Proof of breaking of self-organized criticality in a nonconservative Abelian sandpile model

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By computer simulations, it was reported that the Bak-Tang-Wiesenfeld (BTW) model loses self-organized criticality (SOC) when some particles are annihilated in a toppling process in the bulk of system. We give a rigorous proof that the BTW model loses SOC as soon as the annihilation rate becomes positive. To prove this, a nonconservative Abelian sandpile model is defined on a square lattice, which has a parameter α ($\geqslant 1$) representing the degree of breaking of the conservation law. This model is reduced to be the BTW model when $\alpha=1$. By calculating the average number of topplings in an avalanche $\langle T\rangle$ exactly, it is shown that for any $\alpha>1$, $\langle T\rangle<\infty$ even in the infinite-volume limit. The power-law divergence of $\langle T\rangle$ with an exponent 1 as $\alpha\to 1$ gives a scaling relation $2\nu(2-a)=1$ for the critical exponents ν and a of the distribution function of T. The 1-1 height correlation $C_{11}(r)$ is also calculated analytically and we show that $C_{11}(r)$ is bounded by an exponential function when $\alpha>1$, although $C_{11}(r)\sim r^{-2d}$ was proved by Majumdar and Dhar for the d-dimensional BTW model. A critical exponent ν_{11} characterizing the divergence of the correlation length ξ as $\alpha\to 1$ is defined as $\xi\sim|\alpha-1|^{-\nu_{11}}$ and our result gives an upper bound $\nu_{11}\leqslant 1/2$.

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I. INTRODUCTION

The concept of self-organized criticality (SOC) is fascinating since it is useful to explain the emergence of scaling and fractal behavior widely observed in nature [1,2]. The prototype of the statistical mechanical models exhibiting SOC is the Bak-Tang-Wiesenfeld (BTW) model [3,4] whose time-evolution rules capture some aspects of dynamics of sand grains tumbling on the slope of a sandpile. This cellular automaton model has two processes, external driving and internal relaxation. The separation of two time scales is implicitly required by definition of the model; an external particle should be added in the system *after* an avalanche. There are many models exhibiting the SOC behaviors other than the sandpile model, e.g., the Bak-Sneppen evolution model [5], the forest fire model [6] and the Olami-Feder-Christensen (OFC) earthquake model [7].

However, there has been only a phenomelogical definition of SOC and then the conditions necessary for SOC are still unclear. One of the approaches to solve this problem is to study the robustness of SOC by generalizing the rules of the models, in which the emergence of SOC has been established for their original versions. Since cellular automaton models are suitable to computer simulation rather than to analytic calculations, the study of this line has been done mainly by numerical simulations [8–14]. As explained below, here we will concentrate on the reports of computer simulations [9,10] that showed the breaking of SOC in the BTW model as soon as the rule is generalized so that the conservation of particle numbers is not satisfied even in toppling apart from boundaries.

Almost all of the exactly solved models exhibiting SOC, which are defined on regular lattices with short-range interactions, may fall into only one class called the Abelian sandpile models (ASM) of Dhar [15–24]. Although Manna *et al.* [9] and Ghaffari *et al.* [10] did not take care of the Abelian property when they introduced their models, we have found that we can define nonconservative sandpile models similar to those that are Abelian and then can be solved exactly. In this paper, we generalize the BTW model to make some particles annihilate in a toppling process, and show that, from exact calculations, if a slight annihilation exists, then the model loses criticality. Ghaffari *et al.* concluded by computer simulations that a level of dissipation R (i.e., the breaking of conservation) as small as 1 part in 1000 is enough to destroy criticality [10]. On the other hand, Manna *et al.* [9] reported the increasing fluctuations of simulation data as decreasing the level of dissipation R, which implies the difficulty of studying the system for $R\simeq 0$ by numerical simulations. The exact solutions are very useful to clarify the condition the SOC breaks as shown in the present paper. Our analytical results support the picture for SOC that the conserved BTW models exists at the critical point of the generalized (nonconserved) models [9,10,25]. Our results also give exact values of critical exponents, a scaling relation, and a bound of a critical exponent.

The paper is organized as follows. In Sec. II, we give a precise definition of the nonconservative Abelian sandpile models. In Sec. III, the calculations of the average number of topplings in an avalanche and the height correlation will be given. Concluding remarks are given in Sec. IV.

II. NONCONSERVATIVE ABELIAN SANDPILE MODEL

Consider an $L\times L$ region Λ_L on a square lattice. To each site $\vec{x}=(x,y)\in\Lambda_L$, a variable $z(\vec{x})\in\mathbb{Z}^+=\{1,2,3,\cdots\}$ is assigned, which is the number of sand grains on $\vec{x}$. We define a threshold value z_c and a configuration is stable if $z(\vec{x})\leqslant z_c$ at all sites and otherwise it is unstable. An initial configuration is a random stable configuration, and time-evolution rules consist of the following two rules. (i) Adding a particle at a randomly chosen site $\vec{x}$, corresponds to $z(\vec{x})\to z(\vec{x})+1$, and other $z(\vec{y})$'s ($\vec{y}\neq\vec{x}$) are unchanged. In this procedure, the probability of choosing one site is not necessarily equal. For simplicity, however, we assume that each site is chosen with equal probability from now on. (ii) If any $z(\vec{x})>z_c$, then $z(\vec{y})\to z(\vec{y})-\Delta(\vec{x},\vec{y})$ for all y. where $\Delta(\vec{x},\vec{y})$

is the $(\vec{x},\vec{y})$ element of *the rule matrix* Δ. This process represents a toppling at $\vec{x}$. In our sandpile model we define the rule matrix as

$$\Delta(\vec{x},\vec{y})=\begin{cases} 4\alpha\zeta & \text{if} \quad \vec{x}=\vec{y} \\ -\zeta & \text{if} \quad |\vec{x}-\vec{y}|=1 \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

for $\vec{x},\vec{y}\in\Lambda_L$ and $\Delta(\vec{x},\vec{y})=0$ for $\vec{x}\notin\Lambda_L$ or $\vec{y}\notin\Lambda_L$. This rule means that at a toppling on $\vec{x}$, $4\alpha\zeta$ particles drop from $\vec{x}$ and ζ particles fall onto each of nearest-neighbor sites. When $\alpha>1(\alpha<1)$, $4|\alpha-1|\zeta$ particles are annihilated (created) in a toppling. Without loss of generality, we can assume $z_c=4\alpha\zeta$. We impose the open boundary condition so that if topplings occur at a corner or an edge of Λ_L, particles dropping outside of Λ_L leave the system. Topplings must be continued until the configuration becomes stable again, and this series of topplings is called an avalanche. After an avalanche, return to the perturbation (i).

In this paper, we restrict the cases that particles annihilate in a toppling ($\alpha>1$). It must be noted that $\alpha\zeta$ and ζ must be integers, since we have assumed $z(\vec{x})\in\mathbb{Z}^+$. However α can be chosen any rational number if one choose ζ so that $\alpha\zeta$ is an integer. We can then treat the cases in which the ratio of particle annihilation in a toppling is very small; for example, $\alpha=1.0001$. Such cases are difficult to study by computer simulations.

Let $\mathcal{C}$ be a stable configuration and define an operator $a_{\vec{x}}$ so that $a_{\vec{x}}\mathcal{C}$ is the stable configuration that is reached through an avalanche due to a perturbation at $\vec{x}$. It can be seen that the operators $\{a_{\vec{x}}\}$ are exchangeable: $[a_{\vec{x}},a_{\vec{y}}]=0$. Those models having such property are called ASM [16]. For general Abelian sandpile models, Dhar proved that stable configurations are divided in two classes, *allowed* and *forbidden* configurations. The allowed configurations come out with equal probability in the stationary state, although the forbidden configurations never do. It is proved that the number of allowed configurations N_R is generally given by

$$N_R=\det\Delta. \tag{2}$$

The Green function $G(\vec{x},\vec{y})$ of the sandpile models was defined by Dhar as the average number of topplings at $\vec{y}$ in the avalanche due to a perturbation at $\vec{x}$, which is the solution of the equation

$$\sum_{\vec{y}\in\Lambda_L} G(\vec{x},\vec{y})\Delta(\vec{y},\vec{z})=\delta_{\vec{x},\vec{z}} \quad \text{for} \quad \vec{x}\in\Lambda_L \tag{3}$$

with the boundary condition $G(\vec{x},\vec{y})=0$ if $\vec{x}\notin\Lambda_L$ or $\vec{y}\notin\Lambda_L$. Here $\delta_{\vec{x},\vec{z}}=1$ if $\vec{x}=\vec{z}$ and $\delta_{\vec{x},\vec{z}}=0$ otherwise. The average number of topplings in an avalanche in the steady state $\langle T\rangle_L$ is then given as

$$\langle T\rangle_L=\frac{1}{L^2}\sum_{\vec{x}\in\Lambda_L}\sum_{\vec{y}\in\Lambda_L} G(\vec{x},\vec{y})$$
$$=\frac{1}{L^2}\sum_{\vec{x}\in\Lambda_L}\sum_{\vec{y}\in\Lambda_L}[\Delta^{-1}](\vec{x},\vec{y}). \tag{4}$$

Consider a discrete version $\hat{\Delta}_r$ of the Laplacian operator $\hat{\Delta}=\Sigma_{\mu=1}^2\partial^2/\partial x_\mu^2$,

$$\hat{\Delta}_r f(\vec{x})=\sum_{\mu=1}^{2}[f(\vec{x}+\vec{e}_\mu)+f(\vec{x}-\vec{e}_\mu)-2f(\vec{x})], \tag{5}$$

where $\vec{e}_1$ and $\vec{e}_2$ are the basis vectors of the square lattice. Equation (3), in which $\vec{x}$ and $\vec{y}$ are replaced by $\vec{0}$ and $\vec{x}$, respectively, can be written as

$$[\hat{\Delta}_r-\mu^2]G(\vec{x})=-\delta_{\vec{x},\vec{0}}, \tag{6}$$

for $G(\vec{x})=G(\vec{0},\vec{x})/\xi$ with $\mu^2=4(\alpha-1)$. If $\alpha>1$, $G(\vec{x})$ is the analog of the massive propagator in particle physics. We find that the Green function of the two-dimensional stationary Klein-Gordon equation $[\Delta-\mu^2]\mathcal{G}_2(r)=-\delta(r)$, $r=\sqrt{x^2+y^2}$, is given as $\mathcal{G}_2(r)=(i/4)H_0^{(1)}(i\mu r)$ using the Hankel function of the second kind. Since $\mathcal{G}_2(r)$ gives the two-dimensional Yukawa potential $\mathcal{G}_2(r)\simeq(2\sqrt{2\pi})^{-1}e^{-\mu r}/r^{1/2}$ for $r\gg 1$, Eq. (6) implies that for any $\alpha>1$ the Green function of the present sandpile model shows the exponential decay and the system is out of criticality. Rigorous arguments proving this simple observation will be given in the following sections.

III. ANALYTICAL CALCULATIONS

A. Exact calculations of $\langle T\rangle$

For obtaining $\langle T\rangle_L$ by Eq. (4), we must calculate Δ^{-1}. If we can diagonalize Δ with a matrix P as $\Lambda=P^{-1}\Delta P$, we can easily obtain Δ^{-1} as $P\Lambda^{-1}P^{-1}$. We found that the rule matrix can be diagonalized by the Fourier transformation of $\vec{x}=(x_1,x_2)$ to $\vec{n}=(n_1,n_2)$ as

$$P(\vec{n},\vec{x})=\frac{2}{L+1}\sin\left(\frac{x_1n_1}{L+1}\pi\right)\sin\left(\frac{x_2n_2}{L+1}\pi\right). \tag{7}$$

Then we obtain that the diagonalized matrix Λ is

$$\Lambda(\vec{x},\vec{y})=2\zeta\left\{2\alpha-\cos\left(\frac{x_1}{L+1}\pi\right)-\cos\left(\frac{x_2}{L+1}\pi\right)\right\}\delta_{\vec{x},\vec{y}}. \tag{8}$$

Using Eqs. (7) and (8), $G(\vec{x},\vec{y})$ can be written as

$$G(\vec{x},\vec{y})=\frac{2}{\zeta(L+1)^2}\sum_{n_1=1}^{L}\sum_{n_2=1}^{L}\times\frac{\cos\left(\frac{n_1x}{L+1}\pi\right)\cos\left(\frac{n_2y}{L+1}\pi\right)}{2\alpha-\cos\left(\frac{n_1}{L+1}\pi\right)-\cos\left(\frac{n_2}{L+1}\pi\right)}. \tag{9}$$

Then,

$$\langle T\rangle_L=\frac{2}{\zeta L^2(L+1)^2}\sum_{n_1,n_2,x_1,x_2,y_1,y_2=1}^{L}\frac{\sin\left(\frac{n_1x_1}{L+1}\pi\right)\sin\left(\frac{n_2x_2}{L+1}\pi\right)\sin\left(\frac{n_1y_1}{L+1}\pi\right)\sin\left(\frac{n_2y_2}{L+1}\pi\right)}{2\alpha-\cos\left(\frac{n_1}{L+1}\pi\right)-\cos\left(\frac{n_2}{L+1}\pi\right)}$$

$$=\frac{2}{\zeta L^2(L+1)^2}\sum_{n_1,n_2=1:n_1,n_2\in\{\text{odd}\}}^{L}\frac{\cot^2\left(\frac{n_1\pi}{2(L+1)}\right)\cot^2\left(\frac{n_2\pi}{2(L+1)}\right)}{2\alpha-\cos\left(\frac{n_1\pi}{L+1}\right)-\cos\left(\frac{n_2\pi}{L+1}\right)}. \tag{10}$$

We can take the infinite-volume limit of $\langle T\rangle_L$ and obtain

$$\langle T\rangle=\lim_{L\to\infty}\langle T\rangle_L=\frac{1}{4\zeta(\alpha-1)}<\infty \quad \text{for } \alpha>1. \tag{11}$$

Thus we can conclude the avalanche has a characteristic time scale $\langle T\rangle$ and criticality of the system is destroyed for any $\alpha>1$. It should be noted that $\langle T\rangle$, which is the spatial summation of $G(\vec{x},\vec{y})$ as in Eq. (4), will show a power-law divergence as $\alpha\to 1$ with an exponent 1. It is the analog of the susceptibility and the specific heat that are the spatial summations of the spin-spin and the energy-energy correlations and exhibit the divergence as the temperature goes to its critical value in the usual second-order phase transition of spin systems.

Ghaffari *et al.* found numerically [10] that the distribution of T behaves like $P(T)\sim T^{-a}\exp[-(T/T_0)^b]$, where $T_0\sim(\alpha-1)^{-2\nu}$. Combining this assumption with the definition $\langle T\rangle=\int TP(T)dT$ gives $\langle T\rangle\sim(\alpha-1)^{-2\nu(2-a)}\Gamma[(2-a)/b]$, where $\Gamma(x)$ is the gamma function. Our result (10) gives a scaling relation

$$2\nu(2-a)=1. \tag{12}$$

It is remarkable that the numerical fitting of the data reported by Ghaffari *et al.* gives $\nu\simeq 1/2$ and $a\simeq 1$, which are consistent with Eq. (12).

B. Analytical estimation of the 1-1 height correlation

We study here the 1-1 height correlation $P_{11}(r)$ by applying the method used by Majumdar and Dhar for the BTW model [17]. For preparation, we must go through the *burning algorithm* [16–18] that determines whether a given stable configuration is allowed or not. The procedure of the burning algorithm is the following. For a given configuration, choose a site $\vec{x}$ at random and test whether the inequality $z(\vec{x})>\Sigma_{\vec{y}}\Delta(\vec{y},\vec{x})$ is satisfied or not. The right-hand quantity corresponds to the sum of numbers of particles falling into $\vec{x}$ in topplings at other sites $\vec{y}$. If the inequality is satisfied, then we remove the site $\vec{x}$ from the lattice (the site is burned). Continue the test and the removing processes over all sites until there is no site to be removed. At this time, if the system becomes empty then the initial configuration is allowed, while if there remain unburned sites on the lattice the configuration is forbidden.

Consider a lattice with two separated sites $\vec{O}$ and $\vec{O}'$ and an allowed configuration $\mathcal{C}$ with $z(\vec{O})=z(\vec{O}')=1$ on it. As shown in Fig. 1 we assume that $\vec{N}$, $\vec{W}$, $\vec{S}$ and $\vec{E}$ ($\vec{N'}$, $\vec{W'}$, $\vec{S'}$, and $\vec{E}'$) are the nearest-neighbor sites of $\vec{O}$ ($\vec{O}'$). It is concluded from the burning algorithm that the height of a site whose adjacent site having height 1 must be larger than ζ. Then we can construct a configuration $\{\mathcal{C}'\}$ by reducing the heights at $\vec{N}$, $\vec{W}$, $\vec{S}$, $\vec{N}'$, $\vec{W}'$, and $\vec{S}'$ by ζ. It should be noted that for any allowed configuration $\mathcal{C}$ with $z(\vec{O})=z(\vec{O}')=1$ we have a unique configuration $\mathcal{C}'$ by this procedure and we can make two sets of configurations $\{\mathcal{C}\}$ and $\{\mathcal{C}'\}$ that have one-to-one correspondence to each other.

It is convenient to put a *sink site* out of the lattice and connect each site in the lattice and the sink site by an additional bond. We interpret annihilation of particles in a toppling as a process in which the particles fall into the sink site through the additional bond between the toppling site and the sink site. Particles into the sink site are absorbed and never return to the lattice sites. In this system, in a toppling at $\vec{x}$, $4\alpha\zeta$ particles drop from $\vec{x}$, and then ζ particles fall into every nearest-neighbor site and $(\alpha-1)\zeta$ particles into the sink site. Next we remove the six bonds, $\vec{O}\vec{N}$, $\vec{O}\vec{W}$, $\vec{O}\vec{S}$, $\vec{O}'\vec{N}'$, $\vec{O}'\vec{W}'$, $\vec{O}'\vec{S}'$, and two bonds connecting $\vec{O}$, $\vec{O}'$ and the sink site. The remaining lattice (see Fig. 1) is denoted by Λ_L' instead of the original lattice Λ_L. Now consider our nonconservative ASM on Λ_L', with a slight modification of toppling rules. Since particles cannot flow along the removed bonds, we revise the threshold values of $\vec{O}$, $\vec{N}$, $\vec{W}$, $\vec{S}$, $\vec{E}$, $\vec{O}'$, $\vec{N}'$, $\vec{W}'$, $\vec{S}'$, and $\vec{E}'$ as

$$z_c(\vec{O})=z_c(\vec{O'})=1,$$

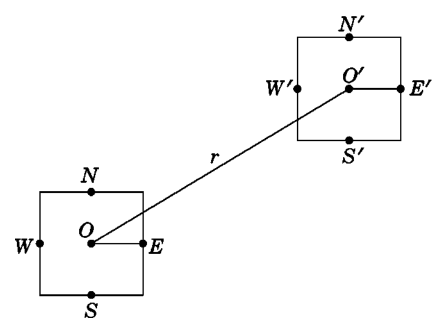

FIG. 1. A lattice Λ_L' obtained by removing six bonds from Λ_L.

$$z_c(\vec{N})=z_c(\vec{W})=z_c(\vec{S})$$
$$=z_c(\vec{N}')=z_c(\vec{W}')=z_c(\vec{S}')=4\alpha\zeta-\zeta,$$
$$z_c(\vec{E})=z_c(\vec{E}')=4\alpha\zeta-(\zeta-1). \tag{13}$$

The new rule matrix Δ' can be written using a modifying matrix B as

$$\Delta'=\Delta+B, \tag{14}$$

where $B(\vec{x},\vec{y})=0$ except for

$$B(\vec{O},\vec{O})=1-4\alpha\zeta,$$
$$B(\vec{N},\vec{N})=B(\vec{W},\vec{W})=B(\vec{S},\vec{S})=-\zeta,$$
$$B(\vec{E},\vec{E})=-\zeta+1,$$
$$B(\vec{O},\vec{N})=B(\vec{O},\vec{W})=B(\vec{O},\vec{S})$$
$$=B(\vec{N},\vec{O})=B(\vec{W},\vec{O})=B(\vec{S},\vec{O})=\zeta,$$
$$B(\vec{O},\vec{E})=B(\vec{E},\vec{O})=\zeta-1, \tag{15}$$

and for other elements obtained by replacing $\vec{O}$, $\vec{N}$, $\vec{W}$, $\vec{S}$, and $\vec{E}$ in Eq. (15) by $\vec{O}'$, $\vec{N}'$, $\vec{W}'$, $\vec{S}'$, and $\vec{E}'$, respectively. By the definitions of Λ'_L and Δ', the set of allowed configurations of this modified ASM on Λ'_L is equal to the set $\{\mathcal{C}'\}$, which is constructed from the set of allowed configurations $\{\mathcal{C}\}$ for the original ASM on Λ_L as mentioned above. This fact can be confirmed by demonstrating the burning algorithm. Let $\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_n$ be the sequence of the burned sites in the algorithm under the rule Δ for $\mathcal{C}$ with $z(\vec{O})=z(\vec{O}')=1$, then we see that $\mathcal{C}'$ passes the burning algorithm under the rule Δ' with the same sequence, $\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_n$. This one-to-one correspondence concludes that the number of allowed configurations with $z(\vec{O})=z(\vec{O}')=1$ is equal to the total number of allowed configurations of the ASM under Δ' on Λ'_L.

Define $P_{11}(\vec{O},\vec{O}')$ to be the probability that the configuration with $z(\vec{O})=z(\vec{O}')=1$ appears in the stationary state of the nonconservative ASM with the rule matrix Δ. Combining the equiprobability property of allowed configurations and Dhar's formula (2) with the above consideration gives

$$P_{11}(\vec{O},\vec{O}')=\frac{\det\Delta'}{\det\Delta}. \tag{16}$$

Substituting Eq. (14) into Eq. (16) gives $P_{11}(\vec{O},\vec{O}')=\det[I+GB]$ with an $L^2\times L^2$ matrix G, where I is a unit matrix. Since the elements of B are zero except the rows and columns labeled by $\vec{O}$, $\vec{N}$, $\vec{W}$, $\vec{S}$, $\vec{E}$, $\vec{O}'$, $\vec{N}'$, $\vec{W}'$, $\vec{S}'$, and $\vec{E}'$, we need to calculate the determinant of only the 10×10 matrix. As explained below, we can define the 10×10 matrices B_1 and G_1 so that

$$P_{11}(\vec{O},\vec{O}')=\det[I_1+G_1B_1], \tag{17}$$

where I_1 is the 10×10 unit matrix. The elements of B_1 such as $(\vec{O},\vec{O}')$, $(\vec{O},\vec{N}')$, $(\vec{O},\vec{W}')$, and so forth are zero, and we have

$$B_1=\begin{pmatrix} B_{11} & 0\\ 0 & B'_{11}\end{pmatrix}, \tag{18}$$

where B_{11} and B'_{11} are 5×5 matrices. The form of G_1 is, accordingly,

$$G_1=\begin{pmatrix} G_{11} & G_{12}\\ G_{12} & G'_{11}\end{pmatrix}, \tag{19}$$

where G_{11}, G_{12}, and G'_{11} are 5×5 matrices. Note that the elements of B_{11} and G_{11} (B'_{11} and G'_{11}) depend only on the location of $\vec{O}$ ($\vec{O}'$) and are independent of the distance between $\vec{O}$ and $\vec{O}'$ although those of G_{12} depend on the locations of both of $\vec{O}$ and $\vec{O}'$.

It can be seen that what we must calculate to determine the elements of B_1 and G_1 are $G(\vec{P}\pm\sigma\vec{e}_1\pm\rho\vec{e}_2,\vec{P}'\pm\sigma'\vec{e}_1\pm\rho'\vec{e}_2)$, where $\vec{P}$ and $\vec{P}'$ are $\vec{O}$ or $\vec{O}'$ and $\sigma,\rho,\sigma',\rho'\in\{-1,0,1\}$, where $G(\vec{x},\vec{y})$ is the Green function given by Eq. (9). Now we fix $\vec{O}$ at the center of the lattice ($[(L+1)/2,(L+1)/2]$), and $\vec{O}'=\vec{O}+x\vec{e}_1+y\vec{e}_2$ with $-L/2<x$, $y<L/2$. Let $r=\sqrt{x^2+y^2}$ and take the $L\to\infty$ limit, then we have

$$\tilde{G}(r)\equiv\lim_{L\to\infty}G(\vec{O},\vec{O}')$$
$$=\frac{1}{2\zeta\pi^2}\int_0^\pi d\theta_1\int_0^\pi d\theta_2\frac{\cos(x\theta_1)\cos(y\theta_2)}{2\alpha-\cos(\theta_1)-\cos(\theta_2)}. \tag{20}$$

Define the truncated correlation function $C_{11}(r)$ as

$$C_{11}(r)=\lim_{L\to\infty}[P_{11}(\vec{O},\vec{O}')-P_1(\vec{O})P_1(\vec{O}')] \tag{21}$$

where $P_1(\vec{O})[P_1(\vec{O}')]$ is the probability that the height at $\vec{O}(\vec{O}')$ is one in the steady state. By Eq. (17) and the properties of matrices G_1 and B_1 explained, we can conclude $C_{11}(r)$ decays as $\tilde{G}^2(r)$.

As shown in the Appendix, we found an upper bound of $|\tilde{G}(r)|$ as

$$|\tilde{G}(r)|<\text{const}\times\frac{e^{-r/2\xi_u}}{r}, \tag{22}$$

where

$$\xi_u=\frac{1}{\sqrt{2}\ln\{\alpha+1-\sqrt{2\alpha}\sqrt{2\alpha-2}\}}. \tag{23}$$

Thus we therefore conclude that $C_{11}(r)$ decays with respect to r faster than $e^{-r/\xi_u}/r^2$. This means that $C_{11}(r)$ is not algebraic but exponential, and the upper bound of the correlation length is given by ξ_u. Note that ξ_u diverges as $\alpha\to1$.

It should be remarked that the $\alpha=1$ limit of the present ASM is the BTW model, and Majumdar and Dhar [17] established that $C_{11}(r) \sim r^{-2d}$ for the d-dimensional BTW models. The power-law divergence of ξ_u of $C_{11}(r)$, $\xi_u \sim |\alpha-1|^{-1/2}$ for $\alpha \simeq 1$, implies that the correlation length ξ of $C_{11}(r)$ will also diverge as $\xi \sim |\alpha-1|^{-\nu_{11}}$ as $\alpha \to 1$ in the nonconservative ASM. Our results (22) and (23) give an upper bound of the correlation length exponent as $\nu_{11} \leqslant 1/2$. We note that the mean-field theory of Vespignani and Zapperi gives the height correlation length exponent as 1/2 for all nonconservative sandpile models [25].

IV. CONCLUDING REMARKS

In the present paper we have introduced a nonconservative Abelian sandpile model on an $L \times L$ square lattice with a parameter $\alpha > 1$, in which at a fraction $(\alpha-1)/\alpha$ particles are annihilated in each toppling process. An exact expression for the average number of toppling in an avalanche in the steady state $\langle T \rangle_L$ is obtained for an arbitrary size of system L and it is shown that $\langle T \rangle = \lim_{L\to\infty} \langle T \rangle_L < \infty$ for any $\alpha > 1$. We have also calculated the 1-1 height correlation function $C_{11}(r)$ and proved that for any $\alpha > 1$ the correlation length ξ is finite and $C_{11}(r)$ decays exponentially for large r. Since $\langle T \rangle$ and ξ are the temporal and spatial characteristic scales of the extension of avalanches, we can conclude that the criticality is lost when $\alpha > 1$. In the limit $\alpha \to 1$, on the other hand, our model is reduced to be the BTW model, for which Dhar [16] proved $\langle T \rangle \sim L^2 \to \infty$ as $L \to \infty$ and Majumdar and Dhar [17] showed $C_{11}(r) \sim r^{-4}$ for $r \gg 1$. It is then concluded that the conservation of particles in the toppling in the bulk of the system is necessary for SOC.

The present exact results show that $\langle T \rangle \sim (\alpha-1)^{-1}$ for any $\alpha > 1$ and $\xi \leqslant \xi_u \sim |\alpha-1|^{-1/2}$ for $0 < \alpha - 1 \ll 1$. The power-law divergence as $\alpha \to 1$ supports the picture that SOC of the BTW model can be considered as a critical phenomenon at the critical point $\alpha=1$ of the present generalized (nonconservative) Abelian sandpile model [9,10,25].

In conclusion we should remark that it is still an open problem for the non-Abelian sandpile models whether or not the conservation of particles in toppling is necessary for the SOC. In the present study, we add a particle at a randomly chosen site (*random drive*). Ghaffari *et al.* [10] claimed that the establishment of SOC depends on how to give perturbation to the system as well as on the method of toppling (e.g., Abelian or non-Abelian). They reported that in a nonconservative and non-Abelian model, which they called the Zhang model [14] but the method of toppling of which is identical to the OFC earthquake model [7], *uniform drive* is necessary (but not sufficient) for SOC. It is noted that their study depends on numerical simulations and approximate renormalization group analysis and it is a challenging future problem to obtain exact and/or rigorous results for non-Abelian models.

APPENDIX: BOUNDS OF $\tilde{G}(r)$

We can immediately perform one of the two integrations of Eq. (20) by the formula

$$\int_0^\pi \frac{\cos a\theta}{A-\cos\theta} d\theta = \pi \frac{(A-\sqrt{A-1}\sqrt{A+1})^a}{\sqrt{A+1}\sqrt{A-1}}. \qquad \text{(A1)}$$

Then $\tilde{G}(r)$ reduces to

$$\tilde{G}(r) = \frac{1}{2\zeta\pi} \int_0^\pi d\theta_1 \cos x\theta_1 \frac{(2\alpha-\cos\theta_1-\sqrt{2\alpha-\cos\theta_1+1}-\sqrt{2\alpha-\cos\theta_1-1})^y}{\sqrt{2\alpha-\cos\theta_1+1}\sqrt{2\alpha-\cos\theta_1-1}}, \qquad \text{(A2)}$$

where $r=\sqrt{x^2+y^2}$.

It may be convenient to change the variable θ_1 for ϕ/x, and then divide the interval $[0, x\pi]$ into x parts as follows. We assume that x is even,

$$\tilde{G}(r) = \frac{1}{2\zeta\pi x} \sum_{k=0}^{x-1} \int_{k\pi}^{(k+1)\pi} \cos\phi\, g\left(\frac{\phi}{x}\right) d\phi, \qquad \text{(A3)}$$

where

$$g(\phi) = \frac{(2\alpha-\cos\phi-\sqrt{2\alpha-\cos\phi-1}\sqrt{2\alpha-\cos\phi+1})^y}{\sqrt{2\alpha-\cos\phi-1}\sqrt{2\alpha-\cos\phi+1}}. \qquad \text{(A4)}$$

Note that $g(\phi)$ is a decreasing function of ϕ in each interval $[k\pi, (k+1)\pi]$. It follows that

$$g\left(\frac{2k+1}{2x}\pi\right) \int_{k\pi}^{(k+1)\pi} \cos\theta d\theta < \int_{k\pi}^{(k+1)\pi} \cos\theta\, g\left(\frac{\theta}{x}\right) d\theta$$
$$< g\left(\frac{k\pi}{x}\right) \int_0^{(k+1/2)\pi} \cos\theta d\theta + g\left(\frac{k+1}{x}\pi\right) \int_{(k+1/2)\pi}^{(k+1)\pi} \cos\theta d\theta \quad \text{if } k \text{ is even} \qquad \text{(A5)}$$

and

$$g\left(\frac{k\pi}{x}\right) \int_0^{(k+1/2)\pi} \cos\theta d\theta + g\left(\frac{k+1}{x}\pi\right) \int_{(k+1/2)\pi}^{(k+1)\pi} \cos\theta d\theta < \int_{k\pi}^{(k+1)\pi} \cos\theta\, g\left(\frac{\theta}{x}\right) d\theta < g\left(\frac{2k+1}{2x}\pi\right) \int_{k\pi}^{(k+1)\pi} \cos(\theta) d\theta \quad \text{if } k \text{ is odd}. \qquad \text{(A6)}$$

Substituting Eqs. (A5) and (A6) into Eq. (A3) gives

$$\tilde{G}_l(r) < \tilde{G}(r) < \tilde{G}_u(r), \quad \text{(A7)}$$

with

$$\tilde{G}_u(r) = \frac{1}{2\zeta\pi x} \sum_{m=0}^{(x-2)/2} \left\{ g\left(\frac{2m\pi}{x}\right) - g\left(\frac{2m+1}{x}\pi\right) \right\},$$

$$\tilde{G}_l(r) = \frac{1}{2\zeta\pi x} \sum_{m=0}^{x/2} \left\{ g\left(\frac{2m\pi}{x}\right) - g\left(\frac{2m-1}{x}\pi\right) \right\}. \quad \text{(A8)}$$

Since $g(x)$ depends on x through $\cos x$, one can expand $g(x)$ with respect to $\cos x$. The summations of $g(x)$ are replaced to the summations of $\cos^l(\theta)$, $l \in \mathbb{Z}$. Then we arrive at

$$\tilde{G}_u(r) < \frac{1}{2\zeta\pi x} \frac{(\alpha+1-\sqrt{2\alpha}\sqrt{2\alpha-2})^y}{\sqrt{2\alpha}\sqrt{2\alpha-2}},$$

$$\tilde{G}_l(r) > -\frac{1}{2\zeta\pi x} \frac{(\alpha+1-\sqrt{2\alpha}\sqrt{2\alpha-2})^y}{\sqrt{2\alpha}\sqrt{2\alpha-2}}, \quad \text{(A9)}$$

where we used the following equalities:

$$\sum_{m=0}^{x/2} \left\{ \cos^l\left(\frac{2m}{x}\pi\right) - \cos^l\left(\frac{2m+1}{x}\pi\right) \right\} = \frac{1}{2}[1-(-1)^l],$$

$$\sum_{m=0}^{(x-2)/2} \left\{ \cos^l\left(\frac{2m}{x}\pi\right) - \cos^l\left(\frac{2m-1}{x}\pi\right) \right\} = \frac{1}{2}[(-1)^l-1]. \quad \text{(A10)}$$

Then the bounds (A7) give Eq. (22).

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