

Master Thesis

Three-Parametric Marcenko-Pastur Law in Random Matrix Theory

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Abstract

The complex Wishart ensemble is the statistical ensemble of $M \times N$ complex random matrices with $M \geq N$ such that the real and imaginary parts of each element are given by independent standard normal variables. The Marcenko–Pastur (MP) density $\rho(x; r)$, $x \geq 0$ describes the distribution for squares of the singular values of the random matrices in this ensemble in the scaling limit $N \rightarrow \infty$, $M \rightarrow \infty$ with a fixed rectangularity $r = N/M \in (0, 1]$. The dynamical extension of the squared-singular-value distribution is realized by the non-colliding squared Bessel process, and its hydrodynamic limit provides the two-parametric MP density $\rho(x; r, t)$ with time $t \geq 0$, whose initial distribution is $\delta(x)$. Recently, Blaizot, Nowak, and Warchoř studied the time-dependent complex Wishart ensemble with an external source and introduced the three-parametric MP density $\rho(x; r, t, a)$ by analyzing the hydrodynamic limit of the process starting from $\delta(x - a)$, $a > 0$. In the present paper, we give useful expressions for $\rho(x; r, t, a)$ and perform a systematic study of dynamic critical phenomena observed at the critical time $t_c(a) = a$ when $r = 1$. The universal behavior in the long-term limit $t \rightarrow \infty$ is also reported. It is expected that the present system having the three-parametric MP density provides a mean-field model for QCD showing spontaneous chiral symmetry breaking.

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Contents

1	Introduction	3
1.1	Marcenko–Pastur law in random matrix theory	3
2	Main Results	4
2.1	Dynamical Extension of MP density	4
2.1.1	Initial condition $\delta(x)$	5
2.1.2	Initial condition $\delta(x - a)$	5
2.2	critical phenomena	10
2.3	Remarks with QCD	12
3	Proof of Results	14
3.1	Solving the algebraic equations for density and its Hilbert transform	14
3.2	Determining the support of density function	19
3.3	Proofs of critical phenomena	20
3.3.1	Proof of Proposition 2.2	20
3.3.2	Proof of Proposition 2.3	21
3.4	Long-Term Scaling and Proof of Proposition 2.4	23
4	Concluding Remarks and Future Perspectives	23

1 Introduction

1.1 Marcenko–Pastur law in random matrix theory

Assume that $M, N \in \mathbb{N} := \{1, 2, \dots\}$, $M \geq N$. Consider $M \times N$ complex random matrices $K = (K_{jk})$ such that the real and the imaginary parts of elements are i.i.d. and normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 1/2$. The normal distribution with mean μ and variance σ^2 is denoted by $N(\mu, \sigma^2)$ and when a random variable X obeys $N(\mu, \sigma^2)$, we write it as $X \sim N(\mu, \sigma^2)$. Then the present setting is described as

$$\Re K_{jk} \sim N(0, 1/2), \quad \Im K_{jk} \sim N(0, 1/2), \quad j = 1, \dots, M, \quad k = 1, \dots, N.$$

We consider a statistical ensemble of $N \times N$ Hermitian random matrices L defined by

$$L = K^\dagger K, \tag{1.1}$$

where K^\dagger denotes the Hermitian conjugate of K . This ensemble of random matrices is called the *complex Wishart random-matrix ensemble* or the *chiral Gaussian unitary ensemble* (chGUE) (see, for instance, [8]). We denote the eigenvalues of L as $X_j^N, j = 1, \dots, N$, which are nonnegative, since L is nonnegative definite by definition; $X_j^N \in \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$. The positive square roots of them, $\sqrt{X_j^N}, j = 1, \dots, N$ are called *singular values* of random rectangular matrices K . In other words, the eigenvalue distribution of the Hermitian random matrices L can be regarded as the distribution of squares of singular values of the rectangular complex random matrices K in the complex Wishart random-matrix ensemble.

Let $\mathcal{C}_c(\mathbb{R})$ be the set of all continuous real-valued function with compact support on \mathbb{R} . We consider the empirical measure defined by

$$\Xi^N(dx) = \frac{1}{N} \sum_{j=1}^N \delta_{X_j^N/M}(dx), \quad x \in \mathbb{R}_{\geq 0}, \tag{1.2}$$

where $\delta_y(dx)$ denotes a Dirac measure concentrated on y such that $\int_{\mathbb{R}} f(x) \delta_y(dx) = f(y)$ for all $f \in \mathcal{C}_c(\mathbb{R})$. Then we take the double limit $N \rightarrow \infty, M \rightarrow \infty$ for each fixed value of the rectangularity

$$r = \lim_{\substack{N \rightarrow \infty \\ M \rightarrow \infty}} \frac{N}{M} \in (0, 1]. \tag{1.3}$$

We can prove that in this scaling limit (1.3), the empirical measure (1.2) converges weakly to a deterministic measure $\rho(x)dx, x \in \mathbb{R}_{\geq 0}$ in the sense that $\int_{\mathbb{R}_{\geq 0}} f(x) \Xi^N(dx) \rightarrow \int_{\mathbb{R}_{\geq 0}} f(x) \rho(x) dx$ as $N \rightarrow \infty$ for any $f \in \mathcal{C}_c(\mathbb{R})$. Moreover, the probability density ρ in the limit measure has a finite support in \mathbb{R} and it is explicitly given as a function of the parameter $r \in (0, 1]$ as [19]

$$\rho(x; r) = \frac{\sqrt{(x - x_L(r))(x_R(r) - x)}}{2\pi r x} \mathbf{1}_{(x_L(r), x_R(r))}(x) \tag{1.4}$$

with

$$x_L(r) := (1 - \sqrt{r})^2, \quad x_R(r) := (1 + \sqrt{r})^2. \tag{1.5}$$

Here $\mathbf{1}_\Lambda(x)$, $\Lambda \subset \mathbb{R}$ is an indicator function such that $\mathbf{1}_\Lambda(x) = 1$ if $x \in \Lambda$, and $\mathbf{1}_\Lambda(x) = 0$ otherwise. This convergence theorem is known as the *Marcenko–Pastur law* for the Wishart random-matrix ensemble [19, 8, 1] and we call (1.4) the *Marcenko–Pastur (MP) density* in this paper.

2 Main Results

2.1 Dynamical Extension of MP density

A dynamical extension of the eigenvalue distribution of the Wishart random-matrix ensemble is realized by the solution $\{X_j^N(t) \in \mathbb{R}_{\geq 0} : t \geq 0, j = 1, 2, \dots, N\}$ of the following system of stochastic differential equations (SDEs),

$$\begin{aligned} dX_j^N(t) &= 2\sqrt{X_j^N(t)}dB_j(t) + 2(\nu + 1)dt \\ &+ 4X_j^N(t) \sum_{\substack{1 \leq k \leq N, \\ k \neq j}} \frac{1}{X_j^N(t) - X_k^N(t)} dt, \quad j = 1, 2, \dots, N, \quad t \geq 0, \end{aligned} \quad (2.6)$$

where $\nu = M - N$ and $B_j(t), t \geq 0$ are independent one-dimensional standard Brownian motions starting from $x_j^N \in \mathbb{R}_{\geq 0}, j = 1, \dots, N$. We assume that $0 \leq x_1^N \leq x_2^N \leq \dots \leq x_N^N < \infty$. This one-parameter family ($\nu > 0$) of N -particle stochastic processes was called (the eigenvalue process of) the *Wishart process* by Bru [5]. It is also called the *Laguerre process* or the *noncolliding squared Bessel process* [16, 14, 15].

We set $\nu = (1 - r)M = (1 - r)N/r, r \in (0, 1]$, and consider the empirical measure of the solution of SDEs (2.6),

$$\Xi_t^N(dx) := \frac{1}{N} \sum_{j=1}^N \delta_{X_j^N(t)/M}(dx), \quad x \in \mathbb{R}_{\geq 0}, \quad t \geq 0. \quad (2.7)$$

If the initial empirical measure satisfies some moment conditions and converges weakly to a measure, $\Xi_0^N(dx) = (1/N) \sum_{j=1}^N \delta_{x_j^N/M}(dx) \rightarrow \xi(dx)$ in the limit $N \rightarrow \infty, M \rightarrow \infty$ with $r = N/M$ fixed in $(0, 1]$, it is proved that $\Xi_t^N(dx)$ converges weakly to a time-dependent deterministic measure, which we denote here as $\rho_\xi(x; r, t), t \geq 0$ [6, 2]. We define the *Green's function* (the resolvent) $G_\xi(z; r, t)$ by the Stieltjes transform of ρ_ξ ,

$$G_\xi(z; r, t) := \int_{\mathbb{R}} \frac{\rho_\xi(x; r, t)}{z - x} dx, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Then we can prove that this solves the following nonlinear partial differential equation (PDE) [6, 2],

$$\frac{\partial G_\xi}{\partial t} = -\frac{\partial G_\xi}{\partial z} + r \left\{ \frac{\partial G_\xi}{\partial z} - 2zG_\xi \frac{\partial G_\xi}{\partial z} - G_\xi^2 \right\}, \quad t \in [0, \infty), \quad (2.8)$$

under the initial condition, $G_\xi(z; r, 0) = \int_{\mathbb{R}} \xi(dx)/(z - x)$, $z \in \mathbb{C} \setminus \mathbb{R}$. Once the Green's function $G_\xi(z; r, 0)$, $z \in \mathbb{C} \setminus \mathbb{R}$ is determined, we can obtain the density function following the Sokhotski-Plemelj theorem,

$$\rho_\xi(x; r, t) = -\Im \left[\lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} G_\xi(x + i\varepsilon; r, t) \right], \quad i := \sqrt{-1}. \quad (2.9)$$

We regard (2.8) as the analogy of the complex Burgers equations in the inviscid limit (*i.e.*, the (complex) one-dimensional Euler equation), and we call the limit process the *hydrodynamic limit* of the Wishart process [2, 3].

2.1.1 Initial condition $\delta(x)$

The simplest case of the dynamical extension of the MP density (1.4) with (1.5) is obtained by setting the initial distribution,

$$\xi(dx) = \delta_0(dx) := \delta(x)dx, \quad (2.10)$$

that is, all particles are concentrated on the origin; $x_j^N = 0$, $j \in \mathbb{N}$. By the method of complex characteristics, Blaizot, Nowak, and Warchoř [2] showed that the Green's function for (2.10), $G_{\delta_0}(z) = G_{\delta_0}(z; r, t)$, is given by the solution of the equation

$$z = \frac{1}{G_{\delta_0}(z)} + \frac{t}{1 - rtG_{\delta_0}(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad r \in (0, 1], \quad t \geq 0. \quad (2.11)$$

They solved (2.11) and using the Sokhotski-Plemelj theorem (2.9) derived the time-dependent extension of (1.4),

$$\rho(x; r, t) := \rho_{\delta_0}(x; r, t) = \frac{\sqrt{(x - x_L(r, t))(x_R(r, t) - x)}}{2\pi r t x} \mathbf{1}_{(x_L(r, t), x_R(r, t))}(x) \quad (2.12)$$

with

$$x_L(r, t) := (1 - \sqrt{r})^2 t, \quad x_R(r, t) := (1 + \sqrt{r})^2 t, \quad t \in (0, \infty). \quad (2.13)$$

Since $\rho(x; r, 1)$ is equal to the original MP density (1.4), the above provides a dynamical derivation of the Marcenko–Pastur law. The dependence on t of $\rho(x; r, t)$ given by (2.12) with (2.13) is very simple, but we regard this as the *two-parametric* MP density in the present paper.

2.1.2 Initial condition $\delta(x - a)$

Blaizot, Nowak, and Warchoř [3] have studied the hydrodynamic limit of the Wishart eigenvalue-process starting from one-parameter family ($a \geq 0$) of the initial distribution,

$$\xi(dx) = \delta_a(dx) := \delta(x - a)dx.$$

They showed that the Green's function, $G(z) = G_{\delta_a}(z; r, t)$, $a \geq 0$, is obtained by the solution of the equation,

$$z = \frac{1}{G_{\delta_a}(z)} + \frac{t}{1 - rtG_{\delta_a}(z)} + \frac{a}{(1 - rtG_{\delta_a}(z))^2}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad r \in (0, 1], \quad t \geq 0, \quad a \geq 0. \quad (2.14)$$

The main theorem of the present paper is the following.

Theorem 2.1 *Let*

$$\begin{aligned} S(x; r, t, a) := & 4ax^3 - \{8a^2 + 4a(3r + 2)t - t^2\}x^2 \\ & + 2[2a^3 - 2a^2(5r - 2)t + a\{r(6r - 1) + 1\}t^2 - (r + 1)t^3]x \\ & + (r - 1)^2t^2\{a^2 - a(4r - 2)t + t^2\}. \end{aligned} \quad (2.15)$$

For $r \in (0, 1], t > 0, a \geq 0$, consider the case such that the cubic equation with respect to x ,

$$S(x; r, t, a) = 0, \quad (2.16)$$

has three real solutions, $x_1 \leq x_2 \leq x_3$, where $x_j = x_j(r, t, a), j = 1, 2, 3$. Define

$$x_L(r, t, a) := x_2(r, t, a), \quad x_R(r, t, a) := x_3(r, t, a). \quad (2.17)$$

Put

$$\begin{aligned} g &= g(x; r, t, a) \\ &:= -2x^3 + 3\{(2r + 1)t + 6a\}x^2 - 3\left[(r - 1)\{(2r + 1)t - 3a\}t - \sqrt{-3S}\right]x + 2(r - 1)^3t^3, \end{aligned} \quad (2.18)$$

with $S = S(x; r, t, a)$ given by (2.15), and define

$$\begin{aligned} \varphi &= \varphi(x; r, t, a) \\ &:= -\frac{2}{3}\{x - (r - 1)t\} - \frac{2^{1/3}x^2 + \{3a - (2r + 1)t\}x + t^2(r - 1)^2}{g^{1/3}} - \frac{g^{1/3}}{3 \times 2^{1/3}}. \end{aligned} \quad (2.19)$$

Then the three-parametric MP density is given by

$$\rho(x; r, t, a) = \frac{\sqrt{(x - f_L(x; r, t, a))(f_R(x; r, t, a) - x)}}{2\pi rxt} \mathbf{1}_{(x_L(r, t, a), x_R(r, t, a))}(x) \quad (2.20)$$

with

$$f_L(x; r, t, a) := \left(\frac{\sqrt{d_-} + \sqrt{d_+}}{2} - \sqrt{d_0} \right)^2, \quad f_R(x; r, t, a) := \left(\frac{\sqrt{d_-} + \sqrt{d_+}}{2} + \sqrt{d_0} \right)^2, \quad (2.21)$$

where

$$\begin{aligned} d_- &= t - a - 2\sqrt{a\varphi}, & d_+ &= t - a + 2\sqrt{a\varphi}, \\ d_0 &= \varphi + x + \frac{t - a}{2} + \frac{1}{2}\sqrt{d_-d_+}. \end{aligned} \quad (2.22)$$

Remark 1 The formula (2.20) for the present three-parametric MP density seems to be similar to the original MP density (1.4) and the two-parametric MP density (2.12). We should note, however, that f_L and f_R appearing in (2.20) are not equal to the endpoints x_L and x_R of the

support of density and they depend on x as shown by (2.21) with (2.18), (2.19) and (2.22). We can see that

$$\varphi(x; r, t, 0) := \lim_{a \rightarrow 0} \varphi(x; r, t, a) = -x + (r - 1)t,$$

and hence $d_{\pm} \rightarrow t, d_0 \rightarrow rt$ as $a \rightarrow 0$. Then as $a \rightarrow 0$, $f_L \rightarrow x_L(r; t)$, $f_R \rightarrow x_R(r; t)$ with (2.13); that is, the dependence of f_L and f_R on x vanishes only in this limit. Theorem 2.1 states that for general $a > 0$, the endpoints x_L and x_R of the support for the three-parametric MP density are given by the suitably chosen solutions (2.17) of the cubic equation (2.16) with (2.15) as proved in Section 3.2 below. That is, the formula (2.20) is universal, but the choice of solutions (2.17) depends on the parameters r, t, a . The equation (2.14) seems to be a simple perturbation of (2.11), but the solution turns out to have rich structures, by which we can describe dynamic critical phenomena at time $t = t_c := a$ for $a > 0$, when $r = 1$, as shown below.

From the view point of the original random matrix theory, Theorem 2.1 gives the limit theorem for the eigenvalue distribution of random matrix L given by (1.1) in the scaling limit (1.3), in which $M \times N$ rectangular complex random matrices $K = (K_{jk})$ are distributed as

$$\begin{aligned} \Re K_{kk} &\sim N(\sqrt{Ma}, t/2), & k = 1, \dots, N, \\ \Re K_{jk} &\sim N(0, t/2), & j = 1, \dots, M, \quad k = 1, \dots, N, \quad j \neq k, \\ \Im K_{jk} &\sim N(0, t/2), & j = 1, \dots, M, \quad k = 1, \dots, N, \end{aligned} \tag{2.23}$$

$t > 0, r \in (0, 1]$. By (1.1), (2.23) gives

$$\mathbf{E}[L_{jk}] = M(a + t)\delta_{jk}, \quad j, k = 1, \dots, N.$$

When $a > 0$, such an ensemble of random matrices will be called the *Wishart ensemble with an external source* or the *non-centered Wishart ensemble*, since even at $t = 0$, the diagonal elements of L have positive means, $\mathbf{E}[L_{jj}] = Ma > 0, j = 1, \dots, N$, [4, 17, 7, 11, 9].

In Figure 1, we compare two histograms for the empirical measures (2.7) of the eigenvalues of matrices $L = K^\dagger K$ given by K of size 1000×300 (with the rectangularity $r = 300/1000 = 0.3$), whose elements are following the probability law (2.23) with different parameters (t, a) . When $(t, a) = (1, 0)$, the distribution of eigenvalues has a maximum at $x \simeq 0.4$. When $(t, a) = (1, 1)$, due to an external source at $x = a = 1$, the distribution is shifted to the positive direction having a maximum at $x \simeq 1$ and becomes broader. The former is well fitted by the original MP density $\rho(x; r = 0.3)$ and the latter is by the three-parametric MP density $\rho(x; r = 0.3, t = 1, a = 1)$ given by (2.20).

For each values of rectangularity $r \in (0, 1]$ and strength of an external source $a \geq 0$, we can show time evolution of the support $(x_L(r, t, a), x_R(r, t, a))$ of $\rho(x; r, t, a)$ on the (x, t) -plane, $(\mathbb{R}_{\geq 0})^2$. Figure 2 shows the domains

$$\mathcal{D}(r, a) := \{(x_L(r, t, a), x_R(r, t, a)) : t \geq 0\} \subset (\mathbb{R}_{\geq 0})^2$$

for $(r, a) = (0.3, 0)$ and $(r, a) = (0.3, 1)$. On the other hand, Figure 3 shows the domains for $(r, a) = (1, 0)$ and $(r, a) = (1, 1)$. As demonstrated by these figures, we can prove the following qualitative change of the domain when $a > 0$ and $r = 1$.

Proposition 2.2 Assume that $a > 0$.

- (i) If and only if $r = 1$, $\mathcal{D}(r, a)$ touches the origin $x = 0$. Otherwise, the left edge of $\text{supp } \rho(x; r, t, a)$ is strictly positive; $x_L(r, t, a) > 0$, $r \in (0, 1)$.
- (ii) When $r = 1$, there is a critical time

$$t_c(a) = a$$

such that $x_L(1, t, a) > 0$ while $0 \leq t < t_c(a)$, and $x_L(1, t, a) \equiv 0$ for $t \geq t_c(a)$. In particular, just before the critical time $t_c(a)$, the left edge of $\text{supp } \rho(x; r, t, a)$ behaves as

$$x_L(1, t, a) \simeq \frac{4}{27a^2}(t_c(a) - t)^\nu \quad \text{with } \nu = 3 \quad \text{as } t \nearrow t_c(a).$$

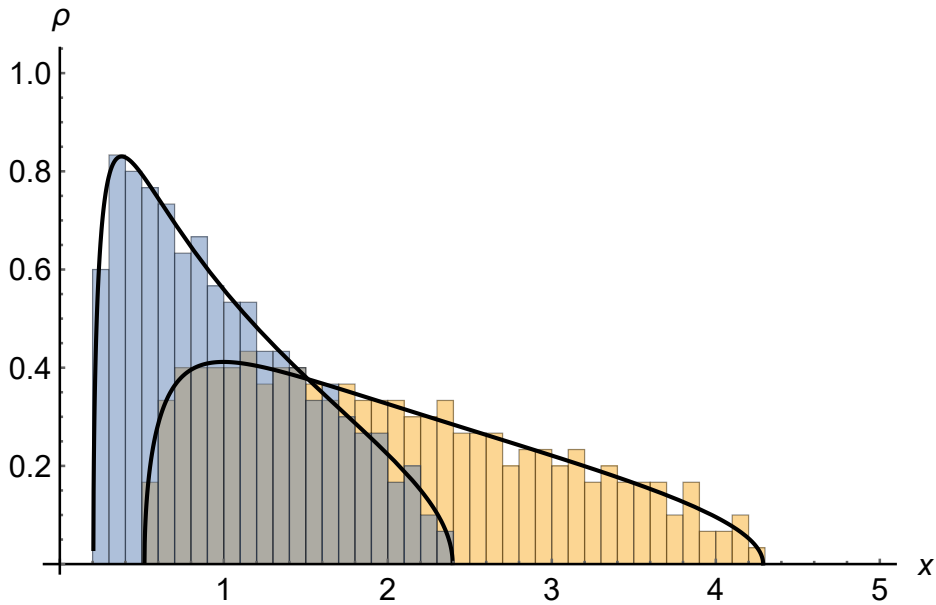


Figure 1: Two histograms for the empirical measures (2.7) with two different sets of parameters $(t, a) = (1, 0)$ and $(1, 1)$ are superposed in order to compare each other. They show the distributions of the eigenvalues of $L = K^\dagger K$ given by K of size 1000×300 , whose elements are randomly generalized following the probability law (2.23) with $(t, a) = (1, 0)$ and $(1, 1)$. The original MP density $\rho(x; r = 0.3)$ and the three-parametric MP density $\rho(x; r = 0.3, t = 1, a = 1)$ are shown by a thin curve and a thick curve, respectively. Due to an external source at $x = a = 1$, the eigenvalue distribution with $(t, a) = (1, 1)$, which is well fitted by the three-parametric MP density $\rho(x; r = 0.3, t = 1, a = 1)$, is shifted to the positive direction and becomes broader compare with the original MP density $\rho(x; r = 0.3)$.

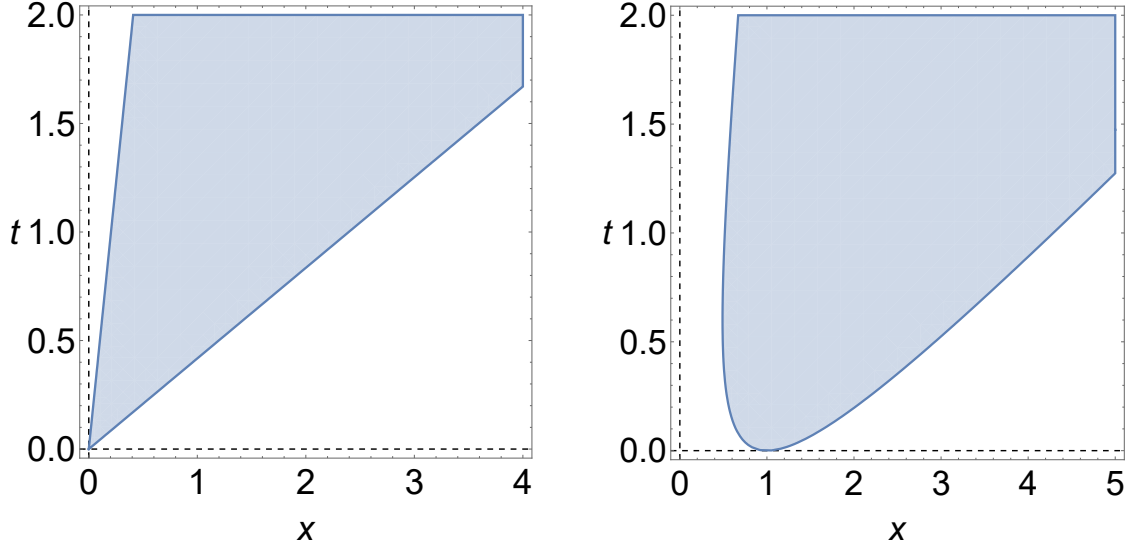


Figure 2: For $r = 0.3$, time evolution of the support (x_L, x_R) is shown on the (x, t) -plane for the two-parametric MP density $\rho(x; r = 0.3, t) := \rho(x; r = 0.3, t, a = 0)$ in the left, and for the three-parametric MP density with $a = 1$, $\rho(x; r = 0.3, t, a = 1)$ in the right. The supports are extended in time, but the left edges of supports are kept to be positive, $x_L > 0$, for all $t > 0$.

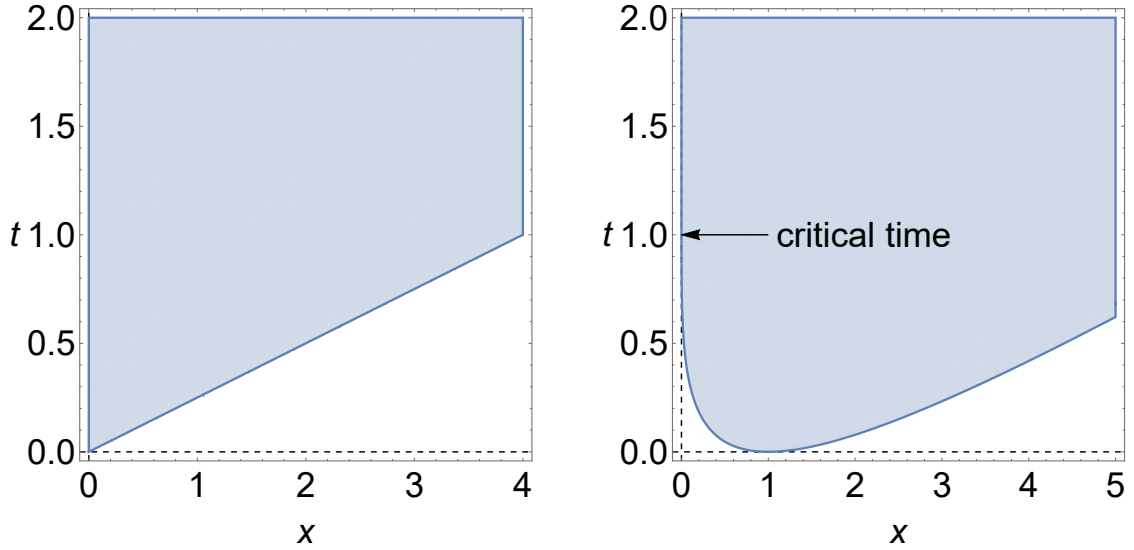


Figure 3: For $r = 1$, time evolution of the support (x_L, x_R) is shown on the (x, t) -plane for the two-parametric MP density $\rho(x; r = 1, t) := \rho(x; r = 1, t, a = 0)$ in the left, and for the three-parametric MP density with $a = 1$, $\rho(x; r = 1, t, a = 1)$ in the right. In the two-parametric MP density, the support starts from the singleton $\{0\}$ at $t = 0$ and the left edge of support x_L is identically zero; $x_L \equiv 1$ for $t \geq 0$. On the other hand, in the three-parametric MP density with $a = 1$, the support starts from the singleton $\{1\}$ at $t = 0$, and $x_L > 0$ when $t < t_c = 1$. As $t \nearrow t_c = 1$, however, $x_L \searrow 0$ continuously, and then $x_L \equiv 1$ for $t \geq t_c = 1$. We regard $t_c = 1$ as a critical time.

2.2 critical phenomena

In the case with $r = 1$, the dynamic critical phenomena at the critical time $t = t_c(a)$ are observed in the vicinity of the origin as follows.

Proposition 2.3 *When $r = 1$, the three-parametric MP density shows the following dynamic critical phenomena at $t = t_c(a)$.*

(i) For $0 < t < t_c(a)$,

$$\rho(x; 1, t, a) \simeq C_1(t, a)(x - x_L(1, t, a))^{\beta_1} \quad \text{with } \beta_1 = \frac{1}{2} \quad \text{as } x \searrow x_L(1, t, a),$$

where

$$C_1(t, a) \simeq \frac{9a}{4\pi}(t_c(a) - t)^{-\gamma_1} \quad \text{with } \gamma_1 = \frac{5}{2} \quad \text{as } t \nearrow t_c(a).$$

(ii) At $t = t_c(a)$,

$$\rho(x; 1, t_c(a), a) \simeq \frac{\sqrt{3}}{2\pi}a^{-2/3}x^{-\gamma_2} \quad \text{with } \gamma_2 = \frac{1}{3} \quad \text{as } x \searrow 0.$$

(iii) For $t > t_c(a)$,

$$\rho(x; 1, t, a) \simeq C_2(t, a)x^{-\gamma_3} \quad \text{with } \gamma_3 = \frac{1}{2} \quad \text{as } x \searrow 0,$$

where

$$C_2(t, a) \simeq \frac{1}{\pi t_c(a)}(t - t_c(a))^{\beta_2} \quad \text{with } \beta_2 = \frac{1}{2} \quad \text{as } t \searrow t_c(a).$$

Remark 2 The *critical exponents* $\nu = 3$ and $\gamma_2 = 1/3$ can be read in the argument given by [3]. Using the expressions given in Theorem 2.1 here we prove them as well as determining other critical exponents and critical amplitudes. The amplitude C_1 of ρ in the subcritical time-region ($0 < t < t_c(a)$) diverges with the critical exponent $\gamma_1 = 5/2$ as $t \nearrow t_c(a)$. Then both at the critical time ($t = t_c(a)$) and in the supercritical time-region ($t > t_c(a)$) ρ diverges as $x \searrow 0$, but the critical exponents are different. In the supercritical time-region, the amplitude C_2 of the diverging ρ with the critical exponent $\gamma_3 = 1/2$ vanishes as $t \searrow t_c(a)$ (with the exponent $\beta_2 = 1/2$). Consequently the divergence of ρ as $x \searrow 0$ is weakened at $t = t_c(a)$ having a smaller value of exponent as $\gamma_2 = 1/3 < \gamma_3 = 1/2$. See Fig.4. The proof of Proposition 2.3 given in Subsection 3.3.2 implies the scaling relation

$$\nu = \beta_2 + \gamma_1.$$

The functions $S(x; r, t, a)$, $g(x; r, t, a)$, $\varphi(x; r, t, a)$, $f_L(x; r, t, a)$, and $f_R(x; r, t, a)$, which appear in Theorem 2.1, are all *homogeneous as multivariate functions of x, t, a* for each fixed value of $r \in (0, 1]$. This fact implies the following scaling property of the three-parametric MP density,

$$\rho(\kappa x; r, \kappa t, \kappa a) = \frac{1}{\kappa} \rho(x; r, t, a), \quad r \in (0, 1], \quad (2.24)$$

for an arbitrary parameter $\kappa > 0$. By this property, the following long-term behavior of the three-parametric MP density is readily concluded.

Proposition 2.4 For $a \geq 0$,

$$\lim_{t \rightarrow \infty} \rho(y; r, t, a) dy \Big|_{y=tx} = \rho(x; r) dx,$$

where $\rho(x; r)$ is given by (1.4) with (1.5).

The long-term behavior of the present three-parametric MP density is given by a dilatation of the original MP density by factor t . In this sense, the original Marcenko–Pastur law is *universal* and it describes the large-scale and long-term behavior of the Wishart ensemble and process.

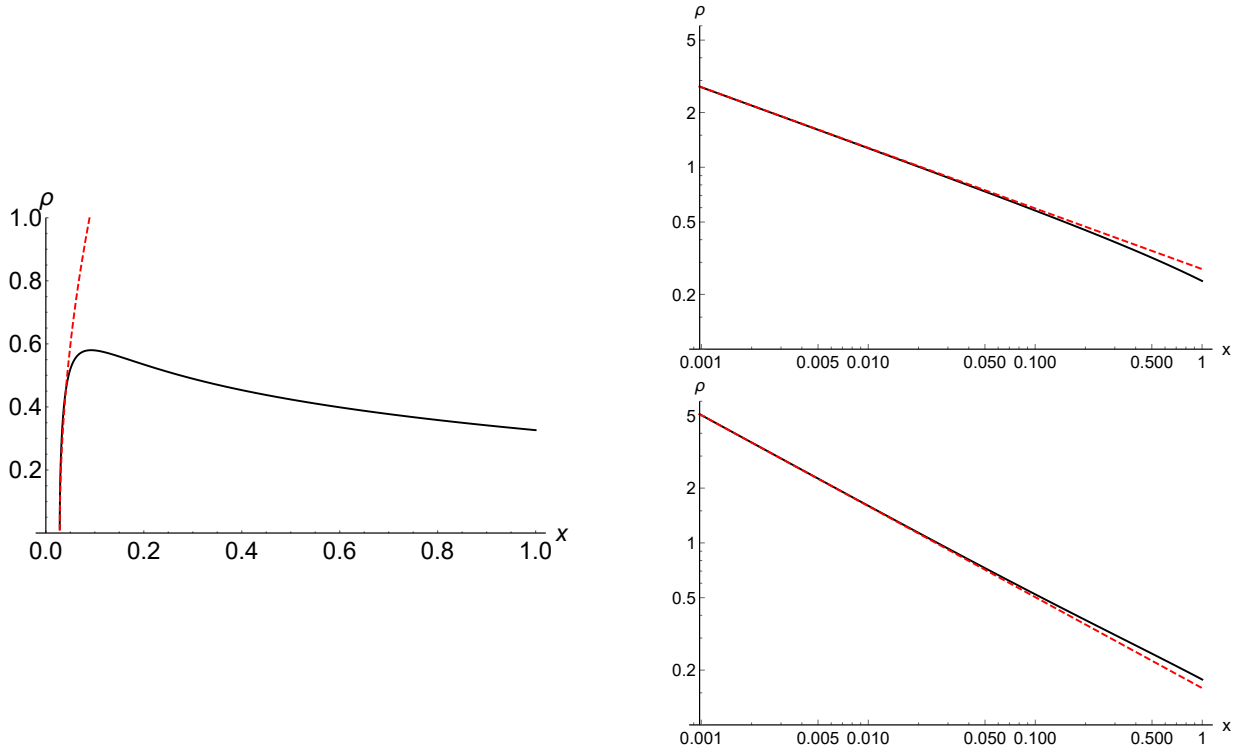


Figure 4: Critical behavior of the three-parametric MP density ρ is shown for $r = 1$ and $a = 1$. The left figure shows the emergence of ρ at $x = x_L \sim 0.028$ with critical exponent $\beta_1 = 1/2$ at a subcritical time ($t = 0.5t_c(1)$). The upper-right figure shows the divergence of ρ as $x \rightarrow 0$ at the critical time ($t = t_c(1)$). The red dotted line shows the asymptotic form given by Proposition 2.3 (ii), which follows the power law with the exponent $\gamma_2 = 1/3$. The lower-right figure shows that at the super critical time ($t = 1.5t_c(1)$). The red dotted line shows the asymptotic form given by Proposition 2.3 (iii), which follows the power law with the exponent $\gamma_3 = 1/2$. In these log-log plots, power-law asymptotics are described by straight lines and their slopes are determined by the exponents. We see that the slope of the red-dotted line in upper-right figure is gentler than that in the lower-right figure, corresponding to the inequality $\gamma_2 = 1/2 < \gamma_3 = 1/2$.

2.3 Remarks with QCD

Quantum chromodynamics (QCD) is the theory of the strong interaction between quarks and gluons. In QCD, the following equation is known as the *Banks-Casher relation*

$$\Sigma := |\langle \bar{q}q \rangle| = \pi \bar{\rho}(0),$$

where Σ denotes the amplitude of *chiral condensation* and $\bar{\rho}$ does the spectrum of the QCD Dirac operator under the hydrodynamic and zero mass limit. The present *non-centered Wishart ensemble* can be provide an effective model for QCD, which is called the *massive chiral GUE* [12, 18, 10, 20]. There the spectrum of QCD Dirac operator in the vicinity of $x = 0$ will be described by the density ρ^{chiral} of the positive-signed and negative-signed singular values of random rectangular matrix K . For the transformation from ρ to ρ^{chiral} , see Eq.(3.34) in [10], for instance. The present three-parametric MP density $\rho(x; r, t, a)$ given by Theorem 2.1 provides the following hydrodynamical description of the time-depending spectrum for the QCD Dirac operator with parameters $r \in (0, 1]$ and $a \geq 0$,

$$\rho^{\text{chiral}}(x; r, t, a) = 2|x|\rho(x^2; r, t, a^2), \quad x \in \mathbb{R}, \quad t > 0, \quad (2.25)$$

under the initial state

$$\rho^{\text{chiral}}(x; r, 0, a) = \delta(x + a) + \delta(x - a), \quad x \in \mathbb{R}.$$

Figure 5 shows the time evolution of (2.25) in the critical case $r = 1$ with $a = 1$. By (2.25), Proposition 2.3 (ii) and (iii) give the following for $r = 1$,

$$\begin{aligned} \rho^{\text{chiral}}(x; 1, t_c(a), a) &\simeq \frac{\sqrt{3}}{\pi} a^{-4/3} |x|^{1/\delta} \quad \text{with } \delta = 3 \text{ as } |x| \rightarrow 0, \\ \rho^{\text{chiral}}(0; 1, t, a) &\simeq \frac{2}{\pi t_c(a^2)} (t - t_c(a^2))^{\beta_2} \quad \text{with } \beta_2 = \frac{1}{2} \text{ as } t \searrow t_c(a). \end{aligned} \quad (2.26)$$

Moreover, Proposition 2.4 implies through (2.25) that, when $r = 1$,

$$\begin{aligned} \lim_{t \rightarrow \infty} \rho^{\text{chiral}}(y; 1, t, a) dy \Big|_{y=\sqrt{tx}} &= \lim_{t \rightarrow \infty} \rho(tx^2; 1, t, a^2) d(tx^2) \\ &= \rho(x^2; 1) dx^2 = 2x\rho(x^2; 1) dx \\ &= \rho^{\text{Wigner}}(x) dx, \quad x \in \mathbb{R}, \end{aligned} \quad (2.27)$$

where

$$\rho^{\text{Wigner}}(x) = \frac{1}{\pi} \sqrt{4 - x^2} \mathbf{1}_{(-2,2)}(x) \quad (2.28)$$

is the density function describing *Wigner's semicircle law* (see, for instance, [8]). As mentioned in [18], the time evolution of ρ^{chiral} from the two-peak shape with zero density at the origin ($0 \leq t \leq t_c(a)$) to the universal shape ρ^{Wigner} after $t \sim 3t_c(a)$ via a critical shape at $t = t_c(a)$ can be interpreted as a transition from an initial state with restored chiral symmetry to a final state with *spontaneous chiral symmetry breaking*. In [12], we find the argument that the present system with the density $\rho^{\text{chiral}}(x; r, t, a)$ give a *mean-field model for QCD* and $\delta = 3$ and $\beta_2 = 1/2$ in (2.26)

are the mean-field values for the scaling exponents describing a condensation of light quarks to create massive constituents.

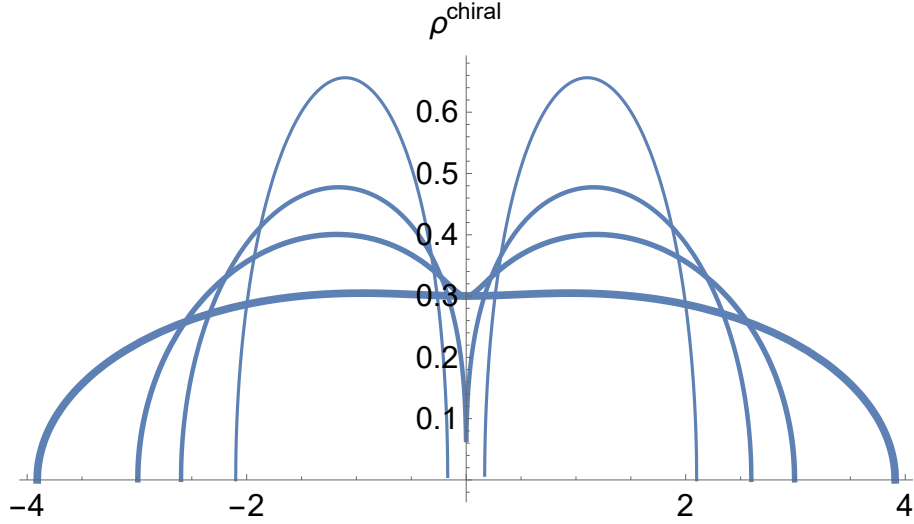


Figure 5: Time evolution of the hydrodynamical density $\rho^{\text{chiral}}(x; r, t, a)$ of the QCD Dirac operator in the critical case, which is obtained as (2.25) from the present three-parametric MP density with $r = 1$. For $a = 1$, ρ^{chiral} is plotted for $t = 0.5$ (the thinnest curve), 1.0, 1.5 and 3 (the thickest curve). When $0 \leq t \leq t_c = 1$, the density at the origin $\rho^{\text{chiral}}(0; r = 1, t, a = 1) = 0$, while it becomes positive for $t > t_c$. For $t > 3t_c$, ρ^{chiral} shows a relaxation in the sense of (2.27) to the universal density following Wigner's semicircle law (2.28).

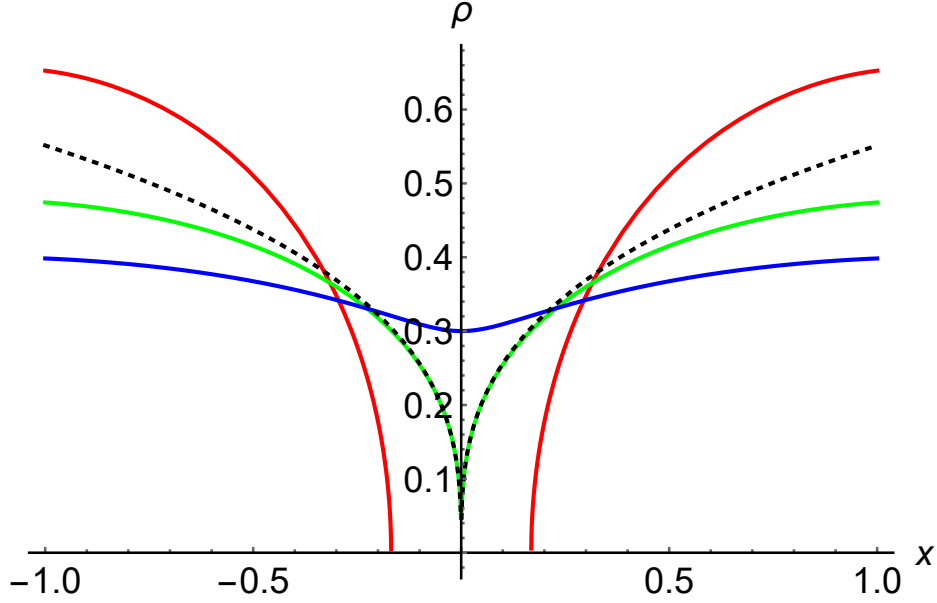


Figure 6: A closer figure in the vicinity of $x = 0$. The hydrodynamical density ρ^{chiral} is plotted for $t = 0.5$ (the red curve) $t = 1$ (the green curve) $t = 1.5$ (the blue curve) with parameters $r = 1, a = 1$. The black dotted line shows the critical curve $(3/\pi)|x|^{1/\delta}$ with the critical exponent $\delta = 3$, which is regarded as the mean-field value for the spontaneous chiral condensation in QCD.

3 Proof of Results

3.1 Solving the algebraic equations for density and its Hilbert transform

For the Green's function $G_{\delta_a}(z) = G_{\delta_a}(z; r, t)$, $a \geq 0$, we put

$$R = R(x) := \lim_{\varepsilon \rightarrow 0} \Re G_{\delta_a}(x + i\varepsilon),$$

$$I = I(x) := -\lim_{\varepsilon \rightarrow 0} \Im G_{\delta_a}(x + i\varepsilon),$$

that is, $\lim_{\varepsilon \rightarrow 0} G(x + i\varepsilon) = R(x) - iI(x)$. For the Green's function $G_{\delta_a}(z) = G_{\delta_a}(z; r, t)$, $a \geq 0$, we put

$$R = R(x) := \lim_{\varepsilon \rightarrow 0} \Re G_{\delta_a}(x + i\varepsilon),$$

$$I = I(x) := -\lim_{\varepsilon \rightarrow 0} \Im G_{\delta_a}(x + i\varepsilon),$$

that is, $\lim_{\varepsilon \rightarrow 0} G(x + i\varepsilon) = R(x) - iI(x)$. For the three-parametric MP density $\rho(x) := \rho(x; r, t, a)$, its *Hilbert transform* is defined by

$$\begin{aligned}\mathcal{H}[\rho](x) &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{\rho(y)}{x - y} dy \\ &:= \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left\{ \int_{-\infty}^{x-\varepsilon} \frac{\rho(y)}{x - y} dy + \int_{x+\varepsilon}^{\infty} \frac{\rho(y)}{x - y} dy \right\}.\end{aligned}$$

The Sokhotski-Plemelj theorem states that

$$\rho(x) = \frac{I(x)}{\pi}, \quad \mathcal{H}[\rho(\cdot)](x) = \frac{R(x)}{\pi}. \quad (3.29)$$

Let

$$A = \frac{1}{R^2 + I^2}, \quad B = \frac{1}{(1 - rtR)^2 + (rtI)^2}. \quad (3.30)$$

By definition, we obtain the equation,

$$A = \frac{(rt)^2}{2rtR - 1 + 1/B}. \quad (3.31)$$

For the equation (2.14), we obtain the following.

Lemma 3.1 *The equation (2.14) for the complex-valued function $G_{\delta_a}(z)$, $a \geq 0$ is equivalent with the following system of equations for the real-valued functions A and B ,*

$$\begin{aligned}x &= RA + (1 - rtR)tB + a [(1 - rtR)^2 - (rtI)^2] B^2, \\ 0 &= [A - rt^2B] - 2a(1 - rtR)rtB^2.\end{aligned} \quad (3.32)$$

Proof We put $z = x + i\varepsilon$, $x, \varepsilon \in \mathbb{R}$ in (2.14) and take the limit $\varepsilon \rightarrow 0$. Then the real part and the imaginary part of the obtained equation give (3.32). ■

Before solving the system of equations (3.32) for general $a \geq 0$, first we solve it for the special case $a = 0$. In this case (3.32) with (3.29) and (3.30) are simplified as

$$\begin{aligned}x &= \frac{R_0}{R_0^2 + (\pi\rho_0)^2} + \frac{(1 - rtR_0)t}{(1 - rtR_0)^2 + (rt)^2(\pi\rho_0)^2} \\ &= \left\{ \frac{1}{R_0^2 + (\pi\rho_0)^2} - \frac{rt^2}{(1 - rtR_0)^2 + (rt)^2(\pi\rho_0)^2} \right\} R_0 + \frac{t}{(1 - rtR_0)^2 + (rt)^2(\pi\rho_0)^2}, \\ 0 &= \frac{1}{R_0^2 + (\pi\rho_0)^2} - \frac{rt^2}{(1 - rtR_0)^2 + (rt)^2(\pi\rho_0)^2},\end{aligned}$$

for $R_0 := R(x; r, t, 0)$ and $\rho_0 := \rho(x; r, t, 0)$. They give two different expressions for x ,

$$x = \frac{1}{rt\{R_0^2 + (\pi\rho_0)^2\}}, \quad (3.33)$$

$$x = \frac{t}{(1 - rtR_0)^2 + (rt)^2(\pi\rho_0)^2}. \quad (3.34)$$

From (3.33), we have the relation

$$(\pi\rho_0)^2 = \frac{1}{rtx} - R_0^2. \quad (3.35)$$

Combining this with (3.34), we have $x = t/(1 - 2rtR_0 + rt/x)$, which is solved as

$$R_0 = \frac{x + (r-1)t}{2rtx}. \quad (3.36)$$

Put (3.36) into (3.35), we obtain

$$(\pi\rho_0)^2 = \frac{1}{(2rtx)^2} \{-x^2 + 2(r+1)tx - (r-1)^2t^2\} = \frac{(x - x_L)(x_R - x)}{(2rtx)^2}, \quad (3.37)$$

where $x_L = x_L(r, t)$ and $x_R = x_R(r, t)$ are given by (2.13) and the two-parametric MP density (2.12) is obtained as the positive square root of (3.37) for $x_L \leq x \leq x_R$.

The above calculation suggests that it will be easier to obtain R than I . By (3.29) and the first equation of (3.30), if

$$\frac{1}{A} - R^2 \geq 0,$$

then

$$\rho(x) = \frac{1}{\pi} \sqrt{\frac{1}{A} - R^2}. \quad (3.38)$$

Hence if we can express A , not using I , but using only R and parameters r, t, a , then the obtained R determines the density function ρ . Actually we will show that this strategy is successful in the following.

By eliminating A in (3.32), we obtain a quadratic equation for B as

$$2a(1 - rtR)B^2 + (t - a)B - x = 0. \quad (3.39)$$

We choose the following solution of (3.39),

$$B = \frac{a - t + \sqrt{D}}{4a(1 - rtR)}, \quad (3.40)$$

with

$$D = D(x; r, t, a) = 8a(1 - rtR)x + (t - a)^2, \quad (3.41)$$

by the following reason. If we put $a = 0$, (3.39) gives $B|_{a=0} = x/t$. On the other hand, (3.41) gives

$$\sqrt{D} = -(a - t) + 4a(1 - rtR_0)\frac{x}{t} + O(a^2),$$

and hence (3.40) has the correct limit in $a \rightarrow 0$; $\lim_{a \rightarrow 0} B = x/t$. If we put (3.40) into (3.31), then we have

$$A = \frac{(rt)^2 \{2(2rtR - 1)x - a + t - \sqrt{D}\}}{2\{(2rtR - 1)^2x + t(2rtR - 1) - a\}}. \quad (3.42)$$

The function A is indeed expressed by R and parameters r, t, a apart from I .

We find that the second equation of (3.32) gives

$$tB = \frac{A}{rt} - 2a(1 - rtR)B^2,$$

and if we use this equation, the quadratic equation (3.39) for B is written as

$$x = \frac{A}{rt} - aB. \quad (3.43)$$

Now we put the expression (3.40) for B and the expression (3.42) for A into (3.43). Then we obtain the following equation for R ,

$$\begin{aligned} & 8x(rtR - 1) \left\{ 4r^2t^2xR^2 + 2rt^2R - 4rtxR + (x - a - t) \right\} \\ & \times \left[8r^3t^3x^2R^3 - 8r^2t^2x\{2x + (r - 1)t\}R^2 \right. \\ & \quad + 2rt[5x^2 + \{(6r - 5)t - a\}x + (r - 1)^2t^2]R \\ & \quad \left. - [2x^2 + \{(4r - 3)t - 2a\}x + (r - 1)t\{(2r - 1)t - a\}] \right] = 0. \end{aligned} \quad (3.44)$$

We want to obtain $R = R(x; r, t, a)$ which solves (3.44) and satisfy the following *continuity condition with respect to a* ,

$$\lim_{a \rightarrow 0} R(x; r, t, a) = R_0 \quad (3.45)$$

with (3.36). This is given as a real solution of the cubic equation obtained from the last factor in (3.44),

$$\begin{aligned} & 8r^3t^3x^2R^3 - 8r^2t^2x\{2x + (r - 1)t\}R^2 \\ & \quad + 2rt[5x^2 + \{(6r - 5)t - a\}x + (r - 1)^2t^2]R \\ & \quad - [2x^2 + \{(4r - 3)t - 2a\}x + (r - 1)t\{(2r - 1)t - a\}] = 0. \end{aligned} \quad (3.46)$$

Applying the Cardano formula, we obtain the solution as

$$R(x; r, t, a) = \frac{2x + (r - 1)t}{3rtx} - \frac{x^2 + \{3a - (2r + 1)t\}x + (r - 1)^2t^2}{3 \times 2^{2/3}rtg^{1/3}x} - \frac{g^{1/3}}{6 \times 2^{1/3}rtx}, \quad (3.47)$$

where $g = g(x; r, t, a)$ is given by (2.18) with (2.15).

Lemma 3.2 *The solution (3.47) of (3.46) satisfies the continuity condition (3.45) with (3.36).*

Proof If we set $a = 0$, (2.18) and (2.15) become

$$g_0 := g(x; r, t, 0) = -2x^3 + 3(2r + 1)tx^2 - 3 \left[(r - 1)(2r + 1)t - \sqrt{-3S_0} \right] x + 2(r - 1)^3t^3,$$

with

$$S_0 := S(x; r, t, 0) = t^2 \{ x^2 - 2(r + 1)tx + (r - 1)^2t^2 \}. \quad (3.48)$$

In this case, the equality

$$g_0 = \frac{1}{4} \left\{ x - (r-1)t + \frac{1}{t} \sqrt{-3S_0} \right\}^3 \quad (3.49)$$

is established. By putting (3.49) with (3.48) into (3.47) with $a = 0$, we can verify (3.45) with (3.36). ■

We set

$$\varphi = \varphi(x; r, t, a) := 2x\{rtR(x; r, t, a) - 1\}. \quad (3.50)$$

Then it is easy to verify that the expression (3.47) for R is written as (2.19) for φ and that (3.42) gives

$$\frac{1}{A} - R^2 = \frac{1}{(2rtx)^2} \left[2(t - a + \sqrt{D})x - \varphi^2 \right],$$

with

$$D = -4a\varphi + (t - a)^2. \quad (3.51)$$

Therefore, if

$$2(t - a + \sqrt{D})x - \varphi^2 \geq 0, \quad (3.52)$$

then (3.38) gives

$$\rho(x) = \frac{\sqrt{2(t - a + \sqrt{D})x - \varphi^2}}{2\pi rtx}. \quad (3.53)$$

For the expression (2.20) for $\rho(x)$ given in Theorem 2.1, we perform the further calculation as follows. It is easy to verify the equality,

$$2(t - a + \sqrt{D}) = \left(\sqrt{d_-} + \sqrt{d_+} \right)^2,$$

where d_{\pm} are defined by (2.22). Therefore, we can see that

$$\begin{aligned} & 2(t - a + \sqrt{D})x - \varphi^2 \\ &= \left\{ \sqrt{x} \left(\sqrt{d_-} + \sqrt{d_+} \right) - \varphi \right\} \left\{ \sqrt{x} \left(\sqrt{d_-} + \sqrt{d_+} \right) + \varphi \right\} \\ &= \left\{ \left(\sqrt{x} + \frac{\sqrt{d_-} + \sqrt{d_+}}{2} \right)^2 - d_0 \right\} \left\{ d_0 - \left(\sqrt{x} - \frac{\sqrt{d_-} + \sqrt{d_+}}{2} \right)^2 \right\} \\ &= \left(\sqrt{x} + \frac{\sqrt{d_-} + \sqrt{d_+}}{2} + \sqrt{d_0} \right) \left(\sqrt{x} + \frac{\sqrt{d_-} + \sqrt{d_+}}{2} - \sqrt{d_0} \right) \\ &\times \left(\sqrt{d_0} + \sqrt{x} - \frac{\sqrt{d_-} + \sqrt{d_+}}{2} \right) \left(\sqrt{d_0} - \sqrt{x} + \frac{\sqrt{d_-} + \sqrt{d_+}}{2} \right) \\ &= (x - f_L)(f_R - x), \end{aligned}$$

where d_0 is given by (2.22), and $f_L = f_L(x; r, t, a)$ and $f_R = f_R(x; r, t, a)$ are given by (2.21). Hence (3.53) is written as (2.20), provided that the condition (3.52) is equivalent with the condition

$$x_L(r, t, a) \leq x \leq x_R(r, t, a), \quad (3.54)$$

where $x_L(r, t, a)$ and $x_R(r, t, a)$ are defined by (2.17).

3.2 Determining the support of density function

Assume that $t > 0, x > 0$. Then if and only if the condition (3.52) is satisfied, ρ given by (3.53) is positive or zero. And if and only if $\rho \geq 0$, its Hilbert transform R/π given by the second equation of (3.29) and φ defined by (3.50) are real valued. By the explicit expression (2.19) with (2.18) for φ , the following is obvious.

Lemma 3.3 *If and only if $S(x; r, t, a) \leq 0$, $\rho(x; r, t, a) \geq 0$.*

Now we prove the following.

Lemma 3.4 *If $S(x; r, t, a) = 0$, then $\rho(x; r, t, a) = 0$.*

Proof The formula (3.53) with (3.51) is written as

$$\rho(x) = \frac{\sqrt{\varphi F(\varphi)}}{2\pi r t x \sqrt{2(a-t+\sqrt{D})x + \varphi^2}}, \quad (3.55)$$

with

$$\begin{aligned} F(\varphi) &:= \frac{1}{\varphi} \{2(a-t+\sqrt{D})x + \varphi^2\} \{2(t-a+\sqrt{D})x - \varphi^2\} \\ &= -\varphi^3 + 4(t-a)x\varphi - 16ax^2. \end{aligned} \quad (3.56)$$

Hence we consider the condition of $F(\varphi) = 0$. For φ defined by (3.50), the cubic equation (3.46) is written as

$$\begin{aligned} H(\varphi) &:= \varphi^3 + 2\{x - (r-1)t\}\varphi^2 + [x^2 + \{(-2r+3)t - a\}x + (r-1)^2t^2]\varphi \\ &\quad + tx\{x + (r-1)(a-t)\} = 0. \end{aligned}$$

Therefore,

$$F(\varphi) = 0 \iff \tilde{F}(\varphi) := F(\varphi) + H(\varphi) = 0.$$

Note that the cubic terms of φ are canceled and $\tilde{F}(\varphi)$ is reduced to be quadratic in φ . We obtain

$$\begin{aligned} \tilde{F}(\varphi) &= 2\{x - (r-1)t\}\varphi^2 + [x^2 + \{(-2r+7)t - 5a\}x + (r-1)^2t^2]\varphi \\ &\quad + x[(t-16a)x + (r-1)t(a-t)] \\ &= 2\{x - (r-1)t\}(\varphi - \varphi_-)(\varphi - \varphi_+), \end{aligned}$$

where

$$\begin{aligned}\varphi_- &= \varphi_-(x; r, t, a) = -\frac{x^2 + \{(7-2r)t - 5a\}x + (r-1)^2t^2 + \sqrt{\Delta}}{4\{x - (r-1)t\}}, \\ \varphi_+ &= \varphi_+(x; r, t, a) = -\frac{x^2 + \{(7-2r)t - 5a\}x + (r-1)^2t^2 - \sqrt{\Delta}}{4\{x - (r-1)t\}},\end{aligned}\quad (3.57)$$

with

$$\begin{aligned}\Delta &= x^4 + 2\{-(2r-3)t + 59a\}x^3 + [\{2r(3r-8) + 35\}t^2 - 2a(58r-33)t + 25a^2]x^2 \\ &\quad - 2(r-1)^2\{(2r-3)t + a\}t^2x + (r-1)^4t^4.\end{aligned}$$

The condition $F(\varphi) = 0$ is thus written as $F(\varphi_+)F(\varphi_-) = 0$. On the other hand, by (3.56) and (3.57), we can show that

$$\begin{aligned}F(\varphi_+)F(\varphi_-) &= H(\varphi_+)H(\varphi_-) \\ &= -\frac{\{4x^3 - \{100a + (12r+13)t\}x^2 + (r-1)t\{(12r+13)t - 25a\}x - 4(r-1)^3t^3\}x^2}{8\{x - (r-1)t\}^3} \\ &\quad \times S(x; r, t, a).\end{aligned}$$

Note that $x - (r-1)t > 0$, if $x > 0$, since $r \in (0, 1], t \geq 0$. Hence the statement of Lemma is concluded. ■

Proof of Theorem 2.1 Assume that $t > 0$. By definition (2.17) of $x_L(r, t, a)$ and $x_R(r, t, a)$, and by Lemmas 3.3 and 3.4, we can conclude that the condition (3.52) is equivalent with the condition (3.54). Under the condition $2(t - a + \sqrt{D})x - \varphi^2 > 0$ (equivalently, $x_L(r, t, a) < x < x_R(r, t, a)$), if $\varphi F(\varphi) \neq 0$, then $2(a - t + \sqrt{D})x + \varphi^2 \neq 0$ by the equality given by the first line of (3.56). Hence $\rho(x)$ given by (3.55) is finite for $x > 0$. The proof of Theorem 2.1 is thus complete. ■

3.3 Proofs of critical phenomena

3.3.1 Proof of Proposition 2.2

The constant term in the cubic function $S(x; r, t, a)$ of x given by (2.15) becomes 0 for arbitrary $t > 0$ and $a \geq 0$, if and only if $r = 1$. This implies Proposition 2.2 (i).

When $r = 1$, the cubic equation (2.16) with (2.15) becomes

$$x\{4ax^2 - (8a^2 + 20at - t^2)x + 4(a-t)^3\} = 0.$$

For $t > 0, a > 0$, this equation has three real solutions; $x = 0$ and

$$\begin{aligned}x = x_{\pm} &:= \frac{1}{8a}\{8a^2 + 20at - t^2 \pm \sqrt{(8a^2 + 20at - t^2)^2 - 64a(a-t)^3}\} \\ &= \frac{1}{8a}\{8a^2 + 20at - t^2 \pm \sqrt{t}(8a+t)^{3/2}\}.\end{aligned}$$

When $0 < t < a$, $8a^2 + 20at - t^2 > 7a^2 + 20at > 0$,

$$\sqrt{t}(8a+t)^{3/2} = \sqrt{(8a^2 + 20at - t^2)^2 - 64a(a-t)^3} < 8a^2 + 20at - t^2,$$

and hence $0 < x_- < x_+$. On the other hand, when $t \geq a$,

$$\sqrt{t}(8a+t)^{3/2} = \sqrt{(8a^2 + 20at - t^2)^2 - 64a(a-t)^3} \geq |8a^2 + 20at - t^2|,$$

and hence $x_- < 0 < x_+$. Then we can conclude the following by the definitions of $x_L(r, t, a)$ and $x_R(r, t, a)$ given by (2.17).

Lemma 3.5 *Assume that $a > 0$.*

(i) *When $0 < t < a$,*

$$\begin{aligned} x_L(1, t, a) &= \frac{1}{8a} \{8a^2 + 20at - t^2 - \sqrt{t}(8a+t)^{3/2}\}, \\ x_R(1, t, a) &= \frac{1}{8a} \{8a^2 + 20at - t^2 + \sqrt{t}(8a+t)^{3/2}\}, \end{aligned} \quad (3.58)$$

and

$$0 < x_L(1, t, a) < x_R(1, t, a).$$

(ii) *When $t \geq a$,*

$$\begin{aligned} x_L(1, t, a) &= 0, \\ x_R(1, t, a) &= \frac{1}{8a} \{8a^2 + 20at - t^2 + \sqrt{t}(8a+t)^{3/2}\} > 0. \end{aligned}$$

Put $t = a - \varepsilon$, $0 < \varepsilon \ll 1$ in x_L given by (3.58). Then it is easy to verify that

$$x_L(1, t, a) = \frac{4}{27} \frac{\varepsilon^3}{a^2} + \frac{8}{81} \frac{\varepsilon^4}{a^3} + \frac{52}{729} \frac{\varepsilon^5}{a^4} + O(\varepsilon^6). \quad (3.59)$$

Hence Proposition 2.2 (ii) is proved.

3.3.2 Proof of Proposition 2.3

In the case with $r = 1$, we have the following expressions from (2.15), (2.18), (2.19), and (3.51),

$$\begin{aligned} S_1(x) &:= S(x; 1, t, a) = x[4ax^2 - (8a^2 + 20at - t^2)x + 4(a-t)^3], \\ g_1(x) &:= g(x; 1, t, a) = x \left[-2x^2 + 9(t+2a)x + 3\sqrt{-3S_1(x)} \right], \\ \varphi_1(x) &:= \varphi(x; 1, t, a) = -\frac{2}{3}x - \frac{2^{1/3}}{3} \frac{\{x + 3(a-t)\}x}{g_1(x)^{1/3}} - \frac{g_1(x)^{1/3}}{3 \times 2^{1/3}}, \\ D_1(x) &:= -4a\varphi_1(x) + (a-t)^2. \end{aligned} \quad (3.60)$$

Here we write $\rho_1(x) := \rho(x; 1, t, a)$.

First assume $0 < t < t_c(a) = a$. Put $x_L := x_L(1, t, a)$ and let $0 < \delta \ll 1$. Since $S_1(x_L) = 0$, we have the expansions in the form,

$$\begin{aligned} S_1(x_L + \delta) &= c_1\delta + c_2\delta^2 + O(\delta^3), \\ g_1(x_L + \delta) &= g_1(x_L) + c_3\delta^{1/2} + c_4\delta + O(\delta^{3/2}), \\ \varphi_1(x_L + \delta) &= \varphi_1(x_L) + c_5\delta + O(\delta^{3/2}), \end{aligned}$$

where $c_j, j = 1, \dots, 5$ are functions of t, a, x_L , but independent of δ . It should be noted that, in the expansion of $\varphi_1(x_L + \delta)$, the coefficient of term $\delta^{1/2}$ is proportional to

$$d := \{x_L + 3(a - t)\}x_L \left(\frac{2}{g_1(x_L)} \right)^{1/3} - \left(\frac{g_1(x_L)}{2} \right)^{1/3},$$

where $g_1(x_L) = \{-2x_L + 9(t + 2a)\}x_L^2$, and we can show that $d \propto S_1(x_L) = 0$. Then, if we note $\rho_1(x_L) = 0$, (3.53) gives

$$\rho_1(x_L + \delta) = \frac{\delta^{1/2}}{\sqrt{2\pi t x_L}} \sqrt{t - a + \sqrt{D_1(x_L)} - \left(\frac{2ax_L}{\sqrt{D_1(x_L)}} + \varphi_1(x_L) \right)} c_5 + O(\delta). \quad (3.61)$$

By (3.59) with $\varepsilon := a - t$, we see that

$$\begin{aligned} \varphi_1(x_L) &= -\frac{4}{9a}\varepsilon^2 + O(\varepsilon^3), \\ D_1(x_L) &= \frac{25}{9}\varepsilon^2 + O(\varepsilon^3), \\ c_5 &= -\frac{5a}{3}\varepsilon^{-1} + O(1). \end{aligned}$$

Hence by (3.61), Proposition 2.3 (i) is proved.

Next assume $t = t_c(a) = a$. Then (3.60) gives

$$\begin{aligned} S_1(x) &= a(4x - 27a)x^2 = -27a^2x^2 + O(x^3), \\ g_1(x) &= 2 \times 3^3ax^2 + O(x^3), \\ \varphi_1(x) &= -a^{1/3}x^{2/3} + O(x), \\ D_1(x) &= -4a\varphi_1(x) = 4a^{4/3}x^{2/3} + O(x). \end{aligned}$$

Then (3.61) proves Proposition 2.3 (ii).

Finally assume $t > t_c(a) = a$. Then (3.60) gives

$$\begin{aligned} S_1(x) &= -4|\varepsilon|^3x + O(x^2), \\ g_1(x) &= 6\sqrt{3}|\varepsilon|^{3/2}x^{3/2} + O(x^2), \\ \varphi_1(x) &= -\frac{t}{|\varepsilon|}x + O(x^{3/2}), \\ D_1(x) &= |\varepsilon|^2 + O(x), \end{aligned}$$

where $|\varepsilon| = -\varepsilon = t - a$. Then (3.61) proves Proposition 2.3 (iii). The proof of Proposition 2.3 is hence complete.

3.4 Long-Term Scaling and Proof of Proposition 2.4

It is obvious that the functions $S(x; r, t, a)$, $g(x; r, t, a)$, $\varphi(x; r, t, a)$, $f_L(x; r, t, a)$, and $f_R(x; r, t, a)$, which appeared in Theorem 2.1, are all homogeneous as multivariate functions of x, t, a for each fixed value of $r \in (0, 1]$. Actually, we see that, for an arbitrary parameter $\kappa > 0$,

$$\begin{aligned} S(\kappa x; r, \kappa t, \kappa a) &= \kappa^4 S(x; r, t, a), \\ g(\kappa x; r, \kappa t, \kappa a) &= \kappa^3 g(x; r, t, a), \quad \varphi(\kappa x; r, \kappa t, \kappa a) = \kappa \varphi(x; r, t, a), \\ f_L(\kappa x; r, \kappa t, \kappa a) &= \kappa f_L(x; r, t, a), \quad f_R(\kappa x; r, \kappa t, \kappa a) = \kappa f_R(x; r, t, a). \end{aligned}$$

Then the scaling property of the three-parametric MP density (2.24) is concluded. If we set $\kappa = 1/t$, replace x by $tx =: y$, and take the limit $t \rightarrow \infty$ for a fixed $a > 0$, Proposition 2.4 is proved.

4 Concluding Remarks and Future Perspectives

In the present paper, we have studied the time-dependent complex Wishart ensemble of random matrices with an external source. Following Blaizot, Nowak, and Warchol [3], we have considered the hydrodynamic limit of the process of the squared-singular-values of random complex rectangular matrices with a rectangularity $r \in (0, 1]$. We solved the algebraic equation (2.14) for the Green's function G_ξ , which is equivalent with the nonlinear PDE for G_ξ (2.8) under the initial distribution $\xi(dx) = \delta_a(dx)$, $a > 0$; a delta measure concentrated at $x = a > 0$. This algebraic equation (2.14) was given and its solution was studied by [3], but explicit expressions for the density function has not been available. In this paper we called the density function of this system the *three-parametric Marcenko–Pastur (MP) density*, $\rho(x; r, t, a)$, $r \in (0, 1]$, $t > 0$, $a \geq 0$, and gave useful expressions to $\rho(x; r, t, a)$ (Theorem 2.1). As an application of the result, the dynamic critical phenomena were clarified (Propositions 2.2 and 2.3), which are observed at the critical time $t_c(a) = a$, if and only if $r = 1$ and $a > 0$. There we have introduced six kinds of *critical exponents*,

$$\nu = 3, \quad \beta_1 = \beta_2 = \frac{1}{2}, \quad \gamma_1 = \frac{5}{2}, \quad \gamma_2 = \frac{1}{\delta} = \frac{1}{3}, \quad \gamma_3 = \frac{1}{2},$$

which represent the singularities of the dynamic critical phenomena.

The present results can be regarded as *macroscopic descriptions* of the system and the critical phenomena. *Microscopic descriptions* have been also studied in several papers for the similar systems and the associated dynamic critical phenomena. Connection between these two kinds of descriptions [3] and the universality of such dynamic critical phenomena will be studied in more detail in the future. As mentioned in Remark 3 given at the end of Section in the context of high energy physics, the present macroscopic description can be regarded as a mean-field approximation for more precise theory of QCD which exhibits spontaneous chiral symmetry breaking.

As emphasized in [2, 3], the MP density of the Wishart random-matrix ensemble has been used in a broad range of mathematical sciences, physics, and information theory (see the references in [2, 3]). It is expected that the non-centered Wishart ensembles/processes and the present three-parametric MP density will be also useful in many applications, where the mean zero condition cannot be assumed.

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