chi_1}{c_1}\right)^k R_k^{(m)}(x).\tag{87}$$

If we set $k\sim N\theta$ as $N\to\infty$, Eq. (B.1) in Appendix B gives

$$\frac{k!T^{\nu}}{\Gamma(k+1+2\mathfrak{a})t_m^{\nu+1}}\exp\left[\left(-1+\frac{t_m}{2T}\right)\frac{x}{c_m}\right]\sim N^{-(2\mathfrak{a}+1)}\theta^{-2\mathfrak{a}},\quad N\to\infty,$$

then we obtain the following from Proposition 5.2.

Proposition 5.3. (1) *Suppose that* $\ell \in \mathbb{N}$ *and* $2\ell \sim N\theta$, $N \to \infty$ *for some* $\theta \in (0,\infty)$. *Then*

$$\widehat{R}_{2\ell}^{(m)}(x) \sim \frac{\theta^{1-\nu}x^{-\kappa/2}}{2}\widetilde{J}_{\nu}^{(-\mathfrak{b}-1)}(\theta,1,x,-s_m), \quad N \to \infty.$$

(2) *Suppose that* $\ell \in \mathbb{N}_0$ *and* $2\ell+1 \sim N\theta$, $N \to \infty$ *for some* $\theta \in (0,\infty)$. *Then*

$$\widehat{R}_{2\ell+1}^{(m)}(x) \sim \frac{\theta^{-\nu}x^{-\kappa/2}}{N}\left[\mathfrak{a}\widetilde{J}_{\nu}^{(-\mathfrak{b}-1)}(\theta,1,x,-s_m) - \widetilde{J}_{\nu}^{(-\mathfrak{b})}(\theta,1,x,-s_m)\right],$$
$$N \to \infty.$$

5.2 Asymptotics of $\Phi_k^{(m)}(x)$

We put

$$Q_{2\ell}(x) = F_{2\ell}(x) \qquad \text{and} \qquad Q_{2\ell+1}(x) = G_{2\ell+1}(x), \tag{88}$$

for $\ell \in \mathbb{N}_0$, and $Q_k \equiv 0$ for $k \in \mathbb{Z}_-$. Lemma 4.1 gives the expansion formula of $Q_k(x)$ in terms of $\{L_j^{\nu}(x)\}$. Here we give the formula to expand $L_j^{\nu}(x)$ in terms of $\{Q_k(x)\}$. In other words, we provide the inverse of the matrix $\boldsymbol{\alpha} = (\alpha_{k,j})$ given by Eq. (64), which is denoted by $\boldsymbol{\beta} = (\beta_{j,k})$.

Let $b(n) = (n+2\mathfrak{a})/n, n \in \mathbb{N}$, and

$$b(m,n) = \begin{cases} b(m)b(m+2)\cdots b(n), & \text{if } m,n \text{ are odd and } m \le n, \\ 1, & \text{if } m,n \text{ are odd and } m > n, \\ 0, & \text{otherwise.} \end{cases} \tag{89}$$

Then the following lemma holds.

Lemma 5.4. *For* $j \in \mathbb{N}_0$

$$L_j^{\nu}(x) = \sum_{k=0}^{j} \beta_{j,k} Q_k(x). \tag{90}$$

where $\beta_{j,k}$, $k, 1, \dots, j$, $j \in \mathbb{N}_0$ *are defined by the following:*
When k is even

$$\beta_{j,k} = \begin{cases} 0, & \text{if } j < k, \\ \binom{j-k-\mathfrak{b}-2}{j-k}, & \text{if } j \ge k, \end{cases} \tag{91}$$

and, when k is odd

$$\beta_{j,k} = \begin{cases} 0, & \text{if } j < k, \\ -\displaystyle\sum_{r=[(k+1)/2]}^{[(j+1)/2]} b(k+2,2r-1)\binom{j-2r-\mathfrak{b}-1}{j-2r+1}, & \text{if } j \ge k. \end{cases} \tag{92}$$

Using Eq. (52), Eq. (78) is rewritten as

$$F^{(m)}(y,x) = \left(\frac{1}{T}\right)^{\kappa} \left(\frac{t_m}{c_m}\right)^{2(\nu+1)} (xy)^{\kappa/2} \exp\left\{\left(-\frac{t_m}{T}\right)\frac{x+y}{2c_m}\right\} \times \sum_{p=0}^{\infty}\sum_{j=0}^{\infty} \langle L_p^\nu, L_j^\nu\rangle_* \widehat{L}_p^\nu\left(\frac{y}{c_m}, s_m\right) \widehat{L}_j^\nu\left(\frac{x}{c_m}, s_m\right).$$

Hence from Eqs. (77) and (86), we have

$$\begin{aligned}\Phi_k^{(m)}(x) &= c_m \int_0^\infty \mathrm{d}\eta\, R_k^{(m)}(c_m\eta) F^{(m)}(c_m\eta, x) \\ &= \frac{k!}{2^{\nu+1}\Gamma(\nu+1)} \left(\frac{c_1}{\chi_1}\right)^k \left(\frac{1}{c_m}\right)^{\nu+1} \frac{t_m^{\nu+1}}{T^\kappa} x^{\kappa/2} \exp\left\{-\frac{t_m x}{2Tc_m}\right\} \\ &\quad \times \sum_{j=0}^{\infty} \left\langle \sum_{p=0}^{k} \alpha_{k,p} L_p^\nu, L_j^\nu \right\rangle_* \widehat{L}_j^\nu\left(\frac{x}{c_m}, s_m\right),\end{aligned} \tag{93}$$

where we have used the orthogonal relation (58) of Laguerre polynomials. Put

$$\widehat{\Phi}_k^{(m)}(x) = \frac{2^\nu T^{-\mathfrak{b}}\Gamma(\nu+1)}{k!} \left(\frac{\chi_1}{c_1}\right)^k \Phi_k^{(m)}(x). \tag{94}$$

Then we have the following proposition.

Proposition 5.5. (1) *Suppose that $\ell \in \mathbb{N}$ and $2\ell \sim N\theta$, $N \to \infty$ for some $\theta \in (0,\infty)$. Then*

$$\widehat{\Phi}_{2\ell}^{(m)}(x) \sim -\theta^\nu x^{\kappa/2} \int_1^\infty d\xi\, \xi^{\mathfrak{a}} \widehat{\mathcal{J}}_\nu^{(\mathfrak{b}+1)}(\theta,\xi,x,s_m), \quad \mathbb{N} \to \infty. \tag{95}$$

(2) *Suppose that $\ell \in \mathbb{N}_0$ and $2\ell+1 \sim N\theta$, $N \to \infty$ for some $\theta \in (0,\infty)$. Then*

$$\widehat{\Phi}_{2\ell+1}^{(m)}(x) \sim -\frac{2\theta^{-1+\nu}x^{\kappa/2}}{N} \widehat{\mathcal{J}}_\nu^{(\mathfrak{b}+1)}(\theta,1,x,s_m), \quad N \to \infty. \tag{96}$$

Proof.

$$\begin{aligned}\widehat{\Phi}_k^{(m)}(x) &\sim \frac{T^{-\nu}x^{\kappa/2}}{2} \sum_{j=0}^{\infty} \left\langle \sum_{p=0}^{k} \alpha_{k,p} L_p^\nu, L_j^\nu \right\rangle_* \widehat{L}_j^\nu\left(\frac{x}{N}, s_m\right), \quad N \to \infty, \\ &= \frac{T^{-\nu}x^{\kappa/2}}{2} \sum_{j=0}^{\infty} \left\langle Q_k, \sum_{q=0}^{j} \beta_{j,q} Q_q \right\rangle_* \widehat{L}_j^\nu\left(\frac{x}{N}, s_m\right).\end{aligned}$$

By the skew orthogonality of $\{Q_k\}$ given by Lemma 4.2, we have

$$\widehat{\Phi}_{2\ell}^{(m)}(x) \sim \frac{T^{-\nu} x^{\kappa/2} r_\ell^*}{2} \sum_{j=0}^{\infty} \beta_{j,2\ell+1} \widehat{L}_j^{\nu} \left(\frac{x}{N}, s_m \right), \quad N \to \infty, \tag{97}$$

$$\widehat{\Phi}_{2\ell+1}^{(m)}(x) \sim -\frac{T^{-\nu} x^{\kappa/2} r_\ell^*}{2} \sum_{j=0}^{\infty} \beta_{j,2\ell} \widehat{L}_j^{\nu} \left(\frac{x}{N}, s_m \right), \quad N \to \infty. \tag{98}$$

By Eqs. (70) and (91), Eq. (98) gives

$$\widehat{\Phi}_{2\ell+1}^{(m)}(x) \sim -2\Gamma(2\mathfrak{a}+1) T^{-\nu} x^{\kappa/2} \binom{2\ell+1+2\mathfrak{a}}{2\ell+1} \times \sum_{j=2\ell}^{\infty} \binom{j-2\ell-\mathfrak{b}-2}{j-2\ell} \widehat{L}_j^{\nu} \left(\frac{x}{N}, s_m \right), \quad N \to \infty. \tag{99}$$

From Eq. (B.8) we have

$$\begin{aligned} \sum_{j=2\ell}^{\infty} \binom{j-2\ell-\mathfrak{b}-2}{j-2\ell} \widehat{L}_j^{\nu} \left(\frac{x}{N}, s_m \right) &= \sum_{r=0}^{\infty} \binom{r-\mathfrak{b}-2}{r} \widehat{L}_{2\ell+r}^{\nu} \left(\frac{x}{N}, s_m \right) \\ &= \sum_{r=0}^{\infty} \binom{r-\mathfrak{b}-2+\alpha}{r} \sum_{p=0}^{\alpha} (-1)^p \binom{\alpha}{p} \widehat{L}_{2\ell+r+p}^{\nu} \left(\frac{x}{N}, s_m \right) \\ &\sim (2\ell)^{-\alpha} \sum_{r=0}^{\infty} \binom{r-\mathfrak{b}-2+\alpha}{r} \widehat{J}_{\nu}^{(\alpha)}(\theta, \eta, x, s_m), \quad N \to \infty, \end{aligned} \tag{100}$$

where Eq. (B.11) of Lemma (B.2) was applied. Setting $j = 2\ell\eta = N\theta\eta$ and using Eq. (B.2) in Appendix B, we conclude that

$$\begin{aligned} \widehat{\Phi}_{2\ell+1}^{(m)}(x) &\sim -2T^{-\nu} (2\ell)^{2\mathfrak{a}-\alpha} x^{\kappa/2} \sum_{j=2\ell}^{\infty} \frac{(j-2\ell+1)^{-\mathfrak{b}-2+\alpha}}{\Gamma(-\mathfrak{b}-1+\alpha)} \widehat{J}_{\nu}^{(\alpha)}(\theta, \eta, x, s_m) \\ &\sim -\frac{2\theta^{-1+\nu} x^{\kappa/2}}{N\Gamma(-\mathfrak{b}-1+\alpha)} \int_1^{\infty} \mathrm{d}\eta \, \frac{\widehat{J}_{\nu}^{(\alpha)}(\theta, \eta, x, s_m)}{(\eta-1)^{\mathfrak{b}+2-\alpha}}, \quad N \to \infty. \end{aligned} \tag{101}$$

Through the expression (19), we obtain Eq. (96).

By Eq. (92) of Lemma 5.4 with Eq. (89)

$$\sum_{j=2\ell+1}^{\infty} \beta_{j,2\ell+1} \widehat{L}_j^{\nu}\left(\frac{x}{N}, s_m\right)$$
$$= \frac{-1}{b(1,2\ell+1)} \sum_{j=2\ell+1}^{\infty} \widehat{L}_j^{\nu}\left(\frac{x}{N}, s_m\right) \sum_{r=\ell+1}^{[(j+1)/2]} b(1,2r-1) \binom{j-2r-\mathfrak{b}-1}{j-2r+1}$$
$$= \frac{-1}{b(1,2\ell+1)} S(\ell),$$

where

$$S(\ell) = \sum_{r=\ell+1}^{\infty} b(1,2r-1) \sum_{j=2r-1}^{\infty} \widehat{L}_j^{\nu}\left(\frac{x}{N}, s_m\right) \binom{j-2r-\mathfrak{b}-1}{j-2r+1}.$$

By this equation with the estimate (B.3) for $b(1,2r-1)$ and Eq. (70), Eq. (97) becomes

$$\widehat{\Phi}_{2\ell}^{(m)}(x) \sim -2T^{-\nu}(2\ell+2)^{-\mathfrak{a}}\Gamma(2\mathfrak{a}+1)\binom{2\ell+1+2\mathfrak{a}}{2\ell+1} x^{\kappa/2} S(\ell)$$
$$\sim -2T^{-\nu}(2\ell+2)^{\mathfrak{a}} x^{\kappa/2} S(\ell), \quad N \to \infty. \tag{102}$$

From Eq. (100) with Eq. (B.3)

$$S(\ell) \sim \sum_{r=\ell+1}^{\infty} (2r)^{\mathfrak{a}} \left(\frac{1}{2\ell}\right)^{\alpha} \sum_{j=2r-1}^{\infty} \frac{(j-2r+1)^{-\mathfrak{b}-2+\alpha}}{\Gamma(-\mathfrak{b}-1+\alpha)} \widehat{\mathcal{J}}_{\nu}^{(\alpha)}(\theta,\eta,x,s_m)$$
$$\sim \frac{(2\ell)^{\kappa/2}}{2\Gamma(-\mathfrak{b}-1+\alpha)} \int_1^{\infty} \mathrm{d}\xi\ \xi^{\mathfrak{a}} \int_{\xi}^{\infty} \mathrm{d}\eta\ \frac{\widehat{\mathcal{J}}_{\nu}^{(\alpha)}(\theta,\eta,x,s_m)}{(\eta-\xi)^{\mathfrak{b}+2-\alpha}}, \quad N \to \infty.$$

Thus

$$\widehat{\Phi}_{2\ell}^{(m)}(x) \sim -\frac{\theta^{\nu} x^{\kappa/2}}{\Gamma(-\mathfrak{b}-1+\alpha)} \int_1^{\infty} \mathrm{d}\xi\ \xi^{\mathfrak{a}} \int_{\xi}^{\infty} \mathrm{d}\eta\ \frac{\widehat{\mathcal{J}}_{\nu}^{(\alpha)}(\theta,\eta,x,s_m)}{(\eta-\xi)^{\mathfrak{b}+2-\alpha}}$$
$$= -\theta^{\nu} x^{\kappa/2} \int_1^{\infty} \mathrm{d}\xi\ \xi^{\mathfrak{a}} \widehat{\mathcal{J}}_{\nu}^{(\mathfrak{b}+1)}(\theta,\xi,x,s_m), \quad N \to \infty,$$

where we used the expression (19). This completes the proof of Proposition 5.5. □

5.3 Asymptotics of $D^{m,n}(x,y)$, $\tilde{I}^{m,n}(x,y)$, $S^{m,n}(x,y)$, and $\tilde{S}^{m,n}(x,y)$

From the expressions (79) with (72) and the definitions (87) and (94) we have

$$D_N^{m,n}(x,y) \sim \sum_{\ell=0}^{(N/2)-1} \left(\frac{2\ell}{N}\right)^{2\mathfrak{a}} \left[\widehat{R}_{2\ell}^{(m)}(x)\widehat{R}_{2\ell+1}^{(n)}(y) - \widehat{R}_{2\ell+1}^{(m)}(x)\widehat{R}_{2\ell}^{(n)}(y)\right],$$
$$\tilde{I}_N^{m,n}(x,y) \sim -\sum_{\ell=(N/2)}^{\infty} \left(\frac{2\ell}{N}\right)^{-2\mathfrak{a}} \left[\widehat{\Phi}_{2\ell}^{(m)}(x)\widehat{\Phi}_{2\ell+1}^{(n)}(y) - \widehat{\Phi}_{2\ell+1}^{(m)}(x)\widehat{\Phi}_{2\ell}^{(n)}(y)\right],$$
$$S_N^{m,n}(x,y) \sim \sum_{\ell=0}^{(N/2)-1} \left[\widehat{\Phi}_{2\ell}^{(m)}(x)\widehat{R}_{2\ell+1}^{(n)}(y) - \widehat{\Phi}_{2\ell+1}^{(m)}(x)\widehat{R}_{2\ell}^{(n)}(y)\right], \quad N \to \infty.$$

From Propositions 5.3 and 5.5 we obtain the following asymptotics:

$$D_N^{m,n}(x,y) \sim \mathcal{D}(s_m,x;s_n,y),$$
$$\tilde{I}_N^{m,n}(x,y) \sim \tilde{\mathcal{I}}(s_m,x;s_n,y),$$
$$S_N^{m,n}(x,y) \sim \mathcal{S}(s_m,x;s_n,y), \quad N \to \infty,$$

where $\mathcal{D}, \tilde{\mathcal{I}}, \mathcal{S}$ are defined by Eq. (21).

Next we study the asymptotic behavior of $\widetilde{p}^{(\nu,\kappa)}(t_n - t_m, y|x)$. From Eq. (51) we have

$$\widetilde{p}^{(\nu,\kappa)}(t_n - t_m, y|x)$$
$$= \left(\frac{t_m}{t_n}\right)^{\nu+1} c_m^{-\mathfrak{a}-1} \left(\frac{x}{c_m}\right)^{\kappa/2} y^{\mathfrak{a}} \exp\left[-\frac{t_m x}{2Tc_m}\right] \exp\left[\left(-2+\frac{t_n}{T}\right)\frac{y}{2c_n}\right]$$
$$\times \sum_{j=0}^{\infty} \widehat{L}_j^{\nu}\left(\frac{x}{c_m}, s_m\right) L_j^{\nu}\left(\frac{y}{c_n}, -s_n\right).$$

Then by simple calculation with Lemma B.2 with $\alpha = 0$, we have

$$\widetilde{p}^{(\nu,\kappa)}(t_n - t_m, y|x) \sim \left(\frac{y}{x}\right)^{\mathfrak{b}/2} \frac{1}{N} \sum_{j=0}^{\infty} e^{2(s_m - s_n)\theta\eta} J_\nu(2\sqrt{\theta\eta x}) J_\nu(2\sqrt{\theta\eta y}),$$
$$\sim \left(\frac{y}{x}\right)^{\mathfrak{b}/2} \mathcal{G}(s_m,x;s_n,y), \quad N \to \infty,$$

where $\mathcal{G}$ is defined by Eq. (23), and then $\widetilde{S}_N^{m,n}(x,y) \sim \widetilde{\mathcal{S}}(s_m,x;s_n,y)$, $N \to \infty$. Then, the proof of Theorem 2.1 is completed.

Appendix A. Proof of Eq. (39)

We assume that the number of particles N is even. Consider the multiple integral

$$Z^{\mathbf{Y}}_{N,T}[\chi] = \left(\frac{1}{N!}\right)^{M+1} \int_{\mathbb{R}_+^{N(M+1)}} \prod_{m=1}^{M+1} d\mathbf{x}^{(m)} \operatorname{sgn}\big(h_N(\mathbf{x}^{(M+1)})\big)$$
$$\times \det_{1\le i,j\le N}\Big[M_{i-1}\big(x_j^{(1)}\big)\widetilde{p}^{(\nu,\kappa)}\big(t_1, x_j^{(1)}|0\big)\big(1+\chi_1\big(x_j^{(1)}\big)\big)\Big]$$
$$\times \prod_{m=1}^{M} \det_{1\le i,j\le N}\Big[\widetilde{p}^{(\nu,\kappa)}\big(t_{m+1}-t_m, x_j^{(m+1)}|x_i^{(m)}\big)\big(1+\chi_{m+1}\big(x_j^{(m+1)}\big)\big)\Big].$$

By the definition (29) with (28) and (26), and by the equality (36), we have

$$\Psi^{\mathbf{Y}}_{N,T}(\mathbf{f};\boldsymbol{\theta}) = \frac{Z^{\mathbf{Y}}_{N,T}[\chi]}{Z^{\mathbf{Y}}_{N,T}[0]}, \tag{A.1}$$

where $Z^{\mathbf{Y}}_{N,T}[0]$ is obtained from $Z^{\mathbf{Y}}_{N,T}[\chi]$ by setting $\chi_m(x) \equiv 0$ for all $m = 1,2,\dots,M+1$.

By repeated applications of the Heine identity

$$\int_{\mathbb{R}^N_{+<}} d\mathbf{x} \det_{1\le i,j\le N}\Big[\phi_i(x_j)\Big] \det_{1\le i,j\le N}\Big[\bar{\phi}_i(x_j)\Big] = \det_{1\le i,j\le N}\left[\int_{\mathbb{R}_+} dx\, \phi_i(x)\bar{\phi}_j(x)\right],$$

for square integrable continuous functions $\phi_i, \bar{\phi}_i, 1 \le i \le N$, we have

$$Z^{\mathbf{Y}}_{N,T}[\chi] = \int_{\mathbb{R}^N_{+<}} d\mathbf{y} \det_{1\le i,j\le N}\Bigg[\int_{\mathbb{R}_+^{M+1}} \prod_{m=1}^{M+1} dx^{(m)}\, \delta\big(y_j - x^{(M+1)}\big)$$
$$\times\Big\{M_{i-1}\big(x^{(1)}\big)\widetilde{p}^{(\nu,\kappa)}\big(t_1, x^{(1)}|0\big)\big(1+\chi_1\big(x^{(1)}\big)\big)\Big\}$$
$$\times \prod_{m=1}^{M}\Big\{\widetilde{p}^{(\nu,\kappa)}\big(t_{m+1}-t_m, x^{(m+1)}|x^{(m)}\big)\big(1+\chi_{m+1}\big(x^{(m+1)}\big)\big)\Big\}\Bigg].$$

Using the notations in Sect. 3.2, it is expressed as

$$Z^{\mathbf{Y}}_{N,T}[\chi] = \int_{\mathbb{R}^N_{+<}} d\mathbf{y} \det_{1\le i,j\le N}\left[\langle i| \left(1+\frac{1}{1-\hat{\chi}\hat{p}_+}\hat{\chi}\hat{p}\right)_N |M+1, y_j\rangle\right]$$

$$= \int_{\mathbb{R}^N_{+<}} d\mathbf{y} \det_{1\le i,j\le N}\left[\langle M+1, y_i| \left(1+\hat{p}\hat{\chi}\frac{1}{1-\hat{p}_-\hat{\chi}}\right)_N |j\rangle\right],$$

since $\langle m,x|(\hat{p}_+)^k|n,y\rangle = \langle n,y|(\hat{p}_-)^k|m,x\rangle \equiv 0$ for $k > n-m \ge 0$. Here we have used the Chapman–Kolmogorov equation, $\int_{\mathbb{R}_+} dy\, \widetilde{p}^{(\nu,\kappa)}(t-s,y|x)\widetilde{p}^{(\nu,\kappa)}(u-t, z|y) = \widetilde{p}^{(\nu,\kappa)}(u-s,z|x)$, $0 \le s \le t \le u \le T$, $x,y \in \mathbb{R}_+$. Next we use the formula of de Bruijn [9]

$$\int_{\mathbb{R}^N_{+<}} d\mathbf{y} \det_{1\le i,j\le N}\left[\phi_i(y_j)\right] = \mathrm{Pf}_{1\le i,j\le N}\left[\int_{\mathbb{R}_+} dy \int_{\mathbb{R}_+} d\tilde{y}\, \mathrm{sgn}(\tilde{y}-y)\phi_i(y)\phi_j(\tilde{y})\right],$$

for integrable continuous functions ϕ_i, $1 \le i \le N$, in which the Pfaffian is defined by Eq. (24). Since $(\mathrm{Pf}(A))^2 = \det A$ for any even-dimensional skew-symmetric matrix A, we have

$$\left(Z^{\mathbf{Y}}_{N,T}[\chi]\right)^2 = \det_{1\le i,j\le N}\left[\langle i|\left(1+\frac{1}{1-\hat{\chi}\hat{p}_+}\hat{\chi}\hat{p}\right)_N \hat{J}\left(1+\hat{p}\hat{\chi}\frac{1}{1-\hat{p}_-\hat{\chi}}\right)_N |j\rangle\right]$$

$$= \det_{1\le i,j\le N}\left[(A_0)_{ij} + (A_1)_{ij} + (A_2)_{ij} + (A_3)_{ij}\right]$$

with

$$(A_0)_{ij} = \langle i|\hat{J}_N|j\rangle,$$

$$(A_1)_{ij} = \langle i|\left(\frac{1}{1-\hat{\chi}\hat{p}_+}\hat{\chi}\hat{p}\hat{J}\right)_N|j\rangle = \langle i|\left(\hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}\hat{p}\hat{J}\right)_N|j\rangle,$$

$$(A_2)_{ij} = \langle i|\left(\hat{J}\hat{p}\hat{\chi}\frac{1}{1-\hat{p}_-\hat{\chi}}\right)_N|j\rangle,$$

$$(A_3)_{ij} = \langle i|\left(\hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}\hat{p}\hat{J}\hat{p}\hat{\chi}\frac{1}{1-\hat{p}_-\hat{\chi}}\right)_N|j\rangle.$$

Since $\left(Z^{\mathbf{Y}}_{N,T}[0]\right)^2 = \det_{1\le i,j\le N}\left[(A_0)_{ij}\right]$, Eq. (A.1) gives

$$\left\{\Psi^{\mathbf{Y}}_{N,T}(\mathbf{f};\boldsymbol{\theta})\right\}^2 = \det_{1\le i,j\le N}\left[\delta_{ij} + (A_0^{-1}A_1)_{ij} + (A_0^{-1}A_2)_{ij} + (A_0^{-1}A_3)_{ij}\right]. \quad \text{(A.2)}$$

By our notation (38), $(A_0^{-1})_{ij} = \langle i|(\hat{J}_N)^{\triangle}|j\rangle$, and it is easy to confirm that Eq. (A.2) is written in the form

$$\left\{\Psi_{N,T}^{\mathbf{Y}}(\mathbf{f};\boldsymbol{\theta})\right\}^2 = \det_{1\le i,j\le N}\left[\delta_{ij} + \langle i|\mathbf{B}|m,x\rangle\langle m,x|\mathbf{C}|j\rangle\right], \tag{A.3}$$

where we have introduced $\mathbf{B}$ and $\mathbf{C}$ as the following two-dimensional row and column vector-valued operators,

$$\mathbf{B} = \left((\hat{J}_N)^{\triangle} \circ \hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}} \quad -(\hat{J}_N)^{\triangle} \circ \left(1 + \hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}\hat{p}\right)\hat{J}\hat{p}\hat{\chi} \right),$$
$$\mathbf{C} = \begin{pmatrix} \hat{p}\hat{J} \\ -\frac{1}{1-\hat{p}_-\hat{\chi}} \end{pmatrix}.$$

The determinant (A.3) is equivalent with the Fredholm determinant,

$$\mathrm{Det}\langle m,x|I_2 + \mathbf{C}\circ\mathbf{B}|n,y\rangle.$$

Introducing matrix-valued operators,

$$\mathbf{K}_+ = \begin{pmatrix} 1-\hat{p}_+\hat{\chi} & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{K}_- = \begin{pmatrix} 1 & 0 \\ 0 & 1-\hat{p}_-\hat{\chi} \end{pmatrix}, \quad \widehat{\mathbf{K}} = \begin{pmatrix} 1 & -\hat{p}\hat{J}\hat{p}\hat{\chi} \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{aligned} I_2 + \mathbf{C}\circ\mathbf{B} &= \mathbf{K}_-{}^{-1}\left[\mathbf{K}_-\mathbf{K}_+ + \begin{pmatrix} \hat{p}\hat{J}\mathcal{J}_N^{\triangle} & \hat{p}\hat{J}\left(1-\mathcal{J}_N^{\triangle}\hat{J}\right)\hat{p} \\ -\mathcal{J}_N^{\triangle} & \mathcal{J}_N^{\triangle}\hat{J}\hat{p} \end{pmatrix}\hat{\chi}\right]\mathbf{K}_+{}^{-1}\widehat{\mathbf{K}} \\ &= \mathbf{K}_-{}^{-1}\left[I_2 + \begin{pmatrix} \hat{p}\hat{J}\mathcal{J}_N^{\triangle} - \hat{p}_+ & \hat{p}\hat{J}\hat{p} - \hat{p}\hat{J}\mathcal{J}_N^{\triangle}\hat{J}\hat{p} \\ -\mathcal{J}_N^{\triangle} & \mathcal{J}_N^{\triangle}\hat{J}\hat{p} - \hat{p}_- \end{pmatrix}\hat{\chi}\right]\mathbf{K}_+{}^{-1}\widehat{\mathbf{K}}, \end{aligned}$$

where $\mathcal{J}_N^{\triangle} = \circ(\hat{J}_N)^{\triangle}\circ$. From the orthogonality (31) and the definitions (34) of the operators $\hat{p}_+$ and $\hat{p}_-$, we have the fact that

$$\mathrm{Det}\langle m,x|\mathbf{K}_+|n,y\rangle = \mathrm{Det}\langle m,x|\mathbf{K}_-|n,y\rangle = \mathrm{Det}\langle m,x|\widehat{\mathbf{K}}|n,y\rangle = 1.$$

Then Eq. (39) is derived.

Appendix B. Elementary calculation for asymptotics estimation

By Stirling's formula $\Gamma(x) \sim \sqrt{2\pi} x^{x-1/2} e^{-x}$, $x \to \infty$, we have

$$\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \sim (n+1)^{\alpha}, \quad n \to \infty, \tag{B.1}$$

$$\binom{n+\alpha}{n} \sim \frac{(n+1)^{\alpha}}{\Gamma(\alpha+1)}, \quad n \to \infty, \tag{B.2}$$

for any $\alpha \in \mathbb{R} \setminus \mathbb{Z}_-$, and

$$b(1, 2\ell-1) = \frac{\Gamma(2\mathfrak{a}+1)}{2^{\mathfrak{a}}\Gamma(\mathfrak{a}+1)} \binom{2\ell+2\mathfrak{a}}{2\ell} \bigg/ \binom{\ell+\mathfrak{a}}{\ell} \sim (2\ell)^{\mathfrak{a}}, \quad \ell \to \infty, \tag{B.3}$$

$$b(2\ell+1, 2p-1) = \frac{b(1,2p-1)}{b(1,2\ell-1)} \sim \left(\frac{p}{\ell}\right)^{\mathfrak{a}}, \quad \ell \to \infty, \tag{B.4}$$

for $\ell, p \in \mathbb{N}$ with $\ell < p$, where $b(m,n)$ is defined by Eq. (89).

From now on, we assume that $T = N$, $t_m = T + s_m$ with $s_m < 0$. We set

$$2\ell = N\theta \quad \text{and} \quad j = 2\ell\eta, \tag{B.5}$$

and consider the limit $N \to \infty$ with some $\eta, \theta \in (0, \infty)$. Then we have

$$\chi_m^j = \left(\frac{2T - t_m}{t_m}\right)^j = \left(1 - \frac{2s_m}{t_m}\right)^{N\theta\eta} \sim \exp\left(-2s_m\theta\eta\right), \quad N \to \infty,$$

and

$$\sum_{p=0}^{\alpha} (-1)^p \binom{\alpha}{p} \chi_m^{j-p} \sim (2\ell)^{-\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}\eta}\right)^{\alpha} \exp\left(-2s_m\theta\eta\right), \quad N \to \infty. \tag{B.6}$$

We use the following identities (see Eq. (54) and pages 8, 201, and 202 in [42]).

(1) Let $\alpha \in \mathbb{N}_0$ and $c \in \mathbb{R}$. Then

$$\sum_{p=0}^{\alpha} (-1)^p \binom{\alpha}{p} \binom{n-p+c}{n-p-j} = \binom{n-\alpha+c}{n-j}. \tag{B.7}$$

(2) Let $\alpha \in \mathbb{N}_0$, $c \in \mathbb{R}$ and a_k, $k = 1, 2, \dots$, be a sequence in $\mathbb{R}$. Then

$$\sum_{r=0}^{\infty} \binom{r+c}{r} a_r = \sum_{r=0}^{\infty} \binom{r+c+\alpha}{r} \sum_{p=0}^{\alpha} (-1)^p \binom{\alpha}{p} a_{r+p}. \tag{B.8}$$

(3) Let $\alpha \in \mathbb{N}_0$, and $a_k, b_k, k = 1, 2, \ldots$, be sequences in $\mathbb{R}$. Then

$$\sum_{k=0}^{\alpha}(-1)^k \binom{\alpha}{k} a_k b_k = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \sum_{p=0}^{\beta}(-1)^p \binom{\beta}{p} a_p \sum_{q=0}^{\alpha-\beta}(-1)^q \binom{\alpha-\beta}{q} b_q. \tag{B.9}$$

Lemma B.1. *For any $\alpha \in \mathbb{N}_0$ and $w \geq 0$ we have*

$$\sum_{p=0}^{\alpha}(-1)^p \binom{\alpha}{p} L_{n-p}^{\nu}\left(\frac{w}{n}\right) \sim \left(\frac{w}{n}\right)^{\alpha-\nu} \left(\frac{\mathrm{d}}{\mathrm{d}w}\right)^{\alpha} \left\{w^{\nu/2} J_\nu(2\sqrt{w})\right\}, \quad n \to \infty.$$

Proof. From the definition of the Laguerre polynomials (56) and (B.7), we have

$$\begin{aligned}\sum_{p=0}^{\alpha}(-1)^p \binom{\alpha}{p} L_{n-p}^{\nu}(y) &= \sum_{p=0}^{\alpha}(-1)^p \binom{\alpha}{p} \sum_{j=0}^{n-p}(-1)^j \binom{n-p+\nu}{n-p-j} \frac{y^j}{j!} \\ &= \sum_{j=0}^{n} \frac{(-1)^j}{j!} \binom{n-\alpha+\nu}{n-j} y^j.\end{aligned}$$

Hence, by Eq. (B.2)

$$\begin{aligned}&\lim_{n\to\infty} \left(\frac{w}{n}\right)^{\nu-\alpha} \sum_{p=0}^{\alpha}(-1)^p \binom{\alpha}{p} L_{n-p}^{\nu}\left(\frac{w}{n}\right) \\ &= \lim_{n\to\infty} \sum_{j=0}^{n} \frac{(-1)^j}{j!} \binom{n-\alpha+\nu}{n-j} \left(\frac{w}{n}\right)^{j+\nu-\alpha} \\ &= \sum_{j=0}^{\infty}(-1)^j \frac{w^{\nu+j-\alpha}}{\Gamma(j-\alpha+\nu+1)j!} \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}w}\right)^{\alpha} \left(\sum_{j=0}^{\infty}(-1)^j \frac{w^{\nu+j}}{\Gamma(\nu+j+1)j!}\right).\end{aligned}$$

Then we obtain the lemma. □

Applying the above lemma, we obtain the following asymptotics, where L_j^{ν} and $\widehat{L}_j^{\nu}$ are defined by Eq. (81).

Lemma B.2. *For any $\alpha \in \mathbb{N}_0$, $\theta, \eta \in (0, \infty)$, and $x \in \mathbb{R}_+$ we have*

$$\sum_{p=0}^{\alpha}(-1)^p \binom{\alpha}{p} L_{j-p}^{\nu}\left(\frac{x}{N}, -s_m\right) \sim \frac{(2\ell)^{\nu-\alpha}}{(\theta x)^{\nu}} \widetilde{\mathcal{J}}_{\nu}^{(\alpha)}(\theta, \eta, x, -s_m), \quad N \to \infty, \tag{B.10}$$

$$\sum_{p=0}^{\alpha}(-1)^p \binom{\alpha}{p} \widehat{L}_{j+p}^{\nu}\left(\frac{x}{N}, s_m\right) \sim (2\ell)^{-\alpha} \widehat{\mathcal{J}}_{\nu}^{(\alpha)}(\theta, \eta, x, s_m), \quad N \to \infty. \tag{B.11}$$

Proof. Since

$$\begin{aligned}
&\sum_{p=0}^{\alpha}(-1)^p \binom{\alpha}{p} L_{j-p}^{\nu}\left(\frac{x}{N}\right) \chi_m^{j-p} \\
&\quad = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \sum_{p=0}^{\beta}(-1)^p \binom{\beta}{p} L_{j-p}^{\nu}\left(\frac{x}{N}\right) \sum_{q=0}^{\alpha-\beta}(-1)^q \binom{\alpha-\beta}{q} \chi_m^{j-q},
\end{aligned}$$

by Eq. (B.9), the asymptotic Eq. (B.10) is derived from Eq. (B.6) and Lemma (B.1) with $n = j = N\theta\eta, w = \theta\eta x$. From Eq. (B.9), we have

$$\begin{aligned}
&\sum_{k=0}^{\alpha}(-1)^k \binom{\alpha}{k} \widehat{L}_{j+k}^{\nu}\left(\frac{x}{N}, s_m\right) \\
&\quad = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \sum_{p=0}^{\beta}(-1)^p \binom{\beta}{p} L_{j+p}^{\nu}\left(\frac{x}{N}\right) \chi_m^{-(j+p)} \\
&\quad\quad \times \sum_{q=0}^{\alpha-\beta}(-1)^q \binom{\alpha-\beta}{q} \frac{\Gamma(j+q+1)}{\Gamma(j+q+1+\nu)}.
\end{aligned} \tag{B.12}$$

By Eq. (B.1), we see

$$\begin{aligned}
&\sum_{q=0}^{\alpha-\beta}(-1)^q \binom{\alpha-\beta}{q} \frac{\Gamma(j+q+1)}{\Gamma(j+q+1+\nu)} \\
&\quad = \nu \sum_{q=0}^{\alpha-\beta-1}(-1)^q \binom{\alpha-\beta-1}{q} \frac{\Gamma(j+q+1)}{\Gamma(j+q+2+\nu)} \\
&\quad = \frac{\nu(\nu+1)\cdots(\nu+\alpha-\beta-1)\Gamma(j+1)}{\Gamma(j+1+\alpha-\beta+\nu)} \\
&\quad \sim (2\ell)^{-(\alpha-\beta+\nu)} \left(-\frac{\mathrm{d}}{\mathrm{d}\eta}\right)^{\alpha-\beta} \eta^{-\nu}, \quad N \to \infty.
\end{aligned}$$

On the other hand, Eq. (B.10) gives

$$\sum_{p=0}^{\beta}(-1)^p \binom{\beta}{p} L_{j+p}^{\nu}\left(\frac{x}{N}\right) \chi_m^{-(j+p)} \sim (2\ell)^{\nu-\beta}\left(-\frac{\mathrm{d}}{\mathrm{d}\eta}\right)^{\beta}\left\{\eta^{\nu}\widehat{J}_{\nu}(\theta,\eta,x,s_m)\right\}.$$

Hence, the asymptotic Eq. (B.11) is derived from Eq. (B.12). □

Appendix C. On temporally homogeneous limit

Lemma C.1. *For any $c \in \mathbb{R}$ and $\eta, \theta, x \geq 0$, we have that as $t \to \infty$*

$$\widetilde{J}_{\nu}^{(c)}(\theta,\eta,x,t) \sim (2t\theta)^c(\theta\eta x)^{\nu/2} J_{\nu}(2\sqrt{\theta\eta x})e^{2t\theta\eta}, \tag{C.1}$$

$$\widehat{J}_{\nu}^{(c)}(\theta,\eta,x,-t) \sim (2t\theta)^c(\theta\eta x)^{-\nu/2} J_{\nu}(2\sqrt{\theta\eta x})e^{-2t\theta\eta}, \tag{C.2}$$

$$\int_1^{\infty} \mathrm{d}\xi\ \xi^{\alpha}\widehat{J}_{\nu}^{(c+1)}(\theta,\xi,x,-t) \sim (2t\theta)^c(\theta x)^{-\nu/2} J_{\nu}(2\sqrt{\theta x})e^{-2t\theta}. \tag{C.3}$$

Proof. From the expression (18) with the definition (14), we have

$$\begin{aligned}\widetilde{J}_{\nu}^{(c)}(\theta,\eta,x,t) &= \frac{e^{2t\theta\eta}}{\Gamma(-c)}\sum_{k=0}^{\infty}\frac{(-1)^k\eta^{k-c}}{k!(k-c)}\sum_{j=0}^{k}\binom{k}{j}\widetilde{J}_{\nu}^{(j)}(\theta,\eta,x,0)(2t\theta)^{k-j}\\ &\sim \frac{e^{2t\theta\eta}}{\Gamma(-c)}(2t\theta)^c\widetilde{J}_{\nu}(\theta,\eta,x,0)\sum_{k=0}^{\infty}\frac{(-1)^k(2t\theta\eta)^{k-c}}{k!(k-c)}, \quad t\to\infty.\end{aligned}$$

From the relation

$$\frac{\mathrm{d}}{\mathrm{d}z}\sum_{k=0}^{\infty}\frac{(-1)^k z^{k-c}}{k!(k-c)} = z^{-c-1}e^{-z},$$

and the equation

$$\Gamma(-c) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k-c)} + \int_1^{\infty}\mathrm{d}z\ z^{-c-1}e^{-z},$$

(see (1.1.19) in [3]), we have

$$\Gamma(-c) = \lim_{t\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(2t\theta\eta)^{k-c}}{k!(k-c)}.$$

Then we conclude

$$\widetilde{J}_\nu^{(c)}(\theta,\eta,x,t) \sim e^{2t\theta\eta}(2t\theta)^c \widetilde{J}_\nu(\theta,\eta,x,0) = (2t\theta)^c \widetilde{J}_\nu(\theta,\eta,x,t), \quad t \to \infty.$$

Hence Eq. (C.1) is derived from Eq. (14).

Let $n = [c+1]_+$ and $\beta > 0$ with $c = n - \beta$. Since

$$\left(-\frac{\mathrm{d}}{\mathrm{d}\eta}\right)^n \widehat{J}_\nu(\theta,\eta,x,-t) \sim (2t\theta)^n \widehat{J}_\nu(\theta,\eta,x,-t), \quad t \to \infty,$$

Eq. (19) gives

$$\begin{aligned}
\widehat{J}_\nu^{(c)}(\theta,\eta,x,-t) &\sim \frac{(2t\theta)^n}{\Gamma(\beta)} \int_0^\infty \mathrm{d}\xi\ \xi^{\beta-1} \widehat{J}_\nu(\theta,\eta+\xi,x,-t) \\
&= \frac{(2t\theta)^{n-\beta}}{\Gamma(\beta)} \int_0^\infty \mathrm{d}\zeta\ \zeta^{\beta-1} \widehat{J}_\nu\left(\theta,\eta+\frac{\zeta}{2t\theta},x,-t\right) \\
&\sim (2t\theta)^c \widehat{J}_\nu(\theta,\eta,x,-t), \quad t \to \infty.
\end{aligned}$$

Then Eq. (C.2) is derived from Eq. (15). From Eq. (C.2) we have

$$\begin{aligned}
\int_1^\infty \mathrm{d}\xi\ \xi^{\mathfrak{a}} \widehat{J}_\nu^{(c+1)}(\theta,\xi,x,-t) &\sim (2t\theta)^{c+1} \int_1^\infty \mathrm{d}\xi\ \xi^{\mathfrak{a}} \widehat{J}_\nu(\theta,\xi,x,0) e^{-2t\theta\xi} \\
&\sim (2t\theta)^c \widehat{J}_\nu(\theta,1,x,-t), \quad t \to \infty.
\end{aligned}$$

This completes the proof. □

Applying the above lemma, we have as $s_m, s_n \to -\infty$ with the difference $s_n - s_m$ fixed

$$\begin{aligned}
\mathcal{D}(s_m,x;s_n,y) &\sim \frac{(xy)^{\mathfrak{b}/2}(s_n-s_m)}{2^{2\mathfrak{b}+3}(s_m s_n)^{\mathfrak{b}+1}} \int_0^1 \mathrm{d}\theta\ \theta^{-\mathfrak{b}} J_\nu(2\sqrt{\theta x}) J_\nu(2\sqrt{\theta y}) e^{-2(s_m+s_n)\theta} \\
&\sim \frac{(xy)^{\mathfrak{b}/2}(s_m-s_n)}{2^{2\mathfrak{b}+4}(s_m+s_n)(s_m s_n)^{\mathfrak{b}+1}} J_\nu(2\sqrt{x}) J_\nu(2\sqrt{y}) e^{-2(s_m+s_n)}, \\
\tilde{\mathcal{I}}(s_m,x;s_n,y) &\sim \frac{2^{2\mathfrak{b}+1}(s_m s_n)^{\mathfrak{b}}(s_n-s_m)}{(xy)^{\mathfrak{b}/2}} \int_1^\infty \mathrm{d}\theta\ \theta^{\mathfrak{b}} J_\nu(2\sqrt{\theta x}) J_\nu(2\sqrt{\theta y}) e^{2(s_m+s_n)\theta} \\
&\sim \frac{2^{2\mathfrak{b}}(s_m s_n)^{\mathfrak{b}}(s_n-s_m)}{(s_m+s_n)(xy)^{\mathfrak{b}/2}} J_\nu(2\sqrt{x}) J_\nu(2\sqrt{y}) e^{2(s_m+s_n)},
\end{aligned}$$

and

$$\mathcal{S}(s_m, x; s_n, y) \sim \left(\frac{y}{x}\right)^{\nu/2} \int_0^1 \mathrm{d}\theta \, J_\nu(2\sqrt{\theta x}) J_\nu(2\sqrt{\theta y}) e^{2(s_m - s_n)\theta}.$$

It is then clear that

$$\lim_{s_m, s_n \to -\infty} \mathcal{D}(s_m, x; s_n, y) \tilde{\mathcal{I}}(s_m, x; s_n, y) = 0.$$

Acknowledgments The authors would like to thank T. Sasamoto and T. Imamura for useful discussion on determinantal and Pfaffian processes. M.K. is supported in part by the Grant-in-Aid for Scientific Research (KIBAN-C, No.17540363) of Japan Society for the Promotion of Science. H.T. is supported in part by the Grant-in-Aid for Scientific Research (KIBAN-C, No.15540106) of Japan Society for the Promotion of Science.

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