Noncolliding Brownian Motion and Determinantal Processes

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Abstract A system of one-dimensional Brownian motions (BMs) conditioned never to collide with each other is realized as (i) Dyson's BM model, which is a process of eigenvalues of hermitian matrix-valued diffusion process in the Gaussian unitary ensemble (GUE), and as (ii) the h-transform of absorbing BM in a Weyl chamber, where the harmonic function h is the product of differences of variables (the Vandermonde determinant). The Karlin–McGregor formula gives determinantal expression to the transition probability density of absorbing BM. We show from the Karlin–McGregor formula, if the initial state is in the eigenvalue distribution of GUE, the noncolliding BM is a determinantal process, in the sense that any multitime correlation function is given by a determinant specified by a matrix-kernel. By taking appropriate scaling limits, spatially homogeneous and inhomogeneous infinite determinantal processes are derived. We note that the determinantal processes related with noncolliding particle systems have a feature in common such that the matrix-kernels are expressed using spectral projections of appropriate effective Hamiltonians. On the common structure of matrix-kernels, continuity of processes in time is proved and general property of the determinantal processes is discussed.

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1 Introduction

In the present paper, we discuss noncolliding Brownian motion (BM) in one-dimension with finite number N of particles and its infinite particle limits $N \to \infty$. The condition imposed to N particles in the model, *not to collide with each other*, causes "entropy forces" between all pairs of particles, which are repulsive long-ranged interactions proportional to the inverse of distances between particles. When we draw sample paths of N particles of the system on the spatio-temporal plane, random sets of nonintersecting N paths are obtained. Viewing them as random patterns of polymers or phase boundaries on a plane, the present system has been used as a model of polymer networks [26, 33], or a model showing wetting (or melting) transitions [36] in statistical physics; see also [50, 92]. Recently many authors have reported that notion of noncolliding BM and its discrete counterpart called *vicious walk* [36] is very useful to analyze the polynuclear growth models [44, 46, 78, 82], time-dependent correlations of quantum spin chains [13], traffic problems [7], and the Chern–Simons theory [6, 27].

1.1 Observations of Three-Dimensional Bessel Process

In order to demonstrate the important connection between the random matrix theory [66] and the noncolliding BM here we show a couple of observations of the three-dimensional Bessel process. The noncolliding BM can be regarded as a multivariate generalization of the three-dimensional Bessel process given below.

Let $B_1(t)$, $B_2(t)$, $B_3(t)$ be one-dimensional standard BMs (see Sect. 2.1 for definition). They are assumed to be independent and we consider a 2×2 traceless hermitian matrix

$$M^{(1)}(t) = \begin{pmatrix} B_1(t) & B_2(t) + \mathrm{i} B_3(t) \\ B_2(t) - \mathrm{i} B_3(t) & -B_1(t) \end{pmatrix}, \tag{1.1}$$

where $\mathrm{i} = \sqrt{-1}$. Since the four entries $(M^{(1)}_{jk}(t))_{1 \leq j,k \leq 2}$ are BMs, $M^{(1)}(t), t \in [0, \infty)$ is regarded as a matrix-valued process, which describe a diffusion process in the space of 2×2 traceless hermitian matrices, which is identified with the three-dimensional real space $\mathbf{R}^3$ ($\mathbf{R}$ denotes the set of all real numbers). At each time $t \in [0, \infty)$, it will be diagonalized by an appropriate unitary matrix and the eigenvalue is given by $\pm X(t)$ with

$$X(t) = \sqrt{(B_1(t))^2 + (B_2(t))^2 + (B_3(t))^2}. \tag{1.2}$$

If we consider a Brownian particle in $\mathbf{R}^3$, $\boldsymbol{B}(t) = (B_1(t), B_2(t), B_3(t)), t \in [0, \infty)$, the distance of the particle from the origin (i.e., the radial coordinate of $\boldsymbol{B}(t)$) is given by (1.2) and thus it equals the eigenvalue process $X(t)$ associated with the matrix-valued process $M^{(1)}(t)$. This is called the three-dimensional Bessel process in probability theory (see, e.g. [18, 48]), and a simple application of the Itô formula gives its stochastic differential equation (SDE) as

$$dX(t) = d\widetilde{B}(t) + \frac{dt}{X(t)}, \quad t \in [0, \infty), \quad X(0) = x > 0, \tag{1.3}$$

where $\widetilde{B}(t)$ is another one-dimensional standard BM than the above $B_j(t)$, $j = 1, 2, 3$. Corresponding to the SDE (1.3), the backward Kolmogorov (Fokker–Planck) equation for the transition probability density $p_X(t, y|x)$, starting from $x > 0$ at time $t = 0$ and arriving at $y > 0$ at time $t > 0$, is given by

$$\frac{\partial}{\partial t} p_X(t, y|x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p_X(t, y|x) + \frac{1}{x} \frac{\partial}{\partial x} p_X(t, y|x), \tag{1.4}$$

and its solution with $\lim_{t \to 0} p_X(t, y|x) = \delta(x - y)$ is obtained as

$$p_X(t, y|x) = \frac{h(y)}{h(x)} p_{\text{abs}}(t, y|x) \tag{1.5}$$

with

$$p_{\text{abs}}(t, y|x) = p(t, y|x) - p(t, y|-x),$$

where $h(x) = x$ and $p(t, y|x)$ is the heat kernel given by (2.1) in Sect. 2.1. The reflection principle of BM can be used to prove that $p_{\text{abs}}(t, y|x)$ is the transition probability density from $x > 0$ to $y > 0$ during time t of the absorbing BM, in which an absorbing wall is set at the origin and any particle is absorbed if it arrives at the wall. It is a matter of course that $h(x) = x$ is harmonic, $\partial^2 h(x)/\partial x^2 = 0$, but we dare to say that p_X is the harmonic transform (h-transform) of p_{abs} looking at (1.5). We introduce another 2×2 matrix

$$M^{(2)}(t, \boldsymbol{y}|\boldsymbol{x}) = \begin{pmatrix} p(t, y_1|x_1) & p(t, y_1|x_2) \\ p(t, y_2|x_1) & p(t, y_2|x_2) \end{pmatrix} \tag{1.6}$$

for $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2) \in \mathbf{R}^2$, and consider its determinant

$$f_2(t, \boldsymbol{y}|\boldsymbol{x}) = \det[M^{(2)}(t, \boldsymbol{y}|\boldsymbol{x})]. \tag{1.7}$$

Then it is easy to see that

$$p_{\text{abs}}(t, y|x) = \sqrt{\frac{\pi t}{2}} f_2(t/2, \{-y/2, y/2\}|\{-x/2, x/2\}), \quad x, y > 0. \tag{1.8}$$

In summary the three-dimensional Bessel process $X(t), t \in [0, \infty)$ has two different realizations; (i) the eigenvalue-process of 2×2 hermitian-matrix valued process (1.1), and (ii) the h-transform of the absorbing BM with a wall at $x = 0$. We also observed that the transition probability density of the absorbing BM has a determinantal expression of a 2×2 matrix (1.6–1.8). If we consider the two-dimensional BM, it is represented by motion of a point in the two-dimensional space $(x_1, x_2) \in \mathbf{R}^2$. We put an absorbing boundary on a line $x_2 = x_1$ and trace the motion of the point in the region $\mathbf{W}_2 = \{\boldsymbol{x} = (x_1, x_2) \in \mathbf{R}^2 : x_1 < x_2\}$. The transition probability density from $\boldsymbol{x} = (x_1, x_2) \in \mathbf{W}_2$ to $\boldsymbol{y} = (y_1, y_2) \in \mathbf{W}_2$ is generally given by the determinant (1.7). As a special case of it (with a time-change $t \to t/2$), (1.8) is given.

1.2 Dyson's BM Model, Karlin–McGregor Formula and Noncolliding BM

The noncolliding BM, $\boldsymbol{X}(t) = (X_1(t), X_2(t), \ldots, X_N(t)), t \in [0, \infty)$ is a conditional diffusion process. It has the following two kinds of realizations.

(i) In order to generate the random matrix ensembles, Dyson introduced $N \times N$ matrix-valued diffusion processes [30]. For the Gaussian unitary ensemble (GUE), N^2 independent one-dimensional BMs are used to assign entries of matrix to satisfy the condition that the matrix is hermitian at any time. Eigenvalues are real and define an N-particle system in one dimension called Dyson's BM model (with the parameter $\beta = 2$ corresponding to GUE). This process solves the SDE (see [55] for the proof using generalized Bru's theorem)

$$dX_j(t) = dB_j(t) + \sum_{1 \le k \le N: k \ne j} \frac{dt}{X_j(t) - X_k(t)}, \quad 1 \le j \le N, \quad t \in [0, \infty), \tag{1.9}$$

where $\boldsymbol{B}(t) = (B_1(t), \ldots, B_N(t))$ is an N-dimensional BM; $B_j(t)$, $1 \leq j \leq N$, are independent one-dimensional standard BMs. The SDE (1.9) is an N-variable generalization of the SDE for the three-dimensional Bessel process (1.3). It was proved that with probability one Dyson's BM is non-colliding [81].

(ii) We consider the following subset of $\mathbf{R}^N$,

$$\mathbf{W}_N = \{\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbf{R}^N : x_1 < x_2 < \cdots < x_N\}. \tag{1.10}$$

It is called the Weyl chamber of type A_{N-1} [41, 55]. The absorbing BM is defined by putting absorbing walls at all boundaries of the region $\mathbf{W}_N$, whose transition probability density, when the process starts from $\boldsymbol{x} \in \mathbf{W}_N$ at time $t = 0$ and arrives at $\boldsymbol{y} \in \mathbf{W}_N$ at time $t > 0$, is given by

$$f_N(t, \boldsymbol{y}|\boldsymbol{x}) = \det_{1 \leq j,k \leq N} [p(t, y_j|x_k)], \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbf{W}_N. \tag{1.11}$$

This determinantal expression is known as the Karlin–McGregor formula [49]. (Such a determinantal formula for nonintersecting paths is known as the Lindströn–Gessel–Viennot formula [40, 62] in the enumerative combinatorics, see also [26, 32, 33, 36, 45, 53, 56, 60, 61, 73, 90].) The noncolliding BM is given by the h-transform of the absorbing BM [41]

$$p_N(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})} f_N(t, \boldsymbol{y}|\boldsymbol{x}), \tag{1.12}$$

where

$$h_N(\boldsymbol{x}) = \prod_{1 \leq j < k \leq N} (x_k - x_j) = \det_{1 \leq j,k \leq N} [x_j^{k-1}] \tag{1.13}$$

is the Vandermonde determinant [39, 64, 89].

We note that there appear three different kinds of matrices. The matrices representing Dyson's matrix-valued process ((1.1) for the simplest case), matrices in the Karlin–McGregor determinants ((1.7) and (1.11)) and that in the Vandermonde determinant (1.13). Of course, the equivalence of Dyson's BM model (the eigenvalue process of the first kind of matrices) with the noncolliding BM implies direct connection between the random matrix theory and stochastic processes. In the present paper, however, we will show that the Karlin–McGregor formula is much more important. In Sect. 3 we will show that the Vandermonde determinant appears in the Schur function expansion of the Karlin–McGregor determinant. With the combination of these two determinants, the orthogonal polynomial method is applicable to study the processes. This method has been developed to analyze multi-matrix models in the random matrix theory [66].

1.3 Matrix-Kernels and Determinantal Processes

In Sect. 2, we will explain that the Hermite orthonormal functions are useful to represent BMs. Precise descriptions of facts briefly mentioned above will be given in Sect. 3. Staring from Karlin–McGregor's determinantal expression of transition probability density, we will prove in Sect. 4 that, if we specify the initial configuration as the GUE-eigenvalue distribution, the generating function of multitime correlation functions is given by a Fredholm determinant for the noncolliding BM, and thus multitime correlation functions are generally given by determinants (Theorem 4.1). The system whose spatial correlations are given by

determinants is usually called a determinantal point field (or a determinantal point process) in probability theory [84, 85]. Theorem 4.1 states that the noncolliding BM is not only a determinantal point field at any fixed time $0 \leq t < \infty$, but it is also a determinantal point field on the spatio-temporal plane. We say that the noncolliding BM is a (finite) *determinantal process* to express this situation.

Determinantal processes are generally determined by their matrix-kernels (see, for example, [96]). The matrix-kernel of our finite noncolliding BM is expressed by the Hermite orthonormal functions $\{\varphi_k\}_{k=0}^{\infty}$, which is called the extended Hermite kernel in [96]. In Sect. 5 we will show that the asymptotic properties of $\{\varphi_k\}_{k=0}^{\infty}$ completely determine infinite particle limits of the matrix-kernel and then the determinantal process. Appropriate scaling limits are performed and two infinite particle systems are derived from the noncolliding BM. One of them is the spatially homogeneous infinite determinantal process with matrix-kernel expressed by trigonometric functions (Theorem 5.1) and another is the spatially inhomogeneous one with matrix-kernel expressed by Airy functions (Theorem 5.2). The former kernel is called the extended sine kernel and the latter the extended Airy kernel in [96]. We will claim in Sect. 6 that these three determinantal processes (one finite and two infinite systems) and others reported in references [46, 51, 57, 78, 95, 96] have a common structure; the matrix-kernels are expressed by spectral projections associated with appropriate self-adjoint operators (effective Hamiltonians) [78]. As explained by Spohn [88] and by Prähofer and Spohn [78], this common feature is shared also with the $1 + 1$ dimensional Fermi field in quantum mechanics (see also [35, 43]). It may be due to the similarity between the Karlin–McGregor formula for noncolliding systems and the Slater determinant for free fermion systems with the Fermi exclusion principle [11]. For finite and infinite determinantal processes with matrix-kernels associated with spectral projections, we will prove that the determinantal processes are continuous in time (Lemma 7.1) and discuss the bilinear forms derived from correlation functions (Proposition 7.2). Future problems are given in Sect. 8.

2 Brownian Motion and Hermite Polynomials

2.1 Diffusion Equation and Hamiltonian of Harmonic Oscillator

Let $(\Omega, \mathcal{F}, P)$ be the probability space. One-dimensional standard BM starting from a point $x_0 \in \mathbf{R}$ is defined as a real-valued stochastic process $\{B(t, \omega) : t \in [0, \infty)\}$, which satisfies the following conditions (see, for example, [42, 48]). Here ω is a label on sample path, $\omega \in \Omega$.

1. $B(0, \omega) = x_0$ with probability 1.
2. For any fixed $\omega \in \Omega$, $B(t)$ is a continuous real-function of t with probability 1. (With this property we say that the paths are continuous in time.)
3. For any series of times $t_0 \equiv 0 < t_1 < \cdots < t_M$, $M = 1, 2, \ldots$, $\{B(t_{m+1}) - B(t_m)\}_{m=0,1,\ldots,M-1}$ are independent and they are normally distributed with mean 0 and variance $t_{m+1} - t_m$.

Then if we introduce an integral kernel (Gaussian kernel)

$$p(t, x|x') = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(x - x')^2}{2t}\right\}, \quad t > 0, x, x' \in \mathbf{R}, \tag{2.1}$$

the probability that the BM stays in an interval $[a_m, b_m]$, $-\infty < a_m < b_m < \infty$, at each time t_m, $m = 1, 2, \ldots, M$, is given by

$$P(B(t_m) \in [a_m, b_m], m = 1, 2, \ldots, M)$$

$$= \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \cdots \int_{a_M}^{b_M} dx_M \prod_{m=0}^{M-1} p(t_{m+1} - t_m, x_{m+1}|x_m).$$

That is, the transition probability density of the BM is given by (2.1). Since it satisfies the one-dimensional diffusion equation (heat equation)

$$\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x) \tag{2.2}$$

with the initial condition $u(0, x) = \delta(x - x')$, it is specially called the heat kernel.

We consider the following transformation of variables, $(t, x) \mapsto (\tau, \zeta)$;

$$\tau = \tau(t) = \log t, \qquad \zeta = \zeta(t, x) = \frac{x}{\sqrt{2t}} \tag{2.3}$$

and set

$$u(t, x) = e^{-(\tau + \zeta^2)/2} U(\tau, \zeta). \tag{2.4}$$

Then the diffusion equation (2.2) is transformed to

$$\frac{\partial}{\partial \tau} U(\tau, \zeta) = -\frac{1}{2} \left(\mathcal{H}_{\mathrm{H}} - \frac{1}{2} \right) U(\tau, \zeta), \tag{2.5}$$

where

$$\mathcal{H}_{\mathrm{H}} = -\frac{1}{2} \frac{\partial^2}{\partial \zeta^2} + \frac{1}{2} \zeta^2. \tag{2.6}$$

Note that $\mathcal{H}_{\mathrm{H}}$ is identified with the Hamiltonian in the coordinate(ζ)-representation of the one-dimensional harmonic oscillator in quantum mechanics, if we set the mass of the oscillator $m = 1$, the circular frequency $\omega = 1$, and $\hbar = 1$.

Following Dirac's description [11, 28], we consider the real-valued Hilbert space with basis $\{|\zeta\rangle : \zeta \in \mathbf{R}\}$, which is orthonormal $\langle \zeta | \zeta' \rangle = \delta(\zeta - \zeta')$ and complete

$$\int_{\mathbf{R}} d\zeta \, |\zeta\rangle\langle\zeta| = 1. \tag{2.7}$$

Let $\widehat{\mathcal{H}}_{\mathrm{H}}$ be the operator such that $\langle \zeta' | \widehat{\mathcal{H}}_{\mathrm{H}} | \zeta \rangle = \delta(\zeta' - \zeta) \mathcal{H}_{\mathrm{H}}$ with (2.6). We consider a state vector $|\Psi(\tau)\rangle$, which follows the equation

$$\frac{\partial}{\partial \tau} |\Psi(\tau)\rangle = -\frac{1}{2} \left(\widehat{\mathcal{H}}_{\mathrm{H}} - \frac{1}{2} \right) |\Psi(\tau)\rangle. \tag{2.8}$$

If we multiply $\langle \zeta |$ to (2.8) from the left, we will have the (2.5) with $U(\tau, \zeta) = \langle \zeta | \Psi(\tau) \rangle$. Given $|\Psi(\tau')\rangle$, $\tau' \in (-\infty, \infty)$, the solution of (2.8) for $\tau \geq \tau'$ is obtained as

$$|\Psi(\tau)\rangle = \exp\left\{ -\frac{1}{2} \left(\widehat{\mathcal{H}}_{\mathrm{H}} - \frac{1}{2} \right)(\tau - \tau') \right\} |\Psi(\tau')\rangle.$$

Then

$$U(\tau, \zeta) = \langle \zeta | \exp\left\{ -\frac{1}{2} \left(\widehat{\mathcal{H}}_{\mathrm{H}} - \frac{1}{2} \right)(\tau - \tau') \right\} |\Psi(\tau')\rangle$$

$$= \int_{-\infty}^{\infty} d\zeta' \, \langle \zeta | \exp\left\{ -\frac{1}{2} \left(\widehat{\mathcal{H}}_{\mathrm{H}} - \frac{1}{2} \right)(\tau - \tau') \right\} |\zeta'\rangle U(\tau', \zeta'),$$

where (2.7) was used. Insert this result into (2.4), we have

$$u(t, x) = \int_{-\infty}^{\infty} d\zeta' \, e^{-(\tau+\zeta^2)/2+((\tau')+(\zeta')^2)/2} \langle\zeta| \exp\left\{-\frac{1}{2}\left(\widehat{\mathcal{H}}_H - \frac{1}{2}\right)(\tau - \tau')\right\}|\zeta'\rangle u(t', x')$$

$$= \int_{-\infty}^{\infty} dx' \, p(t - t', x|x') u(t', x'),$$

where

$$p(t - t', x|x') = \frac{1}{\sqrt{2}} e^{-\tau/2 - \zeta^2/2 + (\zeta')^2/2} \langle\zeta| \exp\left\{-\frac{1}{2}\left(\widehat{\mathcal{H}}_H - \frac{1}{2}\right)(\tau - \tau')\right\}|\zeta'\rangle$$

$$= \frac{1}{\sqrt{2t}} e^{-x^2/4t + (x')^2/4t'} \left\langle\frac{x}{\sqrt{2t}}\middle| \exp\left\{-\frac{1}{2}\left(\widehat{\mathcal{H}}_H - \frac{1}{2}\right)(\tau - \tau')\right\}\middle|\frac{x'}{\sqrt{2t'}}\right\rangle. \quad (2.9)$$

This is the transition probability density previously given as (2.1).

2.2 Hermite Polynomials and Equalities

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. The eigenvalues of $\widehat{\mathcal{H}}_H$ are $n + 1/2$ with $n \in \mathbf{N}_0$; $\widehat{\mathcal{H}}_H|n\rangle = (n + 1/2)|n\rangle$. Here $\{|n\rangle : n \in \mathbf{N}_0\}$ denotes the set of eigenvectors of $\widehat{\mathcal{H}}_H$, which is orthonormal $\langle n|n'\rangle = \delta_{n,n'}$ and complete $\sum_{n=0}^{\infty} |n\rangle\langle n| = 1$. Let $\{H_n(x) : n \in \mathbf{N}_0\}$ be the Hermite polynomials

$$H_n(x) = e^{x^2}\left(-\frac{d}{dx}\right)^n e^{-x^2}$$

$$= n! \sum_{k=1}^{[n/2]} (-1)^n \frac{(2x)^{n-2k}}{k!(n - 2k)!}, \quad (2.10)$$

where $[z]$ denotes the largest number not greater than z. They are orthogonal

$$\int_{\mathbf{R}} dx \, e^{-x^2} H_n(x) H_m(x) = h_n \delta_{n,m}, \quad n, m \in \mathbf{N}_0$$

with $h_n = \sqrt{\pi} 2^n n!$. We set

$$\varphi_n(\zeta) = \frac{1}{\sqrt{h_n}} e^{-\zeta^2/2} H_n(\zeta). \quad (2.11)$$

Then $\{\varphi_n(\zeta) : n \in \mathbf{N}_0\}$ are orthonormal and

$$\langle\zeta|n\rangle = \langle n|\zeta\rangle = \varphi_n(\zeta)$$

is established.

Now we define

$$\widehat{\mathcal{H}}_\varphi = \frac{1}{2}\left(\widehat{\mathcal{H}}_H - \frac{1}{2}\right). \quad (2.12)$$

Then

$$\widehat{\mathcal{H}}_\varphi|n\rangle = \frac{n}{2}|n\rangle, \quad n \in \mathbf{N}_0 \quad (2.13)$$

and (2.9) gives

$$p(t - t', x|x') = \frac{1}{\sqrt{2}} e^{-\tau/2 - \zeta^2/2 + (\zeta')^2/2} \sum_{n=0}^{\infty} \langle \zeta | e^{-\widehat{\mathcal{H}}_\varphi \tau} | n \rangle \langle n | e^{\widehat{\mathcal{H}}_\varphi \tau'} | \zeta' \rangle$$

$$= \frac{1}{\sqrt{2}} e^{-\tau/2 - \zeta^2/2 + (\zeta')^2/2} \sum_{n=0}^{\infty} \langle \zeta | e^{-n\tau/2} | n \rangle \langle n | e^{n\tau'/2} | \zeta' \rangle$$

$$= \frac{1}{\sqrt{2}} e^{-\tau/2 - \zeta^2/2 + (\zeta')^2/2} \sum_{n=0}^{\infty} e^{-n(\tau - \tau')/2} \varphi_n(\zeta) \varphi_n(\zeta'),$$

that is

$$p(t - t', x|x') = \frac{1}{\sqrt{2t}} e^{-x^2/4t + (x')^2/4t'} \sum_{n=0}^{\infty} \left(\frac{t'}{t}\right)^{n/2} \varphi_n\left(\frac{x}{\sqrt{2t}}\right) \varphi_n\left(\frac{x'}{\sqrt{2t'}}\right). \qquad (2.14)$$

This expression for the heat kernel (2.1) is called Mehler's formula [10, 91].

We introduce the vectors

$$|t, x\rangle = e^{\zeta^2/2} e^{\widehat{\mathcal{H}}_\varphi \tau} |\zeta\rangle,$$

$$\langle t, x| = \frac{1}{\sqrt{2}} e^{-(\tau + \zeta^2)/2} \langle \zeta | e^{-\widehat{\mathcal{H}}_\varphi \tau}. \qquad (2.15)$$

It is easy to see that orthonormality and completeness of $\{|\zeta\rangle : \zeta \in \mathbf{R}\}$ imply $\langle t, x|t, x'\rangle = \delta(x - x')$ and

$$\int_{\mathbf{R}} dx |t, x\rangle \langle t, x| = 1. \qquad (2.16)$$

By (2.16), we have the equalities for $0 \leq t_1 \leq t_2 \leq t_3$

$$\int_{\mathbf{R}} dx_2 \langle t_3, x_3|t_2, x_2\rangle \langle t_2, x_2|t_1, x_1\rangle = \langle t_3, x_3|t_1, x_1\rangle, \quad x_1, x_3 \in \mathbf{R},$$

$$\int_{\mathbf{R}} dx_1 \langle t_2, x_2|t_1, x_1\rangle \langle t_1, x_1|n\rangle = \langle t_2, x_2|n\rangle, \quad n \in \mathbf{N}_0, x_2 \in \mathbf{R},$$

$$\int_{\mathbf{R}} dx_1 \int_{\mathbf{R}} dx_2 \langle n|t_2, x_2\rangle \langle t_2, x_2|t_1, x_1\rangle \langle t_1, x_1|n'\rangle = \langle n|n'\rangle = \delta_{n,n'}, \quad n, n' \in \mathbf{N}_0.$$

Though these seem to be trivial, if we note that (2.9) is rewritten as

$$p(t - t', x|x') = \langle t, x|t', x'\rangle, \quad 0 \leq t' \leq t, x, x' \in \mathbf{R}, \qquad (2.17)$$

they become meaningful; for $0 \leq t_1 \leq t_2 \leq t_3$

$$\int_{-\infty}^{\infty} dx_2\, p(t_3 - t_2, x_3|x_2) p(t_2 - t_1, x_2|x_1) = p(t_3 - t_1, x_3|x_1), \quad x_1, x_3 \in \mathbf{R}, \quad (2.18)$$

$$\int_{-\infty}^{\infty} dx_1\, p(t_2 - t_1, x_2|x_1) \phi_n(t_1, x_1) = \phi_n(t_2, x_2), \quad n \in \mathbf{N}_0, x_2 \in \mathbf{R}, \qquad (2.19)$$

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, \widehat{\phi}_n(t_2, x_2) p(t_2 - t_1, x_2|x_1) \phi_{n'}(t_1, x_1) = \delta_{n,n'}, \quad n, n' \in \mathbf{N}_0, \quad (2.20)$$

where

$$\phi_n(t, x) \equiv \langle t, x | n \rangle = \frac{1}{\sqrt{2}} t^{-(n+1)/2} e^{-x^2/4t} \varphi_n \left(\frac{x}{\sqrt{2t}} \right), \tag{2.21}$$

$$\widehat{\phi}_n(t, x) \equiv \langle n | t, x \rangle = t^{n/2} e^{x^2/4t} \varphi_n \left(\frac{x}{\sqrt{2t}} \right), \quad n \in \mathbf{N}_0. \tag{2.22}$$

Equation (2.18) is called the Chapman–Kolmogorov equation. The equalities (2.19) mean invariance of the functions $\phi_n, n \in \mathbf{N}_0$, with respect to the heat kernel.

3 Noncolliding Systems

3.1 Application of Karlin–McGregor Formula

Here we introduce the noncolliding BM in a finite time-period $(0, T)$ with $T > 0$. It is defined as an N-particle system of one-dimensional standard BMs conditioned not to collide with each other in $(0, T)$. As mentioned in Sect. 1.2, the transition probability density of the absorbing BM in the Weyl chamber $\mathbf{W}_N$ of type A_{N-1} is given by (1.11) as an application of Karlin–McGregor formula [49]. The probability that $\mathbf{B}(t)$ starting from $\mathbf{x}' \in \mathbf{W}_N$ stays in $\mathbf{W}_N$ up to at least time $t > 0$ is given by

$$\mathcal{N}_N(t, \mathbf{x}') = \int_{\mathbf{W}_N} f_N(t, \mathbf{x} | \mathbf{x}') d\mathbf{x}, \quad \mathbf{x}' \in \mathbf{W}_N. \tag{3.1}$$

The transition probability density of noncolliding BM is then given by

$$g_{N,T}(t', \mathbf{x}'; t, \mathbf{x}) = \frac{\mathcal{N}_N(T - t, \mathbf{x})}{\mathcal{N}_N(T - t', \mathbf{x}')} f_N(t - t', \mathbf{x} | \mathbf{x}') \tag{3.2}$$

for $0 < t' \le t < T$, $\mathbf{x}, \mathbf{x}' \in \mathbf{W}_N$ [52–54]. It should be noted that this process is in general temporally inhomogeneous. In the following, we will consider the $T \to \infty$ limit to make the process be homogeneous in time.

3.2 Schur Function Expansion

By multilinearity of determinant (1.11) with (2.1),

$$f_N(t, \mathbf{x} | \mathbf{x}') = \left(\frac{1}{2\pi t} \right)^{N/2} e^{-(|\mathbf{x}|^2 + |\mathbf{x}'|^2)/2t} \det_{1 \le j, k \le N} [e^{x_j x_k'/t}],$$

where $|\mathbf{x}|^2 \equiv \sum_{j=1}^N x_j^2$. Consider $F_N(\mathbf{x}, \mathbf{y}) = \det_{1 \le j, k \le N} [e^{x_j y_k}]$ for a pair of multivariates $\mathbf{x} = \{x_j\}_{j=1}^N$ and $\mathbf{y} = \{y_j\}_{j=1}^N$. By definition of determinant, $F(\mathbf{x}, \mathbf{y})$ is skew-symmetric under any exchange of indices of $\{x_j\}$ and also it is for $\{y_j\}$; Let $\mathcal{S}_N$ be the symmetric group of N variables (the set of all permutations of N variables) and for $\sigma \in \mathcal{S}_N$ write $\sigma(\mathbf{x}) = (x_{\sigma(1)}, \ldots, x_{\sigma(N)})$. Then for any $\sigma \in \mathcal{S}_N$, $F_N(\sigma(\mathbf{x}), \mathbf{y}) = F_N(\mathbf{x}, \sigma(\mathbf{y})) = \text{sgn}(\sigma) F_N(\mathbf{x}, \mathbf{y})$. A fundamental skew-symmetric polynomials of multivariate $\mathbf{x}$ is given by a product of differences (the Vandermonde determinant) (1.13). The quotient of $F_N(\mathbf{x}, \mathbf{y})$ divided by $h_N(\mathbf{x}) h_N(\mathbf{y})$ is a symmetric function both of $\mathbf{x}$ and $\mathbf{y}$. The following lemma shows an expansion of the symmetric part using the Schur functions, which are labeled by partitions

$\mu = (\mu_1, \mu_2, \ldots)$, sets of nonnegative integers in decreasing order $\mu_1 \geq \mu_2 \geq \cdots$, and defined by

$$s_\mu(\boldsymbol{x}) = \frac{\det_{1 \leq j,k \leq N}[x_j^{\mu_k + N - k}]}{\det_{1 \leq j,k \leq N}[x_j^{N-k}]}. \tag{3.3}$$

The non-zero μ_j's in a partition μ are called parts of μ and the number of parts is called length of μ and denoted by $\ell(\mu)$ [39, 64, 89]. The Schur function expansion is a special case of character expansions (see [8, 9, 43, 55, 59]). Let $\Gamma(a) = \int_0^\infty e^{-y} y^{a-1} dy$ for $a > 0$ (the Gamma function) and note $\Gamma(a + 1) = a!$ if $a \in \mathbf{N}_0$.

Lemma 3.1

$$\det_{1 \leq j,k \leq N}[e^{x_j y_k}] = h_N(\boldsymbol{x}) h_N(\boldsymbol{y}) \sum_{\mu : \ell(\mu) \leq N} \frac{s_\mu(\boldsymbol{x}) s_\mu(\boldsymbol{y})}{\prod_{k=1}^N \Gamma(\mu_k + N - k + 1)}$$

$$= \frac{h_N(\boldsymbol{x}) h_N(\boldsymbol{y})}{\prod_{k=1}^N \Gamma(k)} \times \{1 + \mathcal{O}(|\boldsymbol{x}|)\} \quad in \ |\boldsymbol{x}| \to 0. \tag{3.4}$$

Proof By multilinearity of determinant, we have

$$\det_{1 \leq j,k \leq N}[e^{x_j y_k}] = \det_{1 \leq j,k \leq N}\left[\sum_{n=0}^\infty \frac{(x_i y_k)^n}{\Gamma(n+1)}\right]$$

$$= \sum_{\boldsymbol{n} = (n_1, n_2, \ldots, n_N) \in \mathbf{N}_0^N} \prod_{m=1}^N \frac{1}{\Gamma(n_m + 1)} \det_{1 \leq j,k \leq N}[(x_j y_k)^{n_j}]. \tag{3.5}$$

We can see that for some symmetric function $f(\boldsymbol{n})$ of $\boldsymbol{n} = (n_1, \ldots, n_N) \in \mathbf{N}_0^N$,

$$\sum_{\boldsymbol{n} \in \mathbf{N}_0^N} f(\boldsymbol{n}) \det_{1 \leq j,k \leq N}[(x_j y_k)^{n_j}] = \sum_{\boldsymbol{n} \in \mathbf{N}_0^N} f(\boldsymbol{n}) \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \det_{1 \leq j,k \leq N}[(x_j y_k)^{n_{\sigma(j)}}],$$

and

$$\sum_{\sigma \in \mathcal{S}_N} \det_{1 \leq j,k \leq N}[(x_j y_k)^{n_{\sigma(j)}}] = \det_{1 \leq j,k \leq N}[x_j^{n_k}] \det_{1 \leq \ell,m \leq N}[y_\ell^{n_m}].$$

Since $\det_{1 \leq j,k \leq N}[x_j^{n_k}] = 0$ if $n_{k_1} = n_{k_2}$ for any pair $1 \leq k_1 \neq k_2 \leq N$, (3.5) equals

$$\sum_{0 \leq n_1 < n_2 < \cdots < n_N} \frac{\det_{1 \leq j,k \leq N}(x_j^{n_k}) \det_{1 \leq \ell,m \leq N}(y_\ell^{n_m})}{\prod_{j=1}^N \Gamma(n_j + 1)}.$$

Here we change the variables in summation from $\{n_j\}$ to $\{\mu_j\}$ by $\mu_j = n_j - N + j, 1 \leq j \leq N$. Using (3.3) we obtain the first equation of (3.4). Since $s_\mu(\boldsymbol{0}) = 0$ unless $\mu = \boldsymbol{0} \equiv (0, 0, \ldots, 0) \in \mathbf{N}_0^N$, and $s_0(\boldsymbol{0}) = 1$, the estimation in $|\boldsymbol{x}| \to 0$ is given as shown by the second equation of (3.4). $\qquad\square$

By this lemma, we have the estimate

$$f_N(t, \boldsymbol{x}|\boldsymbol{x}') = \frac{1}{C_N} t^{-N^2/2} h_N(\boldsymbol{x}) h_N(\boldsymbol{x}') e^{-|\boldsymbol{x}|^2/2t} \times \left\{1 + \mathcal{O}\left(\frac{|\boldsymbol{x}'|}{\sqrt{t}}\right)\right\} \quad in \ \frac{|\boldsymbol{x}'|}{\sqrt{t}} \to 0 \tag{3.6}$$

with $C_N = (2\pi)^{N/2} \prod_{j=1}^N \Gamma(j)$. The integral formula

$$\int_{\mathbf{R}^N} e^{-a|\boldsymbol{x}|^2} |h_N(\boldsymbol{x})|^{2\gamma} d\boldsymbol{x} = (2\pi)^{N/2} (2a)^{-N(\gamma(N-1)+1)/2} \prod_{j=1}^N \frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)}$$

is found in [66] ((17.6.7) p. 321) as a variation of the Selberg integral [83], whose proof was given in [63]. If we set $\gamma = 1/2$, $a = 1/2t$ and note that the integral over $\mathbf{R}^N$ can be replaced by the integral over $\mathbf{W}_N$ multiplied by $N!$, we have

$$\int_{\mathbf{W}_N} e^{-|\boldsymbol{x}|^2/2t} h_N(\boldsymbol{x}) d\boldsymbol{x} = C_N' t^{N(N+1)/4} \tag{3.7}$$

with $C_N' = 2^{N/2} \prod_{j=1}^N \Gamma(j/2)$. Similarly by setting $\gamma = 1$ and $a = 1/2t$, we have

$$\int_{\mathbf{W}_N} e^{-|\boldsymbol{x}|^2/2t} (h_N(\boldsymbol{x}))^2 d\boldsymbol{x} = C_N t^{N^2/2}. \tag{3.8}$$

Using (3.7) with (3.6), we obtain the asymptotics of $\mathcal{N}_N$,

$$\mathcal{N}_N(t, \boldsymbol{x}') = \frac{C_N'}{C_N} t^{-N(N-1)/4} h_N(\boldsymbol{x}') \times \left\{ 1 + \mathcal{O}\left(\frac{|\boldsymbol{x}'|}{\sqrt{t}}\right) \right\} \quad \text{in } \frac{|\boldsymbol{x}'|}{\sqrt{t}} \to 0. \tag{3.9}$$

The integral (3.8) will be used shortly.

3.3 Temporally Homogeneous Limit

By the above estimate (3.9), we can take the $T \to \infty$ limit in (3.2) and obtain the transition probability density, which is homogeneous in time, i.e., a function of time difference $t - t'$,

$$p_N(t - t', \boldsymbol{x}|\boldsymbol{x}') = \lim_{T \to \infty} g_{N,T}(t', \boldsymbol{x}'; t, \boldsymbol{x})$$

$$= h_N(\boldsymbol{x}) f_N(t - t', \boldsymbol{x}|\boldsymbol{x}') \frac{1}{h_N(\boldsymbol{x}')}. \tag{3.10}$$

From now on, we consider the noncolliding BM, which is defined by this transition probability density. It is a temporally homogeneous process and we denote it by $X(t) = (X_1(t), X_2(t), \ldots, X_N(t))$.

Remark 1 The product of differences (the Vandermonde determinant) $h_N(\boldsymbol{x})$ given by (1.13) is a harmonic function in the sense

$$\nabla^2 h_N(\boldsymbol{x}) = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} h_N(\boldsymbol{x}) = 0,$$

which has strictly positive values at interior points of $\mathbf{W}_N$ and zero at the boundary. Equation (3.10) is considered as a transformation from f_N to p_N associated with the harmonic function, which is called the h-transform [29]. That is, the temporally homogeneous noncolliding BM is an h-transform by h_N of the absorbing BM in the Weyl chamber $\mathbf{W}_N$ [41]. It is easy to confirm that $p_N(t, \cdot|\boldsymbol{x})$ satisfies the following backward Kolmogorov equation

$$\frac{\partial}{\partial t} u(t, \boldsymbol{x}) = \frac{1}{2} \nabla^2 u(t, \boldsymbol{x}) + \nabla \log h_N(\boldsymbol{x}) \cdot \nabla u(t, \boldsymbol{x}),$$

which is a multivariate extension of (1.4). It implies that the process $X(t)$ is a diffusion process, which solves the SDE (1.9) of Dyson's BM model (with $\beta = 2$), which describes the time-evolution of eigenvalues of hermitian matrix-valued (GUE) diffusion process [30, 55]. This equivalence between Dyson's BM model for GUE (the eigenvalue part of the matrix-valued BM) and the present noncolliding BM (BM conditioned not to collide) is a multi-dimensional extension of the equivalence between the three-dimensional Bessel process (the radial coordinate of the three-dimensional BM) and the one-dimensional BM conditioned to stay in the positive region $B(t) > 0$, as announced in Sect. 1. See also [53, 55, 58].

Let $v_0(\boldsymbol{x})$ be the probability density at time $t_0 > 0$. Then the probability density of distribution $X(t), t \geq t_0$ is given by

$$
\begin{aligned}
v_t(\boldsymbol{x}) &= \int_{\mathbf{W}_N} p_N(t - t_0, \boldsymbol{x}|\boldsymbol{x}')v_0(\boldsymbol{x}')d\boldsymbol{x}' \\
&= h_N(\boldsymbol{x}) \int_{\mathbf{W}_N} f_N(t - t_0, \boldsymbol{x}|\boldsymbol{x}')\frac{v_0(\boldsymbol{x}')}{h_N(\boldsymbol{x}')}d\boldsymbol{x}'.
\end{aligned} \tag{3.11}
$$

By the estimate (3.6), we will see that

$$
v_t(\boldsymbol{x}) \simeq \frac{1}{C_N} t^{-N^2/2} e^{-|\boldsymbol{x}|^2/2t} (h_N(\boldsymbol{x}))^2 \quad \text{in } t \to \infty, \tag{3.12}
$$

if the distribution v_0 has finite moment. Note that the integral formula (3.8) guarantees that (3.12) is normalized. It should be noted that the distribution (3.12) is equal to the eigenvalue distribution of hermitian random-matrices in GUE with variance $\sigma^2 = t$ [66].

Proposition 3.2 *For any initial distribution having finite moment, the asymptote of probability density of distribution of the noncolliding BM, $X(t)$, in $t \to \infty$ is expressed by the eigenvalue distribution of GUE with variance t.*

4 Determinantal Process

4.1 GUE Initial Distribution

Consider a sequence of times, $0 < t_0 < t_1 < \cdots < t_M, M = 1, 2, \ldots$ for observations of distribution of $X(t)$. Given the initial distribution v_0 at time t_0, multitime probability density is given using (3.10) as

$$
\begin{aligned}
&p_N(t_0, \boldsymbol{x}^{(0)}; t_1, \boldsymbol{x}^{(1)}; \ldots; t_M, \boldsymbol{x}^{(M)}) \\
&= \prod_{m=0}^{M-1} p_N(t_{m+1} - t_m, \boldsymbol{x}^{(m+1)}|\boldsymbol{x}^{(m)})v_0(\boldsymbol{x}^{(0)}) \\
&= h_N(\boldsymbol{x}^{(M)}) \prod_{m=0}^{M-1} f_N(t_{m+1} - t_m, \boldsymbol{x}^{(m+1)}|\boldsymbol{x}^{(m)})\frac{v_0(\boldsymbol{x}^{(0)})}{h_N(\boldsymbol{x}^{(0)})}.
\end{aligned} \tag{4.1}
$$

Now we assume that v_0 is the GUE-eigenvalue distribution with variance t_0,

$$
v_0(\boldsymbol{x}^{(0)}) = \frac{1}{C_N} t_0^{-N^2/2} e^{-|\boldsymbol{x}^{(0)}|^2/2t_0} (h_N(\boldsymbol{x}^{(0)}))^2. \tag{4.2}
$$

Then (4.1) becomes the product of f_N's multiplied by a factor

$$\frac{1}{C_N} t_0^{-N^2/2} e^{-|\boldsymbol{x}^{(0)}|^2/2t_0} h_N(\boldsymbol{x}^{(M)}) h_N(\boldsymbol{x}^{(0)}). \tag{4.3}$$

By multilinearity of determinant and the fact that the coefficient of the highest order term of the Hermite polynomial $H_n(x)$ is $(2x)^n$ (see (2.10)), the following equality is established,

$$\det_{1 \leq j,k \leq N} [H_{j-1}(x_k)] = h_N(2\boldsymbol{x}).$$

Then if we define the multivariate functions

$$\mu_0(\boldsymbol{x}) = \det_{1 \leq j,k \leq N} [\phi_{j-1}(t_0, x_k)],$$

$$\mu_M(\boldsymbol{x}) = \det_{1 \leq j,k \leq N} [\widehat{\phi}_{j-1}(t_M, x_k)], \tag{4.4}$$

where ϕ_n and $\widehat{\phi}_n$ are given by (2.21) and (2.22), respectively, the factor (4.3) is readily shown to be equal to $\mu_0(\boldsymbol{x}^{(0)}) \mu_M(\boldsymbol{x}^{(M)})$. That is, the multitime probability density (4.1) is written as

$$\mu_M(\boldsymbol{x}^{(M)}) \prod_{m=0}^{M-1} f_N(t_{m+1} - t_m, \boldsymbol{x}^{(m+1)} | \boldsymbol{x}^{(m)}) \mu_0(\boldsymbol{x}^{(0)}).$$

This expression shows that the probability law is invariant under any permutation of indices of particles; $x_j^{(m)} \mapsto x_{\sigma(j)}^{(m)}, 0 \leq m \leq M, \sigma \in \mathcal{S}_N$. Then description will be easier if we regard that particles are identical and indistinguishable. We denote by $\mathfrak{X}$ the space of countable subsets ξ of $\mathbf{R}$ satisfying $\sharp(\xi \cap K) < \infty$ for any compact subset K. The space $\mathfrak{X}$ is a Polish space with the vague topology. For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \bigcup_{\ell=1}^{\infty} \mathbf{W}_\ell$, we denote $\{x_j\}_{j=1}^n \in \mathfrak{X}$ simply by $\{\boldsymbol{x}\}$. Then we consider the process on the set $\mathfrak{X}$, $\Xi_N^{\mathbf{X}}(t) = \{X(t)\}, t \in [t_0, \infty)$, such that, for any $M + 1$ times $t_0 < t_1 < \cdots < t_{M-1} < t_M < \infty$ $(M = 0, 1, 2, \ldots)$ the multitime probability density is given of the form

$$\mathfrak{p}_N(t_0, \{\boldsymbol{x}^{(0)}\}; t_1, \{\boldsymbol{x}^{(1)}\}; \ldots; t_{M-1}, \{\boldsymbol{x}^{(M-1)}\}; t_M, \{\boldsymbol{x}^{(M)}\})$$

$$= \mu_M(\{\boldsymbol{x}^{(M)}\}) \prod_{m=0}^{M-1} f_N(t_{m+1} - t_m, \boldsymbol{x}^{(m+1)} | \boldsymbol{x}^{(m)}) \mu_0(\{\boldsymbol{x}^{(0)}\}). \tag{4.5}$$

For $\boldsymbol{x}^{(m)} \in \mathbf{R}^N, 0 \leq m \leq M$, and $N' = 1, 2, \ldots, N$, we put $\boldsymbol{x}_{N'}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_{N'}^{(m)})$. For a sequence $\{N_m\}_{m=0}^M$ of positive integers less than or equal to N, we define the $(N_0, \ldots, N_M)$-multitime correlation function by

$$\rho_N(t_0, \{\boldsymbol{x}_{N_0}^{(0)}\}; t_1, \{\boldsymbol{x}_{N_1}^{(1)}\}; \ldots; t_M, \{\boldsymbol{x}_{N_M}^{(M)}\})$$

$$= \int_{\prod_{m=0}^M \mathbf{R}^{N-N_m}} \mathfrak{p}_N(t_0, \{\boldsymbol{x}^{(0)}\}; \ldots; t_M, \{\boldsymbol{x}^{(M)}\}) \prod_{m=0}^M \frac{1}{(N - N_m)!} \prod_{j=N_m+1}^N dx_j^{(m)}. \tag{4.6}$$

Expectations with respect to the configurations $\{X(t_0)\}, \{X(t_1)\}, \ldots, \{X(t_M)\}$ are denoted by $\mathbf{E}_N$:

$$\mathbf{E}_N[f(\{X(t_0)\}, \{X(t_1)\}, \dots, \{X(t_M)\})]$$

$$= \left(\frac{1}{N!}\right)^{M+1} \int_{\mathbf{R}^{N(M+1)}} f(\{\boldsymbol{x}^{(0)}\}, \dots, \{\boldsymbol{x}^{(M)}\})$$

$$\times \mathfrak{p}_N(t_0, \{\boldsymbol{x}^{(0)}\}; \dots; t_M, \{\boldsymbol{x}^{(M)}\}) \prod_{m=0}^{M} \prod_{j=1}^{N} dx_j^{(m)}. \tag{4.7}$$

Remark 2 Set $t' = 0, t = t_0 > 0, \boldsymbol{x} = \boldsymbol{x}^{(0)}$ in (3.10) and consider the limit $\boldsymbol{x}' \to \boldsymbol{0}$. By (3.6), we can see

$$\lim_{|\boldsymbol{x}'| \to 0} p_N(t_0, \boldsymbol{x}^{(0)} | \boldsymbol{x}') = \nu_0(\boldsymbol{x}^{(0)}). \tag{4.8}$$

This fact is essentially the same thing with that stated as Proposition 3.2 for the scaling property of the process. The GUE-eigenvalue distribution (4.2) adopted here as the initial distribution at time $t_0 > 0$ is immediately realized if we start the present noncolliding system at time 0 from $\{\boldsymbol{0}\}$. The state $\{\boldsymbol{0}\}$, which is a boundary of $\mathbf{W}_N$, is entrance [52–54].

4.2 Generating Function

Let $C_0(\mathbf{R})$ be the set of all continuous real functions with compact supports. For $\boldsymbol{f} = (f_0, f_1, \dots, f_M) \in C_0(\mathbf{R})^M$, and $\boldsymbol{\theta} = (\theta_0, \theta_1, \dots, \theta_M) \in \mathbf{R}^M$, the generating function for multitime correlation functions is defined for the process $\{X(t)\}, t \in [0, T]$ as

$$\Psi_N(\boldsymbol{f}; \boldsymbol{\theta}) = \mathbf{E}_N\left[\exp\left\{\sum_{m=0}^{M} \theta_m \sum_{j_m=1}^{N} f_m(X_{j_m}(t_m))\right\}\right]. \tag{4.9}$$

Let

$$\chi_m(x) = e^{\theta_m f_m(x)} - 1, \quad 0 \le m \le M,$$

and write (4.9) as $\Psi_N[\chi]$. Then by the definition of multitime correlation function (4.6), we have

$$\Psi_N[\chi] = \sum_{N_0=0}^{N} \sum_{N_1=0}^{N} \cdots \sum_{N_M=0}^{N} \prod_{m=0}^{M} \frac{1}{N_m!} \int_{\mathbf{R}^{N_0}} \prod_{j=1}^{N_0} dx_j^{(0)} \int_{\mathbf{R}^{N_1}} \prod_{j=1}^{N_1} dx_j^{(1)} \cdots \int_{\mathbf{R}^{N_M}} \prod_{j=1}^{N_M} dx_j^{(M)}$$

$$\times \prod_{m=0}^{M} \prod_{j=1}^{N_m} \chi_m(x_j^{(m)}) \rho_N(t_0, \{\boldsymbol{x}_{N_0}^{(0)}\}; t_1, \{\boldsymbol{x}_{N_1}^{(1)}\}; \dots; t_M, \{\boldsymbol{x}_{N_M}^{(M)}\}). \tag{4.10}$$

By the definition (4.9) with (4.7) and (4.5), we have

$$\Psi_N[\chi] = \left(\frac{1}{N!}\right)^{M+1} \int_{\mathbf{R}^{N(M+1)}} \prod_{m=0}^{M} \prod_{j=1}^{N} dx_j^{(m)}$$

$$\times \det_{1 \le j,k \le N} [\widehat{\phi}_{j-1}(t_M, x_k^{(M)})(1 + \chi_M(x_k^{(M)}))]$$

$$\times \prod_{m=0}^{M-1} \det_{1 \le j,k \le N} [p(t_{m+1} - t_m, x_k^{(m+1)} | x_j^{(m)})(1 + \chi_m(x_k^{(m)}))]$$

$$\times \det_{1 \le j,k \le N} [\phi_{j-1}(t_0, x_k^{(0)})].$$

By definition of determinant, it is easy to prove the following identity for square integrable continuous functions $g_j, \bar{g}_j, 1 \leq j \leq N$,

$$\frac{1}{N!} \int_{\mathbf{R}^N} \det_{1 \leq j,k \leq N}[g_j(x_k)] \det_{1 \leq j,k \leq N}[\bar{g}_j(x_k)] \prod_{j=1}^{N} dx_j = \det_{1 \leq j,k \leq N} \left[\int_{\mathbf{R}} g_j(x)\bar{g}_k(x)dx \right], \quad (4.11)$$

which is called the Heine identity. By repeated applications of this identity we have

$$\Psi_N[\chi] = \det_{1 \leq j,k \leq N}[F_{jk}[\chi]] \tag{4.12}$$

with

$$F_{jk}[\chi] = \int_{\mathbf{R}^{M+1}} \prod_{m=0}^{M} dx^{(m)} \{\widehat{\phi}_{j-1}(t_M, x^{(M)})(1 + \chi_M(x^{(M)}))\}$$

$$\times \prod_{m=0}^{M-1} \{p(t_{m+1} - t_m, x^{(m+1)}|x^{(m)})(1 + \chi_m(x^{(m)}))\}\phi_{k-1}(t_0, x^{(0)}).$$

By the Chapman–Kolmogorov equation (2.18) and the invariance (2.20), $F_{jk}[0] = \delta_{jk}$ and then $\Psi_N[0] = 1$, which implies that (4.5) is indeed normalized. If we use the notations introduced in Sect. 2, it is written as

$$F_{jk}[\chi] = \int_{\mathbf{R}^M} \prod_{m=0}^{M} dx^{(m)} \langle j - 1|t_M, x^{(M)}\rangle\{1 + \chi_M(x^{(M)})\}$$

$$\times \prod_{m=0}^{M-1} [\langle t_{m+1}, x^{(m+1)}|t_m, x^{(m)}\rangle\{1 + \chi_m(x^{(m)})\}]\langle t_0, x^{(0)}|k - 1\rangle$$

$$= \langle j - 1|k - 1\rangle + \sum_{\ell=0}^{M+1} \sum_{M \geq m_1 > \cdots > m_\ell \geq 0} \int_{\mathbf{R}^\ell} \prod_{n=1}^{\ell} dx^{(m_n)} \langle j - 1|t_{m_1}, x^{(m_1)}\rangle \chi_{m_1}(x^{(m_1)})$$

$$\times \langle t_{m_1}, x^{(m_1)}|t_{m_2}, x^{(m_2)}\rangle \times \cdots$$

$$\times \langle t_{m_{\ell-1}}, x^{(m_{\ell-1})}|t_{m_\ell}, x^{(m_\ell)}\rangle \chi_{m_\ell}(x^{(m_\ell)})\langle t_{m_\ell}, x^{(m_\ell)}|k - 1\rangle. \tag{4.13}$$

Now we introduce an indicator $\widehat{1}_+$ such that

$$\langle t_m, x|\widehat{1}_+|t_n, y\rangle = \begin{cases} \langle t_m, x|t_n, y\rangle & \text{if } m > n, \\ 0, & \text{otherwise.} \end{cases} \tag{4.14}$$

Then (4.13) can be written of the form of an infinite series

$$F_{jk}[\chi] = \langle j - 1|k - 1\rangle + \sum_{\ell=0}^{\infty} \sum_{m_1=0}^{M} \cdots \sum_{m_\ell=0}^{M} \int_{\mathbf{R}^\ell} \prod_{n=1}^{\ell} dx^{(m_n)} \langle j - 1|t_{m_1}, x^{(m_1)}\rangle \chi_{m_1}(x^{(m_1)})$$

$$\times \langle t_{m_1}, x^{(m_1)}|\widehat{1}_+|t_{m_2}, x^{(m_2)}\rangle \times \cdots$$

$$\times \langle t_{m_{\ell-1}}, x^{(m_{\ell-1})}|\widehat{1}_+|t_{m_\ell}, x^{(m_\ell)}\rangle \chi_{m_\ell}(x^{(m_\ell)})\langle t_{m_\ell}, x^{(m_\ell)}|k - 1\rangle,$$

in which all terms with $\ell > M + 1$ are zero by (4.14). Let

$$\widehat{\chi}_+^{m,n}(x, y) = \langle t_m, x | \hat{1}_+ | t_n, y \rangle \chi_n(y), \quad m, n \in \{0, 1, \ldots, M\}, x, y \in \mathbf{R}, \tag{4.15}$$

and define an $M \times M$ matrix-kernel $\widehat{\chi}_+$, whose (m, n)-element is given by the integral kernel (4.15). The (m, n)-element of its qth power $(\widehat{\chi}_+)^q$, $q = 2, 3, \ldots$, will be the following kernel given by $(q - 1)$-multiple integral,

$$[(\widehat{\chi}_+)^q]^{m,n}(x, y) = \sum_{m_1=0}^{M} \int_{\mathbf{R}} dx^{(m_1)} \cdots \sum_{m_{q-1}=0}^{M} \int_{\mathbf{R}} dx^{(m_{q-1})} \widehat{\chi}_+^{m,m_1}(x, x^{(m_1)})$$

$$\times \cdots \times \widehat{\chi}_+^{m_{q-1},n}(x^{(m_{q-1})}, y).$$

Then we have

$$F_{jk}[\chi] = \langle j - 1 | k - 1 \rangle + \sum_{m=0}^{M} \int_{\mathbf{R}} dx \sum_{n=0}^{M} \int_{\mathbf{R}} dy \langle j - 1 | t_m, x \rangle \chi_m(x)$$

$$\times \left\{ \delta_{m,n} \delta(x - y) + \sum_{q=1}^{\infty} [(\widehat{\chi}_+)^q]^{m,n}(x, y) \right\} \langle t_n, y | k - 1 \rangle. \tag{4.16}$$

Let $\hat{1}^{m,n}(x, y) \equiv \delta_{m,n} \delta(x - y)$. Since it is easy to see that

$$\sum_{\ell=0}^{M} \int_{\mathbf{R}} dy [\hat{1} - \widehat{\chi}_+]^{m,\ell}(x, y) \left\{ \delta_{\ell,n} \delta(y - z) + \sum_{q=1}^{\infty} [(\widehat{\chi}_+)^q]^{\ell,n}(y, z) \right\} = \hat{1}^{m,n}(x, z), \tag{4.17}$$

we can write

$$\delta_{m,n} \delta(x - y) + \sum_{q=1}^{\infty} [(\widehat{\chi}_+)^q]^{m,n}(x, y) = \left[\frac{\hat{1}}{\hat{1} - \widehat{\chi}_+} \right]^{m,n}(x, y).$$

Then (4.16) becomes

$$F_{jk}[\chi] = \delta_{jk} + \sum_{m=0}^{M} \int_{\mathbf{R}} dx \, B_j^{(m)}(x) C_k^{(m)}(x), \tag{4.18}$$

where

$$B_j^{(m)}(x) = \sum_{n=0}^{M} \int_{\mathbf{R}} dy \langle j - 1 | t_n, y \rangle \chi_n(y) \left[\frac{\hat{1}}{\hat{1} - \widehat{\chi}_+} \right]^{n,m}(y, x),$$

$$C_k^{(m)}(x) = \langle t_m, x | k - 1 \rangle. \tag{4.19}$$

4.3 Fredholm Determinant and Determinantal Process

Let $\widetilde{B}_j^{(m)}(x)$ and $\widetilde{C}_j^{(m)}(x)$, $0 \le m \le M$, $1 \le j \le N$, be square integrable continuous functions. Then the following formulae can be proved;

$$\det_{1 \le j,k \le N}\left[\delta_{j,k} + \sum_{m=0}^{M} \int_{\mathbf{R}} dx\, \widetilde{B}_j^{(m)}(x)\widetilde{C}_k^{(m)}(x)\right]$$

$$= \det_{1 \le j^{(m)}, j^{(n)} \le N, 0 \le m,n \le M}\left[\delta_{m,n}\delta_{j^{(m)},j^{(n)}} + \int_{\mathbf{R}} dx\, \widetilde{B}_{j^{(m)}}^{(m)}(x)\widetilde{C}_{j^{(n)}}^{(n)}(x)\right]$$

$$= \sum_{N_0=0}^{N} \sum_{N_1=0}^{N} \cdots \sum_{N_M=0}^{N} \prod_{m=0}^{M} \frac{1}{N_m!} \int_{\mathbf{R}^{N_0}} \prod_{j=1}^{N_0} dx_j^{(0)} \int_{\mathbf{R}^{N_1}} \prod_{j=1}^{N_1} dx_j^{(1)} \cdots \int_{\mathbf{R}^{N_M}} \prod_{j=1}^{N_M} dx_j^{(M)}$$

$$\times \det_{1 \le j \le N_m, 1 \le k \le N_n, 0 \le m,n \le M}\left[\sum_{p=1}^{N} \widetilde{C}_p^{(m)}(x_j^{(m)})\widetilde{B}_p^{(n)}(x_k^{(n)})\right]. \qquad (4.20)$$

The expansion formula of the last expression defines the Fredholm determinant, which is abbreviated as

$$\mathrm{Det}\left[\hat{1}^{m,n}(x,y) + \sum_{p=1}^{N} \widetilde{C}_p^{(m)}(x)\widetilde{B}_p^{(n)}(y)\right].$$

The generating function (4.12) with (4.18) and (4.19) is then expressed as the Fredholm determinant,

$$\Psi_N[\chi] = \mathrm{Det}\left[\hat{1}^{m,n}(x,z) + \sum_{\ell=0}^{M} \int_{\mathbf{R}} dy \langle t_m, x | \mathcal{P}^N | t_\ell, y \rangle \chi_\ell(y)\left[\frac{\hat{1}}{\hat{1} - \widehat{\chi}_+}\right]^{\ell,n}(y,z)\right], \quad (4.21)$$

where

$$\mathcal{P}^N = \sum_{p=1}^{N} |p-1\rangle\langle p-1|. \qquad (4.22)$$

Note that (4.17) implies that

$$\hat{1}^{m,n}(x,z) = \sum_{\ell=0}^{M} \int_{\mathbf{R}} dy \{\hat{1}^{m,\ell}(x,y) - \langle t_m, x | \hat{1}_+ | t_\ell, y \rangle \chi_\ell(y)\}\left[\frac{\hat{1}}{\hat{1} - \widehat{\chi}_+}\right]^{\ell,n}(y,z).$$

Plugging this into (4.21), we have

$$\Psi_N[\chi] = \mathrm{Det}\left[\sum_{\ell=0}^{M} \int_{\mathbf{R}} dy[\hat{1}^{m,\ell}(x,y) + \langle t_m, x | (\mathcal{P}^N - \hat{1}_+) | t_\ell, y \rangle \chi_\ell(y)]\left[\frac{\hat{1}}{\hat{1} - \widehat{\chi}_+}\right]^{\ell,n}(y,z)\right]$$

$$= \mathrm{Det}[\hat{1}^{m,n}(x,y) + \langle t_m, x | (\mathcal{P}^N - \hat{1}_+) | t_n, y \rangle \chi_n(y)]\, \mathrm{Det}\left[\left[\frac{\hat{1}}{\hat{1} - \widehat{\chi}_+}\right]^{m,n}(x,y)\right]$$

$$= \mathrm{Det}[\hat{1}^{m,n}(x,y) + \langle t_m, x | (\mathcal{P}^N - \hat{1}_+) | t_n, y \rangle \chi_n(y)]. \qquad (4.23)$$

Here

$$\mathrm{Det}\left[\left[\frac{\hat{1}}{\hat{1} - \widehat{\chi}_+}\right]^{m,n}(x,y)\right] = 1/\mathrm{Det}[[\hat{1} - \widehat{\chi}_+]^{m,n}(x,y)],$$

and we have used $\text{Det}[[\hat{1} - \hat{\chi}_+]^{m,n}(x, y)] = 1$, which is concluded from the fact that $[\hat{1} - \hat{\chi}_+]^{m,n}(x, y) = 0$ for $m < n$ by the definition of $\hat{\chi}_+$, (4.15) with (4.14), and it is $\delta(x - y)$ for $m = n$.

Let

$$S_N^{m,n}(x, y) = \langle t_m, x | \mathcal{P}^N | t_n, y \rangle,$$

$$\widetilde{S}_N^{m,n}(x, y) = -\langle t_m, x | (\hat{1}_+ - \mathcal{P}^N) | t_n, y \rangle.$$

Following the formulae given in Sect. 2, $S_N^{m,n}$ is determined as

$$S_N^{m,n}(x, y) = \sum_{p=1}^{N} \langle t_m, x | p - 1 \rangle \langle p - 1 | t_n, y \rangle$$

$$= \frac{1}{\sqrt{2t_m}} e^{-x^2/4t_m + y^2/4t_n} \sum_{p=1}^{N} \left(\frac{t_n}{t_m}\right)^{(p-1)/2} \left\langle \frac{x}{\sqrt{2t_m}} \middle| p - 1 \right\rangle \left\langle p - 1 \middle| \frac{y}{\sqrt{2t_n}} \right\rangle$$

$$= \frac{1}{\sqrt{2t_m}} e^{-x^2/4t_m + y^2/4t_n} \sum_{k=0}^{N-1} \left(\frac{t_n}{t_m}\right)^{k/2} \varphi_k\left(\frac{x}{\sqrt{2t_m}}\right) \varphi_k\left(\frac{y}{\sqrt{2t_n}}\right).$$

Combination of this with (2.14) gives $\widetilde{S}_N^{m,n}$ as

$$\widetilde{S}_N^{m,n}(x, y) = S_N^{m,n}(x, y) - \mathbf{1}_{\{m>n\}} p(t_m - t_n, x | y)$$

$$= \begin{cases} \dfrac{1}{\sqrt{2t_m}} e^{-x^2/4t_m + y^2/4t_n} \displaystyle\sum_{k=0}^{N-1} \left(\dfrac{t_n}{t_m}\right)^{k/2} \varphi_k\left(\dfrac{x}{\sqrt{2t_m}}\right) \varphi_k\left(\dfrac{y}{\sqrt{2t_n}}\right) & \text{if } m \leq n, \\[3ex] -\dfrac{1}{\sqrt{2t_m}} e^{-x^2/4t_m + y^2/4t_n} \displaystyle\sum_{k=N}^{\infty} \left(\dfrac{t_n}{t_m}\right)^{k/2} \varphi_k\left(\dfrac{x}{\sqrt{2t_m}}\right) \varphi_k\left(\dfrac{y}{\sqrt{2t_n}}\right) & \text{if } m > n, \end{cases}$$

(4.24)

where $\mathbf{1}_{\{\omega\}}$ is the indicator of a condition ω; $\mathbf{1}_{\{\omega\}} = 1$ if ω is satisfied and $\mathbf{1}_{\{\omega\}} = 0$ otherwise.

As shown above, not $\{\widetilde{S}_N^{m,n}(x, y)\}$ themselves, but determinants of matrices made of them are observables. By definition of determinant, factors $\{e^{-x^2/4t_m + y^2/4t_n}\}$ of $\{\widetilde{S}_N^{m,n}(x, y)\}$ in (4.24) are completely cancelled out, when we calculate determinants. So here we define the following matrix-kernel by omitting these factors in $\{\widetilde{S}_N^{m,n}(x, y)\}$,

$$\mathbb{K}_N(t_m, x; t_n, y) = \begin{cases} \dfrac{1}{\sqrt{2t_m}} \displaystyle\sum_{k=0}^{N-1} \left(\dfrac{t_n}{t_m}\right)^{k/2} \varphi_k\left(\dfrac{x}{\sqrt{2t_m}}\right) \varphi_k\left(\dfrac{y}{\sqrt{2t_n}}\right) & \text{if } m \leq n, \\[3ex] -\dfrac{1}{\sqrt{2t_m}} \displaystyle\sum_{k=N}^{\infty} \left(\dfrac{t_n}{t_m}\right)^{k/2} \varphi_k\left(\dfrac{x}{\sqrt{2t_m}}\right) \varphi_k\left(\dfrac{y}{\sqrt{2t_n}}\right) & \text{if } m > n, \end{cases}$$

(4.25)

and rewrite (4.23) as

$$\Psi_N[\chi] = \text{Det}[\delta_{m,n}\delta(x - y) + \mathbb{K}_N(t_m, x; t_n, y)\chi_n(y)].$$

(4.26)

This Fredholm determinant is by definition expanded as

$$
\mathrm{Det}[\delta_{m,n}\delta(x-y) + \mathbb{K}_N(t_m,x;t_n,y)\chi_n(y)]
$$

$$
= \sum_{N_0=0}^{N}\sum_{N_1=0}^{N}\cdots\sum_{N_M=0}^{N}\prod_{m=0}^{M}\frac{1}{N_m!}\int_{\mathbf{R}^{N_0}}\prod_{j=1}^{N_0}dx_j^{(0)}\int_{\mathbf{R}^{N_1}}\prod_{j=1}^{N_1}dx_j^{(1)}\cdots\int_{\mathbf{R}^{N_M}}\prod_{j=1}^{N_M}dx_j^{(M)}
$$

$$
\times\prod_{m=0}^{M}\prod_{j=1}^{N_m}\chi_m(x_j^{(m)})\det_{1\le j\le N_m, 1\le k\le N_n, 0\le m,n\le M}[\mathbb{K}_N(t_m,x_j^{(m)};t_n,x_k^{(n)})]. \tag{4.27}
$$

Comparison of (4.27) and (4.10) determines all of the multitime correlation functions. Now we summarize the above results as a theorem.

Theorem 4.1 *The temporally homogeneous noncolliding BM,* $\Xi_N^{\mathbf{X}}(t) = \{X(t)\}$*, starting from the GUE-eigenvalue distribution* (4.2) *at time* $t_0 > 0$*, is a finite determinantal process in the following sense.*

(i) *The multitime generating function is given by the Fredholm determinant* (4.26)*, where the matrix-kernel* $\mathbb{K}_N$ *is given by* (4.25).

(ii) *Any multitime correlation function is given by a determinant*; *for any* $M \ge 0$*, any sequence* $\{N_m\}_{m=0}^{M}$ *of positive integers less than or equal to* N*, any time sequence* $t_0 < t_1 < \cdots < t_M < \infty$*, the* $(N_0,\ldots,N_M)$*-multitime correlation function is given by*

$$
\rho_N(t_0,\{\mathbf{x}_{N_1}^{(0)}\}; t_1,\{\mathbf{x}_{N_1}^{(1)}\}; \ldots; t_M,\{\mathbf{x}_{N_M}^{(M)}\})
$$

$$
= \det_{1\le j\le N_m, 1\le k\le N_n, 0\le m,n\le M}[\mathbb{K}_N(t_m,x_j^{(m)};t_n,x_k^{(n)})]. \tag{4.28}
$$

The following relations hold for the Hermite polynomials,

$$
H_{k+1}(x) = 2xH_k(x) - 2kH_{k-1}(x), \tag{4.29}
$$

$$
\frac{d}{dx}H_k(x) = 2kH_{k-1}(x), \quad k = 1, 2, 3, \ldots. \tag{4.30}
$$

From (4.29), the Christoffel–Darboux formula is derived for the Hermite orthonormal functions $\{\varphi_k(x)\}_{k\in\mathbf{N}_0}$,

$$
\sum_{k=0}^{N-1}\varphi_k(x)\varphi_k(y) = \sqrt{\frac{N}{2}}\frac{\varphi_N(x)\varphi_{N-1}(y) - \varphi_{N-1}(x)\varphi_N(y)}{x - y} \tag{4.31}
$$

for $x \ne y$. Equation (4.30) can be used to evaluate the limit $y \to x$ in (4.31) and we find

$$
\sum_{k=0}^{N-1}\{\varphi_k(x)\}^2 = N\{\varphi_N(x)\}^2 - \sqrt{N(N+1)}\varphi_{N+1}(x)\varphi_{N-1}(x).
$$

Then the matrix-kernel have the following simpler expressions if $m = n$,

$$\mathbb{K}_N(t_m, x; t_m, y)$$

$$= \begin{cases} \sqrt{\dfrac{N}{2}} \dfrac{\varphi_N(x/\sqrt{2t_m})\varphi_{N-1}(y/\sqrt{2t_m}) - \varphi_{N-1}(x/\sqrt{2t_m})\varphi_N(y/\sqrt{2t_m})}{x - y} \\ \qquad \text{if } x \neq y, \\ \dfrac{1}{\sqrt{2t_m}} \left[N\left\{\varphi_N\left(\dfrac{x}{\sqrt{2t_m}}\right)\right\}^2 - \sqrt{N(N+1)}\varphi_{N-1}\left(\dfrac{x}{\sqrt{2t_m}}\right)\varphi_{N+1}\left(\dfrac{x}{\sqrt{2t_m}}\right) \right] \\ \qquad \text{if } x = y. \end{cases} \quad (4.32)$$

5 Infinite Particle Systems

5.1 Wigner's Semicircle Law

The density of $\Xi_N^{\mathbf{X}}(t)$ is given by

$$\rho_N(t, x) = \frac{1}{\sqrt{2t}} \sum_{k=0}^{N-1} \left\{ \varphi_k\left(\frac{x}{\sqrt{2t}}\right) \right\}^2$$

$$= \frac{1}{\sqrt{2t}} \left[N\left\{\varphi_N\left(\frac{x}{\sqrt{2t}}\right)\right\}^2 - \sqrt{N(N+1)}\varphi_{N-1}\left(\frac{x}{\sqrt{2t}}\right)\varphi_{N+1}\left(\frac{x}{\sqrt{2t}}\right) \right],$$

as a special case ($M = 0$, $N_0 = 1$ with setting $t_0 = t$) of Theorem 4.1 with (4.32). It is easy to confirm that $\int_{-\infty}^{\infty} \rho_N(t, x)dx = N$ by the orthonormality of $\varphi_k(x)$. The following estimations for asymptote in $N \to \infty$ are established [10, 91]. Let ε and ω be the fixed positive numbers. We have

(i) $\varphi_N(\sqrt{2N+1}\cos\phi) = \dfrac{1}{\sqrt{\pi \sin\phi}}\left(\dfrac{2}{N}\right)^{1/4}$

$$\times \left\{ \sin\left[\left(\frac{N}{2} + \frac{1}{4}\right)(\sin 2\phi - 2\phi) + \frac{3}{4}\pi\right] + \mathcal{O}\left(\frac{1}{N}\right) \right\}, \quad \varepsilon \leq \phi \leq \pi - \varepsilon.$$

(ii) $\varphi_N(\sqrt{2N+1}\cosh\phi) = \dfrac{1}{\sqrt{2\pi \sinh\phi}}\left(\dfrac{1}{2N}\right)^{1/4}$

$$\times \exp\left[\left(\frac{N}{2} + \frac{1}{4}\right)(2\phi - \sinh 2\phi) + \frac{3}{4}\pi\right]\left\{1 + \mathcal{O}\left(\frac{1}{N}\right)\right\}, \quad \varepsilon \leq \phi \leq \omega.$$

Using them, we will have the asymptote of the density profile in $N \to \infty$,

$$\rho_N(t, x) \simeq \begin{cases} \dfrac{1}{\pi\sqrt{2t}}\sqrt{2N - \dfrac{x^2}{2t}} & \text{if } -2\sqrt{Nt} \leq x \leq 2\sqrt{Nt}, \\ \\ 0, & \text{otherwise.} \end{cases} \quad (5.1)$$

The distribution of N particles has a finite support, whose interval $\propto \sqrt{N}$, and thus $\rho \sim \sqrt{N} \to \infty$ as $N \to \infty$ for fixed $0 < t < \infty$. If we set $x = 2\sqrt{N}t\xi$, we see

$$\lim_{N\to\infty} \frac{1}{N} \rho_N(t,x)dx = \begin{cases} \dfrac{2}{\pi}\sqrt{1-\xi^2}\,d\xi & \text{if } -1 \leq \xi \leq 1, \\[2mm] 0, & \text{otherwise,} \end{cases} \tag{5.2}$$

which is known as Wigner's semicircle law [66]. See also [81]. In the following we consider scaling limit, in which long-term limit $t \to \infty$ is taken at the same time with $N \to \infty$.

5.2 Bulk Scaling Limit and Homogeneous Infinite System

First we consider the central region $x \simeq 0$ in the semicircle-shaped profile of particle density in the scaling limit

$$t \simeq N \to \infty. \tag{5.3}$$

In this limit the system becomes homogeneous also in space with a constant density $\rho = 1/\pi$. We call this the bulk scaling limit.

Theorem 5.1 *For any $M \geq 0$, any sequence $\{N_m\}_{m=0}^M$ of positive integers, and any strictly increasing sequence $\{s_m\}_{m=1}^M$ of positive numbers*

$$\lim_{N\to\infty} \rho_N(N, \{x_{N_0}^{(0)}\}; N+2s_1, \{x_{N_1}^{(1)}\}; \ldots; N+2s_M, \{x_{N_M}^{(M)}\})$$

$$= \det_{1\leq j\leq N_m, 1\leq k\leq N_n, 0\leq m,n\leq M}[K(s_m, x_j^{(m)}; s_n, x_k^{(n)})]$$

$$\equiv \rho_{\sin}(0, \xi_{N_0}^{(0)}; s_1, \xi_{N_1}^{(1)}; \ldots; s_M, \xi_{N_M}^{(M)}), \tag{5.4}$$

where

$$K(t, y; s, x) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^1 du\, e^{(s-t)u^2}\cos(u(y-x)) & \text{if } t < s, \\[4mm] \dfrac{\sin(y-x)}{\pi(y-x)} & \text{if } t = s, \\[4mm] -\dfrac{1}{\pi}\displaystyle\int_1^\infty du\, e^{-(t-s)u^2}\cos(u(y-x)) & \text{if } t > s. \end{cases} \tag{5.5}$$

Proof For any $u \in \mathbf{R}$, the formula

$$\lim_{\ell\to\infty}(-1)^\ell \ell^{1/4}\varphi_{2\ell}\left(\frac{u}{2\sqrt{\ell}}\right) = \frac{1}{\sqrt{\pi}}\cos u,$$

$$\lim_{\ell\to\infty}(-1)^\ell \ell^{1/4}\varphi_{2\ell+1}\left(\frac{u}{2\sqrt{\ell}}\right) = \frac{1}{\sqrt{\pi}}\sin u \tag{5.6}$$

are known [10, 91]. We note that

$$\left(\frac{t_n}{t_m}\right)^\alpha = \left(\frac{N+2s_n}{N+2s_m}\right)^\alpha$$

$$= \left\{\left(1+\frac{2s_n}{N}\right)^N \left(1+\frac{2s_m}{N}\right)^{-N}\right\}^{\alpha/N}$$

$$\simeq e^{2(s_n-s_m)\alpha/N}$$

for $N \gg 1$ with fixed number α. Then (4.25) with $m \le n$ is evaluated in $N \to \infty$ as

$$\mathbb{K}_N(t_m, y; t_n, x) \simeq \frac{1}{\pi N} \sum_{\ell=0}^{N/2-1} e^{2(s_n-s_m)\ell/N} \sqrt{\frac{N}{2\ell}} \left\{\cos\left(\sqrt{\frac{2\ell}{N}}\, y\right) \cos\left(\sqrt{\frac{2\ell}{N}}\, x\right)\right.$$

$$\left. + \sin\left(\sqrt{\frac{2\ell}{N}}\, y\right) \sin\left(\sqrt{\frac{2\ell}{N}}\, x\right)\right\}$$

$$\simeq \frac{1}{2\pi} \int_0^1 d\lambda\, e^{(s_n-s_m)\lambda} \frac{1}{\sqrt{\lambda}} \{\cos(y\sqrt{\lambda})\cos(x\sqrt{\lambda}) + \sin(y\sqrt{\lambda})\sin(x\sqrt{\lambda})\}$$

$$= \frac{1}{\pi} \int_0^1 du\, e^{(s_n-s_m)u^2} \cos(u(y-x)).$$

In particular, when $m = n$, i.e., $s_n - s_m = 0$, the integration is readily performed to have $\int_0^1 du \cos(u(y-x)) = \sin(y-x)/(y-x)$. Similar evaluation in $N \to \infty$ can be done also for (4.25) with $m > n$. □

In the bulk scaling limit (5.3), the temporally and spatially homogeneous infinite particle system is obtained, whose multitime correlation functions are given by (5.4). The matrix-kernel (5.5) is called the extended sine kernel in [96]. In the present paper, we will call it simply "sine kernel". The system with the sine kernel was studied by Spohn [88], Osada [75, 76], and Nagao and Forrester [70], as an infinite particle limit of Dyson's BM model with $\beta = 2$ [30]. See also [2, 96].

5.3 Soft-Edge Scaling Limit and Spatially Inhomogeneous Infinite System

Next we consider the scaling limit

$$t \simeq N^{1/3} \quad \text{and} \quad x \simeq 2N^{2/3}. \tag{5.7}$$

Since (5.7) gives $x^2/2t \simeq 2N$, the vicinity of the right edge of semicircle-shaped profile (5.1) will be closed up, and we will obtain a spatially inhomogeneous infinite particle system in this scaling limit. Following the random matrix theory [66], we call (5.7) the soft-edge scaling limit.

In order to describe the limit, we introduce the Airy function

$$\text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^\infty dk\, e^{i(xk+k^3/3)}. \tag{5.8}$$

It is a solution of equation

$$\frac{d^2}{dx^2}\text{Ai}(x) = x\text{Ai}(x), \tag{5.9}$$

which behaves as

$$\text{Ai}(x) \simeq \frac{1}{2\sqrt{\pi}x^{1/4}} \exp\left(-\frac{2}{3}x^{3/2}\right),$$

$$\text{Ai}(-x) \simeq \frac{1}{\sqrt{\pi}x^{1/4}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right) \quad \text{in } x \to \infty.$$

In the proof of the following theorem, we will use the formula

$$\lim_{\ell \to \infty} 2^{-1/4}\ell^{1/12}\varphi_\ell\left(\sqrt{2\ell} + \frac{u}{\sqrt{2}}\ell^{-1/6}\right) = \text{Ai}(u) \quad \text{for } u \in \mathbf{R}. \tag{5.10}$$

Let

$$a_N(s) = 2N^{2/3} + 2N^{1/3}s - s^2, \tag{5.11}$$

and $\boldsymbol{x}_{N'}(s) = (a_N(s) + x_1, a_N(s) + x_2, \ldots, a_N(s) + x_{N'})$.

Theorem 5.2 *For any $M \geq 0$, any sequence $\{N_m\}_{m=0}^M$ of positive integers, and any strictly increasing sequence $\{s_m\}_{m=1}^M$ of positive numbers*

$$\lim_{N \to \infty} \rho_N(N^{1/3}, \{\boldsymbol{x}_{N_0}^{(0)}(0)\}; N^{1/3} + 2s_1, \{\boldsymbol{x}_{N_1}^{(1)}(s_1)\}; \ldots; N^{1/3} + 2s_M, \{\boldsymbol{x}_{N_M}^{(M)}(s_M)\})$$

$$= \det_{1 \leq j \leq N_m, 1 \leq k \leq N_n, 0 \leq m,n \leq M}[\mathcal{K}(s_m, x_j^{(m)}; s_n, x_k^{(n)})]$$

$$\equiv \rho_{\text{Ai}}(0, \xi_{N_0}^{(0)}; s_1, \xi_{N_1}^{(0)}; \ldots; s_M, \xi_{N_M}^{(M)}), \tag{5.12}$$

where

$$\mathcal{K}(t, y; s, x) = \begin{cases} \int_{-\infty}^0 d\lambda\, e^{(s-t)\lambda}\text{Ai}(y - \lambda)\text{Ai}(x - \lambda) & \text{if } t \leq s, \\ -\int_0^\infty d\lambda\, e^{-(t-s)\lambda}\text{Ai}(y - \lambda)\text{Ai}(x - \lambda) & \text{if } t > s. \end{cases} \tag{5.13}$$

Proof Putting $k = N - p - 1$ in the summation of (4.25) for $m \leq n$, we have

$$\mathbb{K}_N(t_m, y; t_n, x) = \left(\frac{t_n}{t_m}\right)^{(N-1)/2} \frac{1}{\sqrt{2t_m}} \sum_{p=0}^{N-1} \left(\frac{t_n}{t_m}\right)^{-p/2} \varphi_{N-p-1}\left(\frac{y}{\sqrt{2t_m}}\right) \varphi_{N-p-1}\left(\frac{x}{\sqrt{2t_n}}\right).$$

With the same reason mentioned above (4.25), we can omit the factor $(t_n/t_m)^{(N-1)/2}$. Since, when we set $t_m = N^{1/3} + 2s_m$,

$$\frac{a_N(s_m) + x}{\sqrt{2t_m}} = \sqrt{2N} + \frac{1}{\sqrt{2}}N^{-1/6}x + \mathcal{O}(N^{-1/2}),$$

we can use the formula (5.10);

$$\varphi_{N-p-1}\left(\frac{x}{\sqrt{2t_m}}\right) \simeq \varphi_{N-p-1}\left(\sqrt{2N} + \frac{1}{\sqrt{2}}N^{-1/6}x\right)$$

$$\simeq \varphi_{N-p-1}\left(\sqrt{2(N-p-1)} + \frac{1}{\sqrt{2}}(N-p-1)^{-1/6}\left\{x + \frac{p}{N^{1/3}}\right\}\right)$$

$$\simeq 2^{1/4}N^{-1/12}\mathrm{Ai}\left(x + \frac{p}{N^{1/3}}\right).$$

For

$$\left(\frac{t_n}{t_m}\right)^{-p/2} = \left[\left(\frac{1 + 2s_n/N^{1/3}}{1 + 2s_m/N^{1/3}}\right)^{N^{1/3}/2}\right]^{-p/N^{1/3}} \simeq e^{-(s_n - s_m)p/N^{1/3}} \quad \text{in } N \to \infty,$$

we have

$$\mathbb{K}_N(N^{1/3} + 2s_m, a_N(s_m) + y; N^{1/3} + 2s_n, a_N(s_n) + x)$$

$$\sim \frac{1}{N^{1/3}}\sum_{p=0}^{N-1} e^{-(s_n-s_m)p/N^{1/3}}\mathrm{Ai}\left(y + \frac{p}{N^{1/3}}\right)\mathrm{Ai}\left(x + \frac{p}{N^{1/3}}\right)$$

$$\simeq \int_0^\infty du\, e^{-(s_n-s_m)u}\mathrm{Ai}(y+u)\mathrm{Ai}(x+u) \quad \text{in } N \to \infty.$$

Note that the factor $(t_n/t_m)^{(N-1)/2}$, which is irrelevant in calculating determinants, was omitted in the second line in the above equations. Put $u = -\lambda$ to obtain the expression (5.13). Similar evaluation in $N \to \infty$ of (4.25) can be done also for $m > n$.   □

The infinite system obtained by the soft-edge scaling limit (5.7) is temporally homogeneous, but spatially inhomogeneous as shown by the "Airy kernel" (5.13). Prähofer and Spohn [78] and Johansson [46] studied the right-most path in the present system and called it the Airy process $A(t)$. For a given $t > 0$, $A(t)$ has distribution of the celebrated Tracy–Widom distribution, which is governed by the Painlevé II equation [94]. Recently Tracy and Widom derived a system of partial differential equations (PDE), which govern the Airy kernel (5.13) [95]. They also discussed other determinantal processes by PDE [96]. See also [1, 2].

6 Determinantal Processes Associated with Spectral Projections

6.1 Spectral Projections

First we note that, following the notations in Sect. 2, if we set $\tau = \log t$, $\tau' = \log t'$, $\zeta = y/\sqrt{2t}$, $\zeta' = x/\sqrt{2t'}$, and $\mathbb{K}_N(t, y; s, x)dy = \widetilde{\mathbb{K}}_N(\tau, \zeta; \tau', \zeta')d\zeta$ with $d\zeta = dy/\sqrt{2t}$, the matrix-kernel of the determinantal process of noncolliding BM with finite number of particles $N < \infty$ given by (4.25) is rewritten as

$$\widetilde{\mathbb{K}}_N(\tau, \zeta; \tau', \zeta') = \begin{cases} \langle\zeta|e^{(\tau'-\tau)\widehat{\mathcal{H}}_\varphi}\mathcal{P}_\varphi|\zeta'\rangle & \text{if } \tau \le \tau', \\ -\langle\zeta|e^{-(\tau-\tau')\widehat{\mathcal{H}}_\varphi}(1 - \mathcal{P}_\varphi)|\zeta'\rangle & \text{if } \tau > \tau', \end{cases} \tag{6.1}$$

where $\widehat{\mathcal{H}}_\varphi$ is given by (2.12), and $\mathcal{P}_\varphi$ is a projection operator defined by

$$\mathcal{P}_\varphi = \sum_{0 \le k \le N-1} |k\rangle\langle k|. \tag{6.2}$$

It is called the extended Hermite kernel in [96].

The trigonometric functions of the form

$$S_-(\sqrt{\lambda}x) = \frac{1}{\sqrt{2\pi}\lambda^{1/4}} \sin(\sqrt{\lambda}x),$$

$$S_+(\sqrt{\lambda}x) = \frac{1}{\sqrt{2\pi}\lambda^{1/4}} \cos(\sqrt{\lambda}x),$$

can be regarded as the generalized eigenfunctions of the Hamiltonian

$$\mathcal{H}_{\sin} = -\frac{\partial^2}{\partial x^2} \tag{6.3}$$

with spectrum $\lambda > 0$. Here S_- is an odd function (parity $\mathsf{p} = -$) and S_+ is an even function (parity $\mathsf{p} = +$), respectively. Consider an operator $\widehat{\mathcal{H}}_{\sin}$ such that $\langle x|\widehat{\mathcal{H}}_{\sin}|y\rangle = \delta(x-y)\mathcal{H}_{\sin}$ with (6.3) and introduce a set of its eigenvectors $\{|\lambda, \mathsf{p}; \sin\rangle : \lambda > 0, \mathsf{p} = \pm\}$;

$$\widehat{\mathcal{H}}_{\sin}|\lambda, \mathsf{p}; \sin\rangle = \lambda|\lambda, \mathsf{p}; \sin\rangle.$$

Since $\langle x|\lambda, \mathsf{p}; \sin\rangle = \langle \lambda; \mathsf{p}; \sin|x\rangle = S_{\mathsf{p}}(\sqrt{\lambda}x)$, we can confirm the completeness of the set

$$\sum_{\mathsf{p}=\pm} \int_0^\infty d\lambda \, \langle x|\lambda, \mathsf{p}; \sin\rangle\langle\lambda, \mathsf{p}; \sin|y\rangle = \frac{1}{2\pi} \int_{-\infty}^\infty du \, \cos(u(x-y)) = \delta(x-y).$$

If we change the variable in the Airy differential equation (5.9) by $x \to u - \lambda$, we have

$$-\frac{d^2}{dx^2}\mathrm{Ai}(x-\lambda) + x\,\mathrm{Ai}(x-\lambda) = \lambda\,\mathrm{Ai}(x-\lambda).$$

That is, the Hamiltonian

$$\mathcal{H}_{\mathrm{Ai}} = -\frac{\partial^2}{\partial x^2} + x \tag{6.4}$$

has $\mathbf{R}$ as spectrum and the Airy functions of the form $\mathrm{Ai}(x-\lambda)$ are its generalized eigenfunctions. We can consider the corresponding operator $\widehat{\mathcal{H}}_{\mathrm{Ai}}$ and its eigenvectors $\{|\lambda; \mathrm{Ai}\rangle : \lambda \in \mathbf{R}\}$,

$$\widehat{\mathcal{H}}_{\mathrm{Ai}}|\lambda; \mathrm{Ai}\rangle = \lambda|\lambda; \mathrm{Ai}\rangle,$$

where $\langle x|\lambda; \mathrm{Ai}\rangle = \langle\lambda; \mathrm{Ai}|x\rangle = \mathrm{Ai}(x-\lambda)$. We find the completeness

$$\int_{-\infty}^\infty d\lambda \, \langle x|\lambda; \mathrm{Ai}\rangle\langle\lambda; \mathrm{Ai}|y\rangle = \int_{-\infty}^\infty d\lambda \, \mathrm{Ai}(x-\lambda)\mathrm{Ai}(y-\lambda) = \delta(x-y).$$

Remark 3 The $2(\nu+1)$-dimensional squared Bessel process (BESQ$_\nu$) $Y^{(\nu)}(t)$ is defined as a solution of the SDE

$$dY^{(\nu)}(t) = 2\sqrt{Y^{(\nu)}(t)}dB(t) + 2(\nu+1)dt, \quad \nu > -1, \tag{6.5}$$

where $B(t)$ is a one-dimensional standard BM [18, 80]. Its forward and backward Kolmogorov (Fokker–Planck) equations are given as

$$\frac{\partial}{\partial t}u(t,x) = 2x\frac{\partial^2}{\partial x^2}u(t,x) \mp 2(\nu \mp 1)\frac{\partial}{\partial x}u(t,x), \tag{6.6}$$

where $-$ for the forward and $+$ for the backward equations, respectively. We will see that the Laguerre polynomials $\{L_k^\nu(x)\}$ play for BESQ$_\nu$ the similar role to the Hermite polynomials $\{H_k(x)\}$ for the BM shown in Sect. 2. By considering the noncolliding system of BESQ$_\nu$s instead of BMs [59], we can derive a finite determinantal process whose matrix-kernel is described using the orthonormal Laguerre functions (extended Laguerre kernel [96]). If we take the so-called hard-edge scaling limit, a spatially inhomogeneous infinite particle system is obtained, which is the determinantal process associated with the matrix-kernel

$$
\boldsymbol{K}^{(\nu)}(t, y; s, x) = \begin{cases}
\displaystyle\int_0^1 d\lambda\, e^{(s-t)\lambda} J_\nu(2\sqrt{\lambda y})\, J_\nu(2\sqrt{\lambda x}) & \text{if } t < s, \\[4mm]
\dfrac{J_\nu(2\sqrt{y})\sqrt{x}\, J_\nu'(2\sqrt{x}) - \sqrt{y}\, J_\nu'(2\sqrt{y})\, J_\nu(2\sqrt{x})}{y - x} & \text{if } t = s, \\[4mm]
\displaystyle -\int_1^\infty d\lambda\, e^{-(t-s)\lambda} J_\nu(2\sqrt{\lambda y})\, J_\nu(2\sqrt{\lambda x}) & \text{if } t > s,
\end{cases}
\qquad (6.7)
$$

where $J_\nu(x)$ is the Bessel function, $J_\nu(x) = \sum_{n=0}^\infty (-1)^n (x/2)^{2n+\nu}/\{\Gamma(n+1)\Gamma(n+1+\nu)\}$, and $J_\nu'(x) = dJ_\nu(x)/dx$. This kernel was called the extended Bessel kernel in [96]. See also [57, 76, 96, 98]. We can see that $J_\nu(2\sqrt{\lambda x})$ is the generalized eigenfunction of the Hamiltonian

$$
\mathcal{H}_J = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x} + \frac{\nu^2}{4x}, \qquad (6.8)
$$

with spectrum $\lambda \geq 0$. We introduce the corresponding operator $\widehat{\mathcal{H}}_J$ and its eigenvectors $\{|\lambda; J\rangle : \lambda \geq 0\}$, $\widehat{\mathcal{H}}_J |\lambda; J\rangle = \lambda |\lambda; J\rangle$, where $\langle x|\widehat{\mathcal{H}}_J|y\rangle = \delta(x-y)\mathcal{H}_J$, $\langle x|\lambda; J\rangle = \langle \lambda; J|x\rangle = J_\nu(2\sqrt{\lambda x})$ with the completeness

$$
\int_0^\infty d\lambda \langle x|\lambda; J\rangle\langle \lambda; J|y\rangle = \int_0^\infty d\lambda J_\nu(2\sqrt{\lambda x}) J_\nu(2\sqrt{\lambda y}) = \delta(x - y).
$$

Here we note the fact that the sine kernel $\boldsymbol{K}$ given by (5.5), the Airy kernel $\mathcal{K}$ given by (5.13), and the Bessel kernel given by (6.7) are expressed in the same way as (6.1) for the Hermite kernel $\widetilde{\mathbb{K}}_N$,

$$
K(t, y; s, x) = \begin{cases}
\langle y|e^{(s-t)\widehat{\mathcal{H}}}\mathcal{P}|x\rangle & \text{if } t \leq s, \\[2mm]
-\langle y|e^{-(t-s)\widehat{\mathcal{H}}}(1-\mathcal{P})|x\rangle & \text{if } t > s,
\end{cases}
$$

if we assign the Hamiltonian $\delta(x-y)\mathcal{H} = \langle x|\widehat{\mathcal{H}}|y\rangle$ and projection operator as follows instead of $\widehat{\mathcal{H}}_\varphi$ and (6.2);

(i) For the sine kernel, set $\mathcal{H} = \mathcal{H}_{\sin}$ with (6.3) and $\mathcal{P}_{\sin} = \displaystyle\sum_{p=\pm} \int_0^1 d\lambda\, |\lambda, p; \sin\rangle\langle\lambda, p; \sin|$.

(ii) For the Airy kernel, set $\mathcal{H} = \mathcal{H}_{\mathrm{Ai}}$ with (6.4) and $\mathcal{P}_{\mathrm{Ai}} = \displaystyle\int_{-\infty}^0 d\lambda\, |\lambda; \mathrm{Ai}\rangle\langle\lambda; \mathrm{Ai}|$.

(iii) For the Bessel kernel, set $\mathcal{H} = \mathcal{H}_J$ with (6.8) and $\mathcal{P}_J = \displaystyle\int_0^1 d\lambda\, |\lambda; J\rangle\langle\lambda; J|$.

6.2 Effective Hamiltonians and Matrix-Kernels

In the previous subsection, we claimed that the structure of the matrix-kernel of determinantal correlation functions is common both in finite particle systems and infinite particle systems. It should be noted, however, that even from the same finite system (e.g. noncolliding BM governed by the Hamiltonian $\mathcal{H}_\varphi$), depending on scaling limits, different kinds of infinite determinantal systems are derived, in which the Hamiltonian is replaced by appropriate *effective Hamiltonians* (*e.g.* $\mathcal{H}_{\sin}$ and $\mathcal{H}_{Ai}$) and spectral projection operator is modified.

In the present subsection, we give a possible general consideration on the common structure of determinantal processes. First we note the following fact for a general form of effective Hamiltonian $\mathcal{H} = -a(x)\partial^2/\partial x^2 - b(x)\partial/\partial x - c(x)$, where $a(x), b(x), c(x)$ are sufficiently smooth functions with $a(x) \neq 0$ in an interval $\Lambda \subset \mathbf{R}$. If we change the variable $x \mapsto z$ following

$$z(x) = \int_0^x a(y)^{-1/2} dy,$$

$\mathcal{H} \to \widetilde{\mathcal{H}} = -\partial^2/\partial z^2 - \widetilde{b}(z)\partial/\partial z - c(z)$ with $\widetilde{b}(z(x)) = a(x)z''(x) + b(x)z'(x) = a(x)^{-1/2}\{-a'(x)/2 + b(x)\}$. Then we define

$$r(z) = \exp\left\{-\frac{1}{2}\int_0^z \widetilde{b}(u)du\right\}$$

and if we perform a similarity transformation $\widetilde{\mathcal{H}} \to \overline{\mathcal{H}} = r^{-1}\widetilde{\mathcal{H}}r$, the term of first derivative can be eliminated and we have the form $\overline{\mathcal{H}} = -\partial^2/\partial z^2 + q(z)$. The transformation $\mathcal{H} \to \overline{\mathcal{H}}$ is called the Liouville transformation. Then, without loss of generality, we can assume effective Hamiltonians of the form (the Sturm–Liouville operator [93])

$$\mathcal{H} = -\frac{\partial^2}{\partial x^2} + q(x) \tag{6.9}$$

defined on $\Lambda \subset \mathbf{R}$.

Example 1 The effective Hamiltonians $\mathcal{H}_\varphi, \mathcal{H}_{\sin}, \mathcal{H}_{Ai}$, and $\mathcal{H}_J$ are transformed to the form (6.9) with the following $q(x)$, respectively

$$q(x): \quad \frac{1}{16}(x^2 - 4), \quad 0, \quad x, \quad \left(\nu^2 - \frac{1}{4}\right)\frac{1}{x^2}, \tag{6.10}$$

where $\Lambda = \mathbf{R}$ for the first three cases and $\Lambda = \mathbf{R}_+ \equiv \{x \in \mathbf{R} : x > 0\}$ for the last case.

Let $\widehat{\mathcal{H}}$ be the operator corresponding to (6.9); $\langle x|\widehat{\mathcal{H}}|y\rangle = \delta(x - y)\mathcal{H}$, and define

$$\delta_t(x, y) = \delta_t(y, x) = \langle y|e^{-t\widehat{\mathcal{H}}}|x\rangle. \tag{6.11}$$

By definition, it solves the equation

$$\frac{\partial}{\partial t}\delta_t(x, y) = -\mathcal{H}\delta_t(x, y) \tag{6.12}$$

with

$$\lim_{t\to 0}\delta_t(x, y) = \delta(x - y). \tag{6.13}$$

Assume that $\widehat{\mathcal{H}}$ has a distribution $\omega(d\lambda)$ of spectrum $\sigma(\widehat{\mathcal{H}}) = \{\lambda : \widehat{\mathcal{H}}|\lambda\rangle = \lambda|\lambda\rangle\}$ with the complete set of eigenvectors $\{|\lambda\rangle : \lambda \in \sigma(\widehat{\mathcal{H}})\}$. Then the spectral representation of $\delta_t(x, y)$ is given by

$$\delta_t(x, y) = \langle y|e^{-t\widehat{\mathcal{H}}} \int_{\sigma(\widehat{\mathcal{H}})} \omega(d\lambda)|\lambda\rangle\langle\lambda|x\rangle$$

$$= \int_{\sigma(\widehat{\mathcal{H}})} \omega(d\lambda)e^{-\lambda t}\Phi_\lambda(x)\Phi_\lambda(y)$$

with $\Phi_\lambda(x) = \langle x|\lambda\rangle = \langle\lambda|x\rangle$. We assume that

$$\exists C > 0 \quad \text{s.t.} \quad \int_\Lambda dy\,(x - y)^4\delta_t(x, y) \leq Ct^2, \quad t \in (0, 1], \quad \forall x \in \Lambda, \qquad (6.14)$$

where C does not depend of t.

Example 2 For the four examples (6.10), we have the following explicit expressions of $\delta_t(x, y)$;

$q(x) = \dfrac{1}{16}(x^2 - 4)$:

$$\delta_t(x, y) = \frac{1}{2}\sum_{n=0}^{\infty}\varphi_n\left(\frac{x}{2}\right)\varphi_n\left(\frac{y}{2}\right)e^{-nt/2}$$

$$= \frac{1}{2\sqrt{\pi(1 - e^{-t})}}\exp\left\{-\frac{1}{8}(x - y)^2\coth\left(\frac{t}{2}\right) - \frac{1}{4}xy\tanh\left(\frac{t}{4}\right)\right\},$$

$$x, y \in \Lambda = \mathbf{R},$$

$q(x) = 0$:

$$\delta_t(x, y) = \sum_{\mathsf{p}=\pm}\int_0^{\infty} d\lambda\,\langle y|e^{-t\widehat{\mathcal{H}}_{\sin}}|\lambda, \mathsf{p}; \sin\rangle\langle\lambda, \mathsf{p}; \sin|x\rangle$$

$$= \frac{1}{2\pi}\int_0^{\infty}\frac{d\lambda}{\sqrt{\lambda}}e^{-\lambda t}\{\cos(\sqrt{\lambda}x)\cos(\sqrt{\lambda}y) + \sin(\sqrt{\lambda}x)\sin(\sqrt{\lambda}y)\}$$

$$= \frac{1}{\pi}\int_0^{\infty} du\,e^{-u^2 t}\cos(u(x - y))$$

$$= \frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{(x - y)^2}{4t}\right\}, \quad x, y \in \Lambda = \mathbf{R},$$

$q(x) = x$:

$$\delta_t(x, y) = \int_{-\infty}^{\infty} \mathrm{Ai}(x - \lambda)\mathrm{Ai}(y - \lambda)e^{-\lambda t}d\lambda$$

$$= \frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{(x - y)^2}{4t} - \frac{t(x + y)}{2} + \frac{t^3}{12}\right\}, \quad x, y \in \Lambda = \mathbf{R},$$

$$q(x) = \left(\nu^2 - \frac{1}{4}\right)\frac{1}{x^2}:$$

$$\delta_t(x, y) = \frac{1}{2}\int_0^\infty \sqrt{x}\, J_\nu(\sqrt{\lambda}x)\sqrt{y}\, J_\nu(\sqrt{\lambda}y)e^{-\lambda t}\, d\lambda$$

$$= \frac{\sqrt{xy}}{2t}e^{-(x^2+y^2)/4t}I_\nu\left(\frac{xy}{2t}\right), \quad \nu > -1, \; x, y \in \Lambda = \mathbf{R}_+,$$

where $I_\nu(z)$ is the modified Bessel function given by $I_\nu(z) = \sum_{n=0}^\infty (z/2)^{2n+\nu}/ \{n!\Gamma(\nu + n + 1)\}$. We can confirm that (6.14) is satisfied in these four cases.

Now we consider a subset of spectrum $\sigma(\widehat{\mathcal{H}})$, $\sigma_-(\widehat{\mathcal{H}}) = \{\lambda \in \sigma(\widehat{\mathcal{H}}) : \lambda \leq \lambda_*\}$ with a specified level $\lambda_* \in \sigma(\widehat{\mathcal{H}})$, and define the projection operator onto $\sigma_-(\widehat{\mathcal{H}})$

$$\mathcal{P} = \int_{\sigma_-(\widehat{\mathcal{H}})} \omega(d\lambda)|\lambda\rangle\langle\lambda|. \tag{6.15}$$

so that

$$\mathcal{G}_t(x, y) = \mathcal{G}_t(y, x) = \langle y|e^{t\widehat{\mathcal{H}}}\mathcal{P}|x\rangle,$$
$$\overline{\mathcal{G}}_t(x, y) = \overline{\mathcal{G}}_t(y, x) = -\langle y|e^{-t\widehat{\mathcal{H}}}(1 - \mathcal{P})|x\rangle, \tag{6.16}$$

and

$$K(x, y) = \lim_{t \to 0}\mathcal{G}_t(x, y) = \lim_{t \to 0}\mathcal{G}_{-t}(x, y) = \langle y|\mathcal{P}|x\rangle, \tag{6.17}$$

$$\rho(x) = K(x, x) = \langle x|\mathcal{P}|x\rangle. \tag{6.18}$$

By definition

$$\overline{\mathcal{G}}_t(x, y) = \mathcal{G}_{-t}(x, y) - \delta_t(x, y), \tag{6.19}$$

$$\frac{\partial}{\partial t}\mathcal{G}_t(x, y) = \mathcal{H}\mathcal{G}_t(x, y),$$
$$\frac{\partial}{\partial t}\overline{\mathcal{G}}_t(x, y) = -\mathcal{H}\overline{\mathcal{G}}_t(x, y), \tag{6.20}$$

and

$$\lim_{t \to 0}\frac{\partial^2}{\partial y^2}\mathcal{G}_t(x, y) = \lim_{t \to 0}\frac{\partial^2}{\partial y^2}\mathcal{G}_{-t}(x, y) = \frac{\partial^2}{\partial y^2}K(x, y). \tag{6.21}$$

Moreover, by the completeness of $\{|x\rangle : x \in \Lambda\}$, $\int_\Lambda dx|x\rangle\langle x| = 1$, we have the relations

$$\int_\Lambda dy\, \delta_t(x, y)\mathcal{G}_t(y, x) = \rho(x), \tag{6.22}$$

$$\int_\Lambda dy\, \delta_t(x, y)\mathcal{G}_t(y, z) = K(x, z). \tag{6.23}$$

We define matrix-kernel K by

$$K(t, y; s, x) = \begin{cases} \mathcal{G}_{s-t}(y, x) & \text{if } t \leq s, \\ \overline{\mathcal{G}}_{t-s}(y, x) & \text{if } t > s, \end{cases} \qquad (6.24)$$

and consider the determinantal process $\Xi(t) = \{\mathbf{X}(t)\}$, whose multitime correlation function is given by

$$\rho(0, \xi_{N_0}^{(0)}; t_1, \xi_{N_1}^{(1)}; \ldots; t_M, \xi_{N_M}^{(M)}) = \det_{1 \leq j \leq N_m, 1 \leq k \leq N_n, 0 \leq m, n \leq M} [K(t_m, x_j^{(m)}; t_n, x_k^{(n)})] \qquad (6.25)$$

for any $M \geq 0$, any sequence $\{N_m\}_{m=0}^{M}$ of positive integers, and any series of observation times $0 < t_1 < \cdots < t_M$.

The invariant measure of the process is the determinantal point field μ associated with $K(x, y)$ given by (6.17).

The two-time correlation function $\rho(0, \{\boldsymbol{x}_m\}; t, \{\boldsymbol{y}_n\})$ of the system is given by

$$\rho(0, \{\boldsymbol{x}_m\}; t, \{\boldsymbol{y}_n\}) = \det \overline{\mathbf{M}}(t, \boldsymbol{y}_n | \boldsymbol{x}_m), \quad t > 0, \; m, n \geq 0,$$

where

$$\overline{\mathbf{M}}(t, \boldsymbol{y}_n | \boldsymbol{x}_m) = \begin{pmatrix} \rho(x_1) & \cdots & K(x_1, x_m) & \mathcal{G}_t(x_1, y_1) & \cdots & \mathcal{G}_t(x_1, y_n) \\ K(x_2, x_1,) & \cdots & K(x_2, x_m) & \mathcal{G}_t(x_2, y_1) & \cdots & \mathcal{G}_t(x_2, y_n) \\ & \cdots & & & \cdots & \\ K(x_m, x_1) & \cdots & \rho(x_m) & \mathcal{G}_t(x_m, y_1) & \cdots & \mathcal{G}_t(x_m, y_n) \\ \overline{\mathcal{G}}_t(y_1, x_1) & \cdots & \overline{\mathcal{G}}_t(y_1, x_m) & \rho(y_1) & \cdots & K(y_1, y_n) \\ \overline{\mathcal{G}}_t(y_2, x_1) & \cdots & \overline{\mathcal{G}}_t(y_2, x_m) & K(y_2, y_1) & \cdots & K(y_2, y_n) \\ & \cdots & & & \cdots & \\ \overline{\mathcal{G}}_t(y_n, x_1) & \cdots & \overline{\mathcal{G}}_t(y_n, x_m) & K(y_n, y_1) & \cdots & \rho(y_n) \end{pmatrix}. \qquad (6.26)$$

For a matrix $A = (a_{ij})_{i \in I, j \in J}$ with index sets I, J, we denote its submatrix as $A_{I'J'} \equiv (a_{ij})_{i \in I', j \in J'}$ for $I' \subset I$, $J' \subset J$, and the complementary submatrix as $A^{I'J'} \equiv (a_{ij})_{i \in I \setminus I', j \in J \setminus J'}$. By using the relation (6.19), $\rho(0, \{\boldsymbol{x}_m\}; t, \{\boldsymbol{y}_n\})$ is expanded as

$$\rho(0, \{\boldsymbol{x}_m\}; t, \{\boldsymbol{y}_n\}) = \det \mathbf{M}(t, \boldsymbol{y}_n | \boldsymbol{x}_m)$$

$$+ \sum_{\ell=1}^{m \wedge n} \sum_{1 \leq a_1 < \cdots < a_\ell \leq m} \sum_{1 \leq b_1 < \cdots < b_\ell \leq n} (-1)^{\ell + \sum_{i=1}^{\ell}(a_i + m + b_i)}$$

$$\times \det \mathbf{D}(t, \boldsymbol{y}_n | \boldsymbol{x}_m)_{\{a_\ell\}\{b_\ell\}} \det \mathbf{M}(t, \boldsymbol{y}_n | \boldsymbol{x}_m)^{\{m+b_\ell\}\{a_\ell\}}, \qquad (6.27)$$

where

$$\mathbf{M}(t, \boldsymbol{y}_n | \boldsymbol{x}_m) = \begin{pmatrix} \rho(x_1) & \cdots & K(x_1, x_m) & \mathcal{G}_t(x_1, y_1) & \cdots & \mathcal{G}_t(x_1, y_n) \\ K(x_2, x_1) & \cdots & K(x_2, x_m) & \mathcal{G}_t(x_2, y_1) & \cdots & \mathcal{G}_t(x_2, y_n) \\ & \cdots & & & \cdots & \\ K(x_m, x_1) & \cdots & \rho(x_m) & \mathcal{G}_t(x_m, y_1) & \cdots & \mathcal{G}_t(x_m, y_n) \\ \mathcal{G}_{-t}(y_1, x_1) & \cdots & \mathcal{G}_{-t}(y_1, x_m) & \rho(y_1) & \cdots & K(y_1, y_n) \\ \mathcal{G}_{-t}(y_2, x_1) & \cdots & \mathcal{G}_{-t}(y_2, x_m) & K(y_2, y_1) & \cdots & K(y_2, y_n) \\ & \cdots & & & \cdots & \\ \mathcal{G}_{-t}(y_n, x_1) & \cdots & \mathcal{G}_{-t}(y_n, x_m) & K(y_n, y_1) & \cdots & \rho(y_n) \end{pmatrix},$$

$$\mathbf{D}(t, \boldsymbol{y}_n | \boldsymbol{x}_m) = (\delta_t(x_i, y_j))_{1 \leq i \leq m, 1 \leq j \leq n}.$$

It was shown by Shirai and Takahashi [84] that the Palm measure μ^z coincides with the determinantal point field associated with the kernel K^z defined by

$$K^z(x, y) = \frac{1}{K(z, z)} \det \begin{pmatrix} K(x, y) & K(x, z) \\ K(z, y) & K(z, z) \end{pmatrix}.$$

Note that

$$\rho^z(\{\boldsymbol{x}_m\}) = \frac{1}{\rho(z)} \rho(\{\boldsymbol{x}_m\} \cup \{z\}). \tag{6.28}$$

For $f, g \in C_0(\mathbf{R})$, let

$$\langle f, \xi \rangle = \sum_{x \in \xi} f(x) \quad \text{and} \quad \langle g, \eta \rangle = \sum_{y \in \eta} g(y), \tag{6.29}$$

and set $\alpha, \beta \in \mathbf{C}$. Define $F(\xi) = e^{\alpha \langle f, \xi \rangle}$, $G(\xi) = e^{\beta \langle g, \xi \rangle}$. Then the two-time generating function is defined by

$$\Phi_t(f, g; \alpha, \beta) \equiv \mathbf{E}[F(\Xi(0)) G(\Xi(t))] = \mathbf{E}[e^{\alpha \langle f, \Xi(0) \rangle + \beta \langle g, \Xi(t) \rangle}], \tag{6.30}$$

for $t > 0$. Let

$$\chi_1(x) = e^{\alpha f(x)} - 1, \qquad \chi_2(y) = e^{\beta g(y)} - 1,$$

then we can see

$$\Phi_t(f, g; \alpha, \beta)$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{j=1}^{m} \chi_1(x_j) \prod_{k=1}^{n} \chi_2(y_k) \rho(0, \{\boldsymbol{x}_m\}; t, \{\boldsymbol{y}_n\}). \tag{6.31}$$

7 Characterization of Determinantal Processes

In the previous section, we introduced a class of determinantal processes associated with spectral projections defined by effective Hamiltonians. Here we give properties of determinantal processes of this class, which can be derived from the common structure of correlation functions.

7.1 Continuity

Let $C_0^{\infty}(\mathbf{R})$ be the set of all infinitely differentiable real functions with compact supports and set $\langle f, \xi \rangle \equiv \sum_{x \in \xi} f(x)$ for $f \in C_0^{\infty}(\mathbf{R})$, $\xi \in \mathfrak{X}$. By a criterion of Kolmogorov (see, for example, [12, 47]), the following lemma implies that $\sum_{x \in \Xi(t)} f(x)$ is continuous in time with probability one for any $f \in C_0^{\infty}(\mathbf{R})$. Since $\mathfrak{X}$ is separable with the vague topology, we can choose a countable set $\{f_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\mathbf{R})$ such that $\xi_n \to \xi$ in $n \to \infty$ on $\mathfrak{X}$ if and only if $\langle \xi_n, f_j \rangle \to \langle \xi, f_j \rangle$ in $n \to \infty$, $\forall j \geq 1$. Then it implies that the determinantal process $\Xi(t)$ is continuous in the vague topology with probability one. That is, if the condition (6.14) is satisfied, it can be expressed by

$$\Xi(t) = \sum_{j=1}^{\infty} \delta_{X_j(t)}$$

with some real-valued continuous processes $X_j(t)$, $j \in \mathbf{N}$.

Lemma 7.1 *Let* $\Xi(t)$ *be the determinantal process, whose multitime correlation functions are given by* (6.25) *with* (6.16) *and* (6.24) *associated with an effective Hamiltonian* (6.9) *defined on* $\Lambda \subset \mathbf{R}$. *Assume that* (6.14) *is satisfied. Then for any* $f \in C_0^\infty(\mathbf{R})$

$$\mathbf{E}[|\langle f, \Xi(t)\rangle - \langle f, \Xi(0)\rangle|^4] \leq Ct^2, \quad t \in (0, 1], \tag{7.1}$$

where C does not depend on t.

Remark 4 Theorems 5.1 and 5.2 are the limit theorems of the processes

$$\Xi_N^{\text{bulk}}(t) \equiv \{X_1(N + 2t), \ldots, X_N(N + 2t)\} \Longrightarrow \Xi^{\sin}(t),$$

$$\Xi_N^{\text{edge}}(t) \equiv \{X_1(N^{1/3} + 2t) - a_N(t), \ldots, X_N(N^{1/3} + 2t) - a_N(t)\} \Longrightarrow \Xi^{\text{Ai}}(t), \quad N \to \infty$$

with $a_N(t)$ defined by (5.11), in the sense of finite-dimensional distributions, where the multitime correlation functions of $\Xi^{\sin}(t)$ and $\Xi^{\text{Ai}}(t)$ are given by $\rho_{\sin}$ of (5.4) and ρ_{Ai} of (5.12), respectively. When we apply this lemma to the N-particle system of noncolliding BM, $\Xi_N^{\mathbf{X}}(t)$, which is a determinantal process associated with the Hermite kernel $\widetilde{\mathbb{K}}_N$ given by (6.1), C depends on N. Careful estimation gives an upper bound on C, which is uniform in N, and tightness is established both in $\{\Xi_N^{\text{bulk}}(t)\}_{N\in\mathbf{N}}$ and $\{\Xi_N^{\text{edge}}(t)\}_{N\in\mathbf{N}}$.

Proof of Lemma 7.1 Here we use the notation

$$\rho_{m,n}(\boldsymbol{x}_{m+n}) = \rho(0, \{x_1, \ldots, x_m\}; t, \{x_{m+1}, \ldots, x_{m+n}\}).$$

The left-hand-side of (7.1) is given by $\partial^4 \Phi_t(-f, f; \alpha, \alpha)/\partial^4\alpha|_{\alpha=0}$ from the two-time generating function given by (6.30) and (6.31), which equals

$$\int_{\Lambda^4} d\boldsymbol{x}_4 \prod_{i=1}^4 f(x_i)\{\rho_{4,0}(\boldsymbol{x}_4) - 4\rho_{3,1}(\boldsymbol{x}_4) + 6\rho_{2,2}(\boldsymbol{x}_4) - 4\rho_{1,3}(\boldsymbol{x}_4) + \rho_{0,4}(\boldsymbol{x}_4)\}$$

$$+ \int_{\Lambda^3} d\boldsymbol{x}_3 \prod_{i=1}^3 f(x_i)\left\{2\sum_{j=1}^3 f(x_j)\rho_{3,0}(\boldsymbol{x}_3) + 6(-f(x_1) - f(x_2) + f(x_3))\rho_{2,1}(\boldsymbol{x}_3)\right.$$

$$\left. + 6(f(x_1) - f(x_2) - f(x_3))\rho_{1,2}(\boldsymbol{x}_3) + 2\sum_{j=1}^3 f(x_j)\rho_{0,3}(\boldsymbol{x}_3)\right\}$$

$$+ \int_{\Lambda^2} d\boldsymbol{x}_2 \prod_{i=1}^2 f(x_i)\{(2f(x_1)^2 + 3f(x_1)f(x_2) + 2f(x_2)^2)(\rho_{2,0}(\boldsymbol{x}_2) + \rho_{0,2}(\boldsymbol{x}_2))$$

$$+ (-4f(x_1)^2 + 6f(x_1)f(x_2) - 4f(x_2)^2)\rho_{1,1}(\boldsymbol{x}_2)\} + \int_\Lambda dx_1 f(x_1)^4\{\rho_{1,0}(x_1) + \rho_{0,1}(x_1)\}.$$

Since $\rho_{3,0}(x_1, x_2, x_3) = \rho_{0,3}(x_1, x_2, x_3)$, $\rho_{2,1}(x_1, x_2, x_3) = \rho_{1,2}(x_3, x_2, x_1)$, etc., the above

quantity is twice of

$$I \equiv \int_{\Lambda^4} d\boldsymbol{x}_4 \prod_{i=1}^{4} f(x_i) \{\rho_{4,0}(\boldsymbol{x}_4) - 4\rho_{3,1}(\boldsymbol{x}_4) + 3\rho_{2,2}(\boldsymbol{x}_4)\}$$

$$+ \int_{\Lambda^3} d\boldsymbol{x}_3 \prod_{i=1}^{3} f(x_i) \left\{ 2\sum_{j=1}^{3} f(x_j)\rho_{3,0}(\boldsymbol{x}_3) + 6(-f(x_1) - f(x_2) + f(x_3))\rho_{2,1}(\boldsymbol{x}_3) \right\}$$

$$+ \int_{\Lambda^2} d\boldsymbol{x}_2 \prod_{i=1}^{2} f(x_i) \{(2f(x_1)^2 + 3f(x_1)f(x_2) + 2f(x_2)^2)\rho_{2,0}(\boldsymbol{x}_2)$$

$$+ (-2f(x_1)^2 + 3f(x_1)f(x_2) - 2f(x_2)^2)\rho_{1,1}(\boldsymbol{x}_2)\}$$

$$+ \int_{\Lambda} dx_1 \, f(x_1)^4 \rho_{1,0}(x_1).$$

We put $\mathbf{M}_{m,n}(\boldsymbol{x}_{m+n}) = \mathbf{M}(t, x_{m+1}, \ldots, x_{m+n}|x_1, \ldots, x_m)$, $\mathbf{D}(\boldsymbol{x}_n) = \mathbf{D}(t, \boldsymbol{x}_n|\boldsymbol{x}_n)$ and

$$D_{1,1}(\boldsymbol{x}_2) = -\delta_t(x_2, x_1)\mathcal{G}_t(x_1, x_2),$$

$$D_{2,1}(\boldsymbol{x}_3) = \sum_{i=1}^{2} (-1)^{i+3} \delta_t(x_3, x_i) \det \mathbf{M}_{2,1}(\boldsymbol{x}_3)^{\{3\}\{i\}},$$

$$D_{3,1}(\boldsymbol{x}_4) = \sum_{i=1}^{3} (-1)^{i+4} \delta_t(x_4, x_i) \det \mathbf{M}_{3,1}(\boldsymbol{x}_4)^{\{4\}\{i\}},$$

$$D_{2,2}(\boldsymbol{x}_4) = \sum_{i=1}^{2} \sum_{j=3}^{4} (-1)^{i+j} \delta_t(x_j, x_i) \det \mathbf{M}_{2,2}(\boldsymbol{x}_4)^{\{j\}\{i\}},$$

$$\widehat{D}_{2,2}(\boldsymbol{x}_4) = \det \mathbf{D}(\boldsymbol{x}_4)_{\{3,4\}\{1,2\}} \det \mathbf{M}_{2,2}(\boldsymbol{x}_4)^{\{3,4\}\{1,2\}}.$$

From (6.27) we have

$$\rho_{1,0}(x_1) = \det \mathbf{M}_{1,0}(x_1),$$

$$\rho_{1,1}(\boldsymbol{x}_2) = \det \mathbf{M}_{1,1}(\boldsymbol{x}_2) - D_{1,1}(\boldsymbol{x}_2),$$

$$\rho_{2,1}(\boldsymbol{x}_3) = \det \mathbf{M}_{2,1}(\boldsymbol{x}_3) - D_{2,1}(\boldsymbol{x}_3),$$

$$\rho_{3,1}(\boldsymbol{x}_4) = \det \mathbf{M}_{3,1}(\boldsymbol{x}_4) - D_{3,1}(\boldsymbol{x}_4),$$

$$\rho_{2,2}(\boldsymbol{x}_4) = \det \mathbf{M}_{2,2}(\boldsymbol{x}_4) - D_{2,2}(\boldsymbol{x}_4) + \widehat{D}_{2,2}(\boldsymbol{x}_4).$$

We divide I into four terms $I = \sum_{j=1}^{4} I_j$ with

$$I_1 \equiv \int_{\Lambda} dx_1 \, f(x_1)^4 \det \mathbf{M}_{1,0}(x_1)$$

$$- \int_{\Lambda^2} d\boldsymbol{x}_2 \prod_{i=1}^{2} f(x_i)(-2f(x_1)^2 + 3f(x_1)f(x_2) - 2f(x_2)^2)D_{1,1}(\boldsymbol{x}_2),$$

$$I_2 \equiv \int_{\Lambda^2} d\boldsymbol{x}_2 \prod_{i=1}^{2} f(x_i)\{(2f(x_1)^2 + 3f(x_1)f(x_2) + 2f(x_2)^2)\det\mathbf{M}_{2,0}(\boldsymbol{x}_2)$$

$$+ (-2f(x_1)^2 + 3f(x_1)f(x_2) - 2f(x_2)^2)\det\mathbf{M}_{1,1}(\boldsymbol{x}_2)\}$$

$$- \int_{\Lambda^3} d\boldsymbol{x}_3 \prod_{i=1}^{3} f(x_i)6(-f(x_1) - f(x_2) + f(x_3))\mathrm{D}_{2,1}(\boldsymbol{x}_3)$$

$$+ \int_{\Lambda^4} d\boldsymbol{x}_4 \prod_{i=1}^{4} f(x_i)3\widehat{\mathrm{D}}_{2,2}(\boldsymbol{x}_4),$$

$$I_3 \equiv \int_{\Lambda^3} d\boldsymbol{x}_3 \prod_{i=1}^{3} f(x_i)\left\{2\sum_{j=1}^{3} f(x_j)\det\mathbf{M}_{3,0}(\boldsymbol{x}_3)\right.$$

$$\left. + 6(-f(x_1) - f(x_2) + f(x_3))\det\mathbf{M}_{2,1}(\boldsymbol{x}_3)\right\}$$

$$- \int_{\Lambda^4} d\boldsymbol{x}_4 \prod_{i=1}^{4} f(x_i)\{-4\mathrm{D}_{3,1}(\boldsymbol{x}_4) + 3\mathrm{D}_{2,2}(\boldsymbol{x}_4)\},$$

$$I_4 \equiv \int_{\Lambda^4} d\boldsymbol{x}_4 \prod_{i=1}^{4} f(x_i)\{\det\mathbf{M}_{4,0}(\boldsymbol{x}_4) - 4\det\mathbf{M}_{3,1}(\boldsymbol{x}_4) + 3\det\mathbf{M}_{2,2}(\boldsymbol{x}_4)\}.$$

Using (6.22) we have

$$I_1 = -2\int_{\Lambda^2} f(x_1)f(x_2)^3\mathcal{G}_t(x_1, x_2)\delta_t(x_1, x_2)dx_1dx_2$$

$$- 2\int_{\Lambda^2} f(x_1)^3 f(x_2)\mathcal{G}_t(x_1, x_2)\delta_t(x_1, x_2)dx_1dx_2$$

$$+ 3\int_{\Lambda^2} f(x_1)^2 f(x_2)^2\mathcal{G}_t(x_1, x_2)\delta_t(x_1, x_2)dx_1dx_2$$

$$+ \int_{\Lambda} f(x_1)^4\rho(x_1)dx_1$$

$$= \frac{1}{2}\int_{\Lambda^2} \{f(x_1) - f(x_2)\}^4\mathcal{G}_t(x_1, x_2)\delta_t(x_1, x_2)dx_1dx_2.$$

By the assumption (6.14) we have

$$I_1 \le C_1 t^2, \quad t \in (0, 1], \tag{7.2}$$

where C_1 does not depend on t.

Since $\mathcal{G}_t(x, y) = K(x, y) + \partial\mathcal{G}_t(x, y)/\partial t|_{t=0} t + \mathcal{O}(t^2)$, we have for any $k, \ell \in \mathbf{N}$

$$\det\mathbf{M}_{k,\ell}(\boldsymbol{x}_{k+\ell}) = \det\mathbf{M}_{k-1,\ell+1}(\boldsymbol{x}_{k+\ell}) + \mathcal{O}(t^2).$$

Then

$$I_j = \tilde{I}_j + \mathcal{O}(t^2), \quad j = 2, 3, 4,$$

where

$$\tilde{I}_2 = \int_{\Lambda^2} d\boldsymbol{x}_2 \prod_{i=1}^{2} f(x_i) 6 f(x_1) f(x_2) \det \mathbf{M}_{2,0}(\boldsymbol{x}_2)$$

$$- \int_{\Lambda^3} d\boldsymbol{x}_3 \prod_{i=1}^{3} f(x_i) 6(-f(x_1) - f(x_2) + f(x_3)) \mathrm{D}_{2,1}(\boldsymbol{x}_3)$$

$$+ \int_{\Lambda^4} d\boldsymbol{x}_4 \prod_{i=1}^{4} f(x_i) 3 \widehat{\mathrm{D}}_{2,2}(\boldsymbol{x}_4),$$

$$\tilde{I}_3 = - \int_{\Lambda^4} d\boldsymbol{x}_4 \prod_{i=1}^{4} f(x_i) \{-4\mathrm{D}_{3,1}(\boldsymbol{x}_4) + 3\mathrm{D}_{2,2}(\boldsymbol{x}_4)\},$$

and $\tilde{I}_4 = 0$. Since the estimate (7.2) was obtained, it is enough to show that

$$\tilde{I}_j \le C_j t^2, \quad t \in (0, 1], \ j = 2, 3 \tag{7.3}$$

for the proof, where C_j's do not depend on t. Using (6.22) and (6.23), we obtain

$$\tilde{I}_2 = \int_{\Lambda^4} d\boldsymbol{x}_4 \begin{vmatrix} \mathcal{G}_t(x_4, x_1) & \mathcal{G}_t(x_4, x_2) \\ \mathcal{G}_t(x_3, x_1) & \mathcal{G}_t(x_3, x_2) \end{vmatrix} \left\{ 6 f(x_1)^2 f(x_2)^2 \delta_t(x_1, x_4) \delta_t(x_2, x_3) \right.$$

$$+ 3 f(x_1) f(x_2) f(x_3) f(x_4) \begin{vmatrix} \delta_t(x_4, x_1) & \delta_t(x_4, x_2) \\ \delta_t(x_3, x_1) & \delta_t(x_3, x_2) \end{vmatrix}$$

$$\left. + 6 f(x_1) f(x_2) f(x_3) (-f(x_1) - f(x_2) + f(x_3)) \begin{vmatrix} \delta_t(x_4, x_1) & \delta_t(x_4, x_2) \\ \delta_t(x_3, x_1) & \delta_t(x_3, x_2) \end{vmatrix} \right\}$$

$$= \int_{\Lambda^4} d\boldsymbol{x}_4 \begin{vmatrix} \mathcal{G}_t(x_4, x_1) & \mathcal{G}_t(x_4, x_2) \\ \mathcal{G}_t(x_3, x_1) & \mathcal{G}_t(x_3, x_2) \end{vmatrix} \begin{vmatrix} \delta_t(x_4, x_1) & \delta_t(x_4, x_2) \\ \delta_t(x_3, x_1) & \delta_t(x_3, x_2) \end{vmatrix} 3F(\boldsymbol{x}_4),$$

where

$$F(\boldsymbol{x}_4) = f(x_1) f(x_2) \{f(x_1) f(x_2) + f(x_3) f(x_4) - 2 f(x_1) f(x_3) - 2 f(x_2) f(x_3) + 2 f(x_3)^2\}.$$

By simple calculation we see that

$$F(x_1, x_2, x_3, x_4) + F(x_1, x_2, x_4, x_3) + F(x_3, x_4, x_1, x_2) + F(x_3, x_4, x_2, x_1)$$

$$= -2(f(x_1) - f(x_3))(f(x_3) - f(x_2))(f(x_2) - f(x_4))(f(x_4) - f(x_1)).$$

Then

$$\int_{\Lambda^4} d\boldsymbol{x}_4 \begin{vmatrix} \mathcal{G}_t(x_4, x_1) & \mathcal{G}_t(x_4, x_2) \\ \mathcal{G}_t(x_3, x_1) & \mathcal{G}_t(x_3, x_2) \end{vmatrix} \begin{vmatrix} \delta_t(x_4, x_1) & \delta_t(x_4, x_2) \\ \delta_t(x_3, x_1) & \delta_t(x_3, x_2) \end{vmatrix} F(\boldsymbol{x}_4)$$

$$= \int_{\Lambda^4} d\boldsymbol{x}_4 \delta_t(x_4, x_1) \delta_t(x_3, x_2)(f(x_1) - f(x_4))(f(x_2) - f(x_3))$$

$$\times \begin{vmatrix} \mathcal{G}_t(x_4, x_1) & \mathcal{G}_t(x_4, x_2) \\ \mathcal{G}_t(x_3, x_1) & \mathcal{G}_t(x_3, x_2) \end{vmatrix} (f(x_1) - f(x_3))(f(x_2) - f(x_4))$$

$$= \int_{\Lambda^4} d\boldsymbol{x}_4 \, \delta_t(x_4, x_1)\delta_t(x_3, x_2)(f(x_1) - f(x_4))(f(x_2) - f(x_3))$$

$$\times \{(\mathcal{G}_t(x_4, x_1)\mathcal{G}_t(x_3, x_2) - \mathcal{G}_t(x_1, x_2)\mathcal{G}_t(x_3, x_4))(f(x_1) - f(x_4))(f(x_2) - f(x_3))$$

$$- \mathcal{G}_t(x_4, x_1)\mathcal{G}_t(x_3, x_2)(f(x_1)f(x_4) + f(x_2)f(x_3))$$

$$+ \mathcal{G}_t(x_2, x_4)\mathcal{G}_t(x_1, x_3)(f(x_1)f(x_4) + f(x_2)f(x_3))\}$$

$$= \int_{\Lambda^4} d\boldsymbol{x}_4 \delta_t(x_4, x_1)\delta_t(x_3, x_2)(f(x_1) - f(x_4))(f(x_2) - f(x_3))$$

$$\times \{(\mathcal{G}_t(x_4, x_1)\mathcal{G}_t(x_3, x_2) - \mathcal{G}_t(x_1, x_2)\mathcal{G}_t(x_3, x_4))(f(x_1) - f(x_4))(f(x_2) - f(x_3))$$

$$+ (\mathcal{G}_t(x_2, x_4)\mathcal{G}_t(x_1, x_3) - \mathcal{G}_t(x_2, x_1)\mathcal{G}_t(x_4, x_3))(f(x_1)f(x_4) + f(x_2)f(x_3))\}$$

$$= \int_{\Lambda^4} d\boldsymbol{x}_4 \, \delta_t(x_4, x_1)\delta_t(x_3, x_2)(f(x_1) - f(x_4))(f(x_2) - f(x_3))$$

$$\times \Bigg[(\mathcal{G}_t(x_4, x_1)\mathcal{G}_t(x_3, x_2) - \mathcal{G}_t(x_1, x_2)\mathcal{G}_t(x_3, x_4))(f(x_1) - f(x_4))(f(x_2) - f(x_3))$$

$$+ \left(\frac{\partial}{\partial x_1}\mathcal{G}_t(x_1, x_2)\frac{\partial}{\partial x_2}\mathcal{G}_t(x_1, x_2) - \mathcal{G}_t(x_2, x_1)\frac{\partial^2}{\partial x_1 \partial x_2}\mathcal{G}_t(x_1, x_2) \right)$$

$$\times (f(x_1)f(x_4) + f(x_2)f(x_3))(x_4 - x_1)(x_3 - x_2) \Bigg] + \mathcal{O}(t^2).$$

By the assumption (6.14) we obtain (7.3) for $j = 2$.

Since

$$(-1)^{1+4}\delta_t(x_1, x_4) \det \mathbf{M}_{3,1}(x_1, x_2, x_3, x_4)^{\{1\}\{4\}}$$

$$= (-1)^{2+4}\delta_t(x_2, x_4) \det \mathbf{M}_{3,1}(x_2, x_1, x_3, x_4)^{\{2\}\{4\}}$$

$$= (-1)^{3+4}\delta_t(x_3, x_4) \det \mathbf{M}_{3,1}(x_3, x_2, x_1, x_4)^{\{3\}\{4\}},$$

$$(-1)^{1+4}\delta_t(x_1, x_4) \det \mathbf{M}_{2,2}(x_1, x_2, x_3, x_4)^{\{1\}\{4\}}$$

$$= (-1)^{1+3}\delta_t(x_1, x_3) \det \mathbf{M}_{2,2}(x_1, x_2, x_4, x_3)^{\{1\}\{3\}}$$

$$= (-1)^{2+3}\delta_t(x_2, x_3) \det \mathbf{M}_{2,2}(x_2, x_1, x_4, x_3)^{\{2\}\{3\}}$$

$$= (-1)^{2+4}\delta_t(x_2, x_4) \det \mathbf{M}_{2,2}(x_2, x_1, x_3, x_4)^{\{2\}\{4\}},$$

we have

$$\tilde{I}_3 = 12 \int_{\Lambda^4} d\boldsymbol{x}_4 \prod_{i=1}^4 f(x_i)\delta_t(x_1, x_4) \left[- \begin{vmatrix} K(x_2, x_1) & \rho(x_2) & K(x_2, x_3) \\ K(x_3, x_1) & K(x_3, x_2) & \rho(x_3) \\ \mathcal{G}_t(x_4, x_1) & \mathcal{G}_t(x_4, x_2) & \mathcal{G}_t(x_4, x_3) \end{vmatrix} \right.$$

$$\left. + \begin{vmatrix} K(x_2, x_1) & \rho(x_2) & \mathcal{G}_{-t}(x_2, x_3) \\ \mathcal{G}_t(x_3, x_1) & \mathcal{G}_t(x_3, x_2) & \rho(x_3) \\ \mathcal{G}_t(x_4, x_1) & \mathcal{G}_t(x_4, x_2) & K(x_4, x_3) \end{vmatrix} \right].$$

Then we have (7.3) for $j = 3$. This completes the proof. $\qquad \square$

7.2 Bilinear Forms

Since the Fredholm determinant of the dual operator coincides with that of the original operator, the reversibility (and also the stationarity) of the present determinantal processes is guaranteed.

Osada constructed $\mathfrak{X}$-valued reversible processes, which have determinantal point fields as their reversible stationary measures by Dirichlet form approach. With the Palm measure μ^z the Dirichlet form of Osada can be written as

$$\mathcal{E}(F, G) = \int_\Lambda dz \rho(z) \int_{\mathfrak{X}} \mu^z(d\eta) \frac{\partial}{\partial z} F(\{z\} \cup \eta) \frac{\partial}{\partial z} G(\{z\} \cup \eta) \qquad (7.4)$$

for local smooth functions F, G [76] (see also [88]). By the general theory of Dirichlet forms [38], his processes are diffusion processes (i.e., continuous strong Markov processes). For our class of determinantal processes, we found the following fact (Proposition 7.2). It suggests that our determinantal processes are identified with Osada's processes.

A function F on the configuration space $\mathfrak{X}$ is said to be polynomial, if it is written of the form $F(\xi) = \widetilde{F}(\langle f_1, \xi \rangle, \langle f_2, \xi \rangle, \ldots, \langle f_k, \xi \rangle)$ with a polynomial function $\widetilde{F}$ on $\mathbf{R}^k$, $k \in \mathbf{N}_0$, where $f_j \in C_0^\infty(\mathbf{R})$, $1 \leq j \leq k$. Let $\wp$ be the set of all polynomial functions on $\mathfrak{X}$, which is a dense subset of $L^2(\mathfrak{X}, \mu)$; the space of square integrable functions on $\mathfrak{X}$ with the determinantal point field μ.

Proposition 7.2 *Let* $\Xi(t)$ *be the determinantal process, whose multitime correlation functions are given by* (6.25). *Then for* $F, G \in \wp$ *we have*

$$-\frac{\partial}{\partial t} \mathbf{E}[F(\Xi(0)) G(\Xi(t))] \Big|_{t=0} = \mathcal{E}(F, G), \qquad (7.5)$$

where $\mathcal{E}$ *is given by* (7.4).

Markov property of determinantal processes has been studied by Borodin and Olshanski [14, 16]. But we can not prove that the infinite determinantal processes in our class are Markovian. The equivalence between our processes and Osada's is not yet established.

To show this proposition, it is enough to consider the case that F and G are of the form

$$F(\xi) = \exp\left(\sum_{m=1}^k \alpha_m \sum_{x \in \xi} f_m(x)\right), \qquad G(\xi) = \exp\left(\sum_{m=1}^k \beta_m \sum_{x \in \xi} g_m(x)\right)$$

with $k \in \mathbf{N}_0$, $\alpha_m, \beta_m \in \mathbf{R}$, $f_m, g_m \in C_0^\infty(\mathbf{R})$, $1 \leq m \leq k$. Then if we set $\chi_1(x) = \exp(\sum_{m=1}^k \alpha_m f_m(x)) - 1$ and $\chi_2(x) = \exp(\sum_{m=1}^k \beta_m g_m(x)) - 1$,

$$F(\xi) = \prod_{x \in \xi}(1 + \chi_1(x)), \qquad G(\xi) = \prod_{x \in \xi}(1 + \chi_2(x)).$$

The left-hand-side of (7.5) equals

$$-\frac{\partial}{\partial t} \Phi_t(\{f_m\}, \{g_m\}; \{\alpha_m\}, \{\beta_m\}) \Big|_{t=0}$$

$$= -\sum_{m=0}^\infty \sum_{n=0}^\infty \frac{1}{m!n!} \int_{\Lambda^m} d\boldsymbol{x} \int_{\Lambda^n} d\boldsymbol{y} \prod_{j=1}^m \chi_1(x_j) \prod_{k=1}^n \chi_2(y_k) \frac{\partial}{\partial t} \rho(0, \{\boldsymbol{x}_m\}; t, \{\boldsymbol{y}_n\}) \Big|_{t=0}.$$

Since the Palm measure μ^z is a determinantal point field associated with $\rho^z(\{\boldsymbol{x}_m\})$ (see (6.28)) the right-hand-side of (7.5) equals

$$
\int_\Lambda dz \rho(z) \frac{\partial \chi_1(z)}{\partial z} \frac{\partial \chi_2(z)}{\partial z} \int_{\mathfrak{X}} \mu^z(d\eta) F(\eta) G(\eta)
$$

$$
= \int_\Lambda dz \rho(z) \frac{\partial \chi_1(z)}{\partial z} \frac{\partial \chi_2(z)}{\partial z}
$$

$$
\times \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} d\boldsymbol{x}_m \prod_{j=1}^{m} \{\chi_1(x_j)\chi_2(x_j) + \chi_1(x_j) + \chi_2(x_j)\} \rho^z(\{\boldsymbol{x}_m\}).
$$

Hence Proposition 7.2 can be derived from the following lemma.

Lemma 7.3 *For any $m, n \in \mathbf{N}$ we have*

$$
\lim_{t \to 0} \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{j=1}^{m} \chi_1(x_j) \prod_{k=1}^{n} \chi_2(y_k) \frac{\partial}{\partial t} \rho(0, \{\boldsymbol{x}_m\}; t, \{\boldsymbol{y}_n\})
$$

$$
= -\sum_{\ell=1}^{m \wedge n} \frac{m! n!}{(m-\ell)!(n-\ell)!(\ell-1)!} \int_\Lambda dz \rho(z) \frac{\partial \chi_1(z)}{\partial z} \frac{\partial \chi_2(z)}{\partial z}
$$

$$
\times \int_{\Lambda^{m-\ell}} d\boldsymbol{x}_{m-\ell} \int_{\Lambda^{n-\ell}} d\boldsymbol{y}_{n-\ell} \int_{\Lambda^{\ell-1}} d\boldsymbol{w}_{\ell-1} \prod_{j=1}^{m-\ell} \chi_1(x_j) \prod_{k=1}^{n-\ell} \chi_2(y_k) \prod_{i=1}^{\ell-1} \chi_1(w_i)\chi_2(w_i)
$$

$$
\times \rho^z(\{\boldsymbol{x}_{m-\ell}\} \cup \{\boldsymbol{y}_{n-\ell}\} \cup \{\boldsymbol{w}_{\ell-1}\}).
$$

Proof of Lemma 7.3 We use the expansion formula (6.27) of the two-time correlation function to calculate the integral

$$
I = \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^{m} \chi_1(x_i) \prod_{j=1}^{n} \chi_2(y_j) \rho(0, \{\boldsymbol{x}_m\}; t, \{\boldsymbol{y}_n\}).
$$

By permutation invariance of the integrand we put $a_i = m - \ell + i, b_i = i$ with $i = 1, 2, \ldots, \ell$, and then

$$
I = \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^{m} \chi_1(x_i) \prod_{j=1}^{n} \chi_2(y_j) \det \mathbf{M}(t, \boldsymbol{y}_n | \boldsymbol{x}_m)
$$

$$
+ \sum_{\ell=1}^{m \wedge n} \binom{m}{\ell}\binom{n}{\ell} \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^{m} \chi_1(x_i) \prod_{j=1}^{n} \chi_2(y_j)
$$

$$
\times \det \mathbf{D}(t, \boldsymbol{y}_n | \boldsymbol{x}_m)_{\{a_\ell\}\{b_\ell\}} \det \mathbf{M}(t, \boldsymbol{y}_n | \boldsymbol{x}_m)^{\{m+b_\ell\}\{a_\ell\}}
$$

$$
= \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^{m} \chi_1(x_i) \prod_{j=1}^{n} \chi_2(y_j) \det \mathbf{M}(t, \boldsymbol{y}_n | \boldsymbol{x}_m)
$$

$$
+ \sum_{\ell=1}^{m \wedge n} \frac{m! n!}{\ell!(m-\ell)!(n-\ell)!} \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n
$$

$$
\times \prod_{i=1}^{m} \chi_1(x_i) \prod_{j=1}^{n} \chi_2(y_j) \prod_{k=1}^{\ell} \delta_t(x_{m-\ell+k}, y_k) \times \det \mathbf{M}_t^\ell(\boldsymbol{x}_m; \boldsymbol{y}_n),
$$

where $\mathbf{M}_t^\ell(\boldsymbol{x}_m; \boldsymbol{y}_n) = \mathbf{M}(t, \boldsymbol{y}_n | \boldsymbol{x}_m)^{\{m+1,\dots,m+\ell\}\{m-\ell+1,\dots,m\}}$. By using (6.13), (6.17), (6.20) and (6.21), we see that

$$\lim_{t \to 0} \frac{\partial}{\partial t} \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^m \chi_1(x_i) \prod_{j=1}^n \chi_2(y_j) \det \mathbf{M}(t, \boldsymbol{y}_n | \boldsymbol{x}_m) = 0. \qquad (7.6)$$

From (6.12) and (6.20)

$$\frac{\partial}{\partial t} \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^m \chi_1(x_i) \prod_{j=1}^n \chi_2(y_j) \prod_{k=1}^\ell \delta_t(x_{m-\ell+k}, y_k) \det \mathbf{M}_t^\ell(\boldsymbol{x}_m; \boldsymbol{y}_n)$$

$$= \sum_{p=1}^\ell \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^m \chi_1(x_i) \prod_{j=1}^n \chi_2(y_j) \frac{\partial^2}{\partial y_p^2} \delta_t(x_{m-\ell+p}, y_p)$$

$$\times \prod_{k=1, k \neq p}^\ell \delta_t(x_{m-\ell+k}, y_k) \det \mathbf{M}_t^\ell(\boldsymbol{x}_m; \boldsymbol{y}_n)$$

$$+ \sum_{p=1}^{m-\ell} \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^m \chi_1(x_i) \prod_{j=1}^n \chi_2(y_j) \prod_{k=1}^\ell \delta_t(x_{m-\ell+k}, y_k)$$

$$\times \det \begin{pmatrix} K(x_1, x_1) & \cdots & 0 & \cdots & K(x_1, x_{m-\ell}) & \mathcal{G}_t(x_1, y_1) & \cdots & \mathcal{G}_t(x_1, y_n) \\ K(x_2, x_1) & \cdots & 0 & \cdots & K(x_2, x_{m-\ell}) & \mathcal{G}_t(x_2, y_1) & \cdots & \mathcal{G}_t(x_2, y_n) \\ & & \cdots & & & & & \\ K(x_m, x_1) & \cdots & 0 & \cdots & K(x_m, x_{m-\ell}) & \mathcal{G}_t(x_m, y_1) & \cdots & \mathcal{G}_t(x_m, y_n) \\ \mathcal{G}_{-t}(y_{\ell+1}, x_1) & \cdots & \partial_{y_{\ell+1}}^2 \mathcal{G}_{-t}(y_{\ell+1}, x_p) & \cdots & \mathcal{G}_{-t}(y_{\ell+1}, x_{m-\ell}) & K(y_{\ell+1}, y_1) & \cdots & K(y_{\ell+1}, y_n) \\ \mathcal{G}_{-t}(y_{\ell+2}, x_1) & \cdots & \partial_{y_{\ell+2}}^2 \mathcal{G}_{-t}(y_{\ell+2}, x_p) & \cdots & \mathcal{G}_{-t}(y_{\ell+2}, x_{m-\ell}) & K(y_{\ell+2}, y_1) & \cdots & K(y_{\ell+2}, y_n) \\ & & \cdots & & & & & \\ \mathcal{G}_{-t}(y_n, x_1) & \cdots & \partial_{y_n}^2 \mathcal{G}_{-t}(y_n, x_p) & \cdots & \mathcal{G}_{-t}(y_n, x_{m-\ell}) & K(y_n, y_1) & \cdots & K(y_n, y_n) \end{pmatrix}$$

$$- \sum_{p=1}^n \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^m \chi_1(x_i) \prod_{j=1}^n \chi_2(y_j) \prod_{k=1}^\ell \delta_t(x_{m-\ell+k}, y_k)$$

$$\times \det \begin{pmatrix} K(x_1, x_1) & \cdots & K(x_1, x_{m-\ell}) & \mathcal{G}_t(x_1, y_1) & \cdots & \partial_{y_p}^2 \mathcal{G}_t(x_1, y_p) & \cdots & \mathcal{G}_t(x_1, y_n) \\ K(x_2, x_1) & \cdots & K(x_2, x_{m-\ell}) & \mathcal{G}_t(x_2, y_1) & \cdots & \partial_{y_p}^2 \mathcal{G}_t(x_2, y_p) & \cdots & \mathcal{G}_t(x_2, y_n) \\ & \cdots & & & & & & \\ K(x_m, x_1) & \cdots & K(x_m, x_{m-\ell}) & \mathcal{G}_t(x_m, y_1) & \cdots & \partial_{y_p}^2 \mathcal{G}_t(x_m, y_p) & \cdots & \mathcal{G}_t(x_m, y_n) \\ \mathcal{G}_{-t}(y_{\ell+1}, x_1) & \cdots & \mathcal{G}_{-t}(y_{\ell+1}, x_{m-\ell}) & K(y_{\ell+1}, y_1) & \cdots & 0 & \cdots & K(y_{\ell+1}, y_n) \\ \mathcal{G}_{-t}(y_{\ell+2}, x_1) & \cdots & \mathcal{G}_{-t}(y_{\ell+2}, x_{m-\ell}) & K(y_{\ell+2}, y_1) & \cdots & 0 & \cdots & K(y_{\ell+2}, y_n) \\ & \cdots & & & & & & \\ \mathcal{G}_{-t}(y_n, x_1) & \cdots & \mathcal{G}_{-t}(y_n, x_{m-\ell}) & K(y_n, y_1) & \cdots & 0 & \cdots & K(y_n, y_n) \end{pmatrix}$$

$$\equiv I_1(t) + I_2(t) - I_3(t), \qquad (7.7)$$

where we have used the abbreviation $\partial_y^2 = \partial^2/\partial y^2$. By partial integration

$$I_1(t) = \sum_{p=1}^\ell \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^m \chi_1(x_i) \prod_{j=1, j \neq p}^n \chi_2(y_j) \prod_{k=1}^\ell \delta_t(x_{m-\ell+k}, y_k)$$

$$\times \frac{\partial^2}{\partial y_p^2} \{ \chi_2(y_p) \det \mathbf{M}_t^\ell(\boldsymbol{x}_m; \boldsymbol{y}_n) \}$$

$$= \sum_{p=1}^\ell \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^m \chi_1(x_i) \prod_{j=1}^n \chi_2(y_j) \prod_{k=1}^\ell \delta_t(x_{m-\ell+k}, y_k) \frac{\partial^2}{\partial y_p^2} \det \mathbf{M}_t^\ell(\boldsymbol{x}_m; \boldsymbol{y}_n)$$

$$+ \sum_{p=1}^{\ell} \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^{m} \chi_1(x_i) \prod_{j=1, j\neq p}^{n} \chi_2(y_j) \prod_{k=1}^{\ell} \delta_t(x_{m-\ell+k}, y_k)$$

$$\times \left\{ \frac{\partial^2 \chi_2(y_p)}{\partial y_p^2} \det \mathbf{M}_t^{\ell}(\boldsymbol{x}_m; \boldsymbol{y}_n) + 2 \frac{\partial \chi_2(y_p)}{\partial y_p} \frac{\partial \det \mathbf{M}_t^{\ell}(\boldsymbol{x}_m; \boldsymbol{y}_n)}{\partial y_p} \right\}$$

$$\equiv I_{11}(t) + I_{12}(t).$$

By simple calculation

$$I_{11}(t) + I_2(t) - I_3(t)$$

$$= \sum_{p=1}^{\ell} \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^{m} \chi_1(x_i) \prod_{j=1}^{n} \chi_2(y_j) \prod_{k=1}^{\ell} \delta_t(x_{m-\ell+k}, y_k)$$

$$\times \det \begin{pmatrix}
K(x_1, x_1) & \cdots & K(x_1, x_{m-\ell}) & \mathcal{G}_t(x_1, y_1) & \cdots & 0 & \cdots & \mathcal{G}_t(x_1, y_n) \\
K(x_2, x_1) & \cdots & K(x_2, x_{m-\ell}) & \mathcal{G}_t(x_2, y_1) & \cdots & 0 & \cdots & \mathcal{G}_t(x_2, y_n) \\
& \cdots & & & \cdots & & & \\
K(x_m, x_1) & \cdots & K(x_m, x_{m-\ell}) & \mathcal{G}_t(x_m, y_1) & \cdots & 0 & \cdots & \mathcal{G}_t(x_m, y_n) \\
\mathcal{G}_{-t}(y_{\ell+1}, x_1) & \cdots & \mathcal{G}_{-t}(y_{\ell+1}, x_{m-\ell}) & K(y_{\ell+1}, y_1) & \cdots & \partial_{y_p}^2 K(y_{\ell+1}, y_p) & \cdots & K(y_{\ell+1}, y_n) \\
\mathcal{G}_{-t}(y_{\ell+2}, x_1) & \cdots & \mathcal{G}_{-t}(y_{\ell+2}, x_{m-\ell}) & K(y_{\ell+2}, y_1) & \cdots & \partial_{y_p}^2 K(y_{\ell+2}, y_p) & \cdots & K(y_{\ell+2}, y_n) \\
& \cdots & & & \cdots & & & \\
\mathcal{G}_{-t}(y_n, x_1) & \cdots & \mathcal{G}_{-t}(y_n, x_{m-\ell}) & K(y_n, y_1) & \cdots & \partial_{y_p}^2 K(y_n, y_p) & \cdots & K(y_n, y_n)
\end{pmatrix}$$

$$+ \sum_{p=1}^{m-\ell} \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^{m} \chi_1(x_i) \prod_{j=1}^{n} \chi_2(y_j) \prod_{k=1}^{\ell} \delta_t(x_{m-\ell+k}, y_k)$$

$$\times \det \begin{pmatrix}
K(x_1, x_1) & \cdots & 0 & \cdots & K(x_1, x_{m-\ell}) & \mathcal{G}_t(x_1, y_1) & \cdots & \mathcal{G}_t(x_1, y_n) \\
K(x_2, x_1) & \cdots & 0 & \cdots & K(x_2, x_{m-\ell}) & \mathcal{G}_t(x_2, y_1) & \cdots & \mathcal{G}_t(x_2, y_n) \\
& \cdots & & & \cdots & & & \\
K(x_m, x_1) & \cdots & 0 & \cdots & K(x_m, x_{m-\ell}) & \mathcal{G}_t(x_m, y_1) & \cdots & \mathcal{G}_t(x_m, y_n) \\
\mathcal{G}_{-t}(y_{\ell+1}, x_1) & \cdots & \partial_{y_{\ell+1}}^2 \mathcal{G}_{-t}(y_{\ell+1}, x_p) & \cdots & \mathcal{G}_{-t}(y_{\ell+1}, x_{m-\ell}) & K(y_{\ell+1}, y_1) & \cdots & K(y_{\ell+1}, y_n) \\
\mathcal{G}_{-t}(y_{\ell+2}, x_1) & \cdots & \partial_{y_{\ell+2}}^2 \mathcal{G}_{-t}(y_{\ell+2}, x_p) & \cdots & \mathcal{G}_{-t}(y_{\ell+2}, x_{m-\ell}) & K(y_{\ell+2}, y_1) & \cdots & K(y_{\ell+2}, y_n) \\
& \cdots & & & \cdots & & & \\
\mathcal{G}_{-t}(y_n, x_1) & \cdots & \partial_{y_n}^2 \mathcal{G}_{-t}(y_n, x_p) & \cdots & \mathcal{G}_{-t}(y_n, x_{m-\ell}) & K(y_n, y_1) & \cdots & K(y_n, y_n)
\end{pmatrix}$$

$$- \sum_{p=\ell+1}^{n} \int_{\Lambda^m} d\boldsymbol{x}_m \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^{m} \chi_1(x_i) \prod_{j=1}^{n} \chi_2(y_j) \prod_{k=1}^{\ell} \delta_t(x_{m-\ell+k}, y_k)$$

$$\times \det \begin{pmatrix}
K(x_1, x_1) & \cdots & K(x_1, x_{m-\ell}) & \mathcal{G}_t(x_1, y_1) & \cdots & \partial_{y_p}^2 \mathcal{G}_t(x_1, y_p) & \cdots & \mathcal{G}_t(x_1, y_n) \\
K(x_2, x_1) & \cdots & K(x_2, x_{m-\ell}) & \mathcal{G}_t(x_2, y_1) & \cdots & \partial_{y_p}^2 \mathcal{G}_t(x_2, y_p) & \cdots & \mathcal{G}_t(x_2, y_n) \\
& \cdots & & & \cdots & & & \\
K(x_m, x_1) & \cdots & K(x_m, x_{m-\ell}) & \mathcal{G}_t(x_m, y_1) & \cdots & \partial_{y_p}^2 \mathcal{G}_t(x_m, y_p) & \cdots & \mathcal{G}_t(x_m, y_n) \\
\mathcal{G}_{-t}(y_{\ell+1}, x_1) & \cdots & \mathcal{G}_{-t}(y_{\ell+1}, x_{m-\ell}) & K(y_{\ell+1}, y_1) & \cdots & 0 & \cdots & K(y_{\ell+1}, y_n) \\
\mathcal{G}_{-t}(y_{\ell+2}, x_1) & \cdots & \mathcal{G}_{-t}(y_{\ell+2}, x_{m-\ell}) & K(y_{\ell+2}, y_1) & \cdots & 0 & \cdots & K(y_{\ell+2}, y_n) \\
& \cdots & & & \cdots & & & \\
\mathcal{G}_{-t}(y_n, x_1) & \cdots & \mathcal{G}_{-t}(y_n, x_{m-\ell}) & K(y_n, y_1) & \cdots & 0 & \cdots & K(y_n, y_n)
\end{pmatrix}.$$

By using (6.13) and (6.21) we have

$$\lim_{t\to 0}\{I_{11}(t) + I_2(t) - I_3(t)\} = 0. \tag{7.8}$$

Suppose that $x_{m-\ell+i} \to y_i$, $i = 1, 2, \ldots \ell$,

$$\det \mathbf{M}_t^{\ell}(\boldsymbol{x}_m; \boldsymbol{y}_n) \to \rho(\{\boldsymbol{x}_{m-\ell}\} \cup \{\boldsymbol{y}_n\}),$$

$$\frac{\partial \det \mathbf{M}_t^{\ell}(\boldsymbol{x}_m; \boldsymbol{y}_n)}{\partial y_p} \to \frac{1}{2} \frac{\partial \rho(\{\boldsymbol{x}_{m-\ell}\} \cup \{\boldsymbol{y}_n\})}{\partial y_p} \quad \text{in } t \to 0$$

for any $p = 1, 2, \ldots, \ell$. Hence we have

$$
\begin{aligned}
\lim_{t \to 0} I_{12}(t) &= \sum_{p=1}^{\ell} \int_{\Lambda^{m-\ell}} d\boldsymbol{x}_{m-\ell} \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^{m-\ell} \chi_1(x_i) \prod_{k=1, k \neq p}^{\ell} \chi_1(y_k) \chi_2(y_k) \prod_{j=\ell+1}^{n} \chi_2(y_j) \\
&\quad \times \chi_1(y_p) \left\{ \frac{\partial^2 \chi_2(y_p)}{\partial y_p^2} \rho(\{\boldsymbol{x}_{m-\ell}\} \cup \{\boldsymbol{y}_n\}) + \frac{\partial \chi_2(y_p)}{\partial y_p} \frac{\partial \rho(\{\boldsymbol{x}_{m-\ell}\} \cup \{\boldsymbol{y}_n\})}{\partial y_p} \right\} \\
&= \sum_{p=1}^{\ell} \int_{\Lambda^{m-\ell}} d\boldsymbol{x}_{m-\ell} \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^{m-\ell} \chi_1(x_i) \prod_{k=1, k \neq p}^{\ell} \chi_1(y_k) \chi_2(y_k) \prod_{j=\ell+1}^{n} \chi_2(y_j) \\
&\quad \times \chi_1(y_p) \frac{\partial}{\partial y_p} \left\{ \frac{\partial \chi_2(y_p)}{\partial y_p} \rho(\{\boldsymbol{x}_{m-\ell}\} \cup \{\boldsymbol{y}_n\}) \right\} \\
&= -\sum_{p=1}^{\ell} \int_{\Lambda^{m-\ell}} d\boldsymbol{x}_{m-\ell} \int_{\Lambda^n} d\boldsymbol{y}_n \prod_{i=1}^{m-\ell} \chi_1(x_i) \prod_{k=1, k \neq p}^{\ell} \chi_1(y_k) \chi_2(y_k) \prod_{j=\ell+1}^{n} \chi_2(y_j) \\
&\quad \times \frac{\partial \chi_1(y_p)}{\partial y_p} \frac{\partial \chi_2(y_p)}{\partial y_p} \rho(\{\boldsymbol{x}_{m-\ell}\} \cup \{\boldsymbol{y}_n\}) \\
&= -\ell \int_{\Lambda^{m-\ell}} d\boldsymbol{x}_{m-\ell} \int_{\Lambda} dz \int_{\Lambda^{\ell-1}} d\boldsymbol{w}_{\ell-1} \int_{\Lambda^{n-\ell}} d\boldsymbol{y}_{n-\ell} \\
&\quad \times \prod_{i=1}^{m-\ell} \chi_1(x_i) \prod_{k=1}^{\ell-1} \chi_1(y_k) \chi_2(y_k) \prod_{j=1}^{n} \chi_2(w_j) \\
&\quad \times \frac{\partial \chi_1(z)}{\partial z} \frac{\partial \chi_2(z)}{\partial z} \rho(\{\boldsymbol{x}_{m-\ell}\} \cup \{z\} \cup \{\boldsymbol{w}_{\ell-1}\} \cup \{\boldsymbol{y}_{n-\ell}\}) \\
&= -\ell \int_{\Lambda} dz \rho(z) \frac{\partial \chi_1(z)}{\partial z} \frac{\partial \chi_2(z)}{\partial z} \int_{\Lambda^{m-\ell}} d\boldsymbol{x}_{m-\ell} \int_{\Lambda^{\ell-1}} d\boldsymbol{w}_{\ell-1} \int_{\Lambda^{n-\ell}} d\boldsymbol{y}_{n-\ell} \\
&\quad \times \prod_{i=1}^{m-\ell} \chi_1(x_i) \prod_{k=1}^{\ell-1} \chi_1(y_k) \chi_2(y_k) \prod_{j=1}^{n} \chi_2(w_j) \rho^z(\{\boldsymbol{x}_{m-\ell}\} \cup \{\boldsymbol{w}_{\ell-1}\} \cup \{\boldsymbol{y}_{n-\ell}\}). \quad (7.9)
\end{aligned}
$$

Combining (7.6), (7.7), (7.8) and (7.9), we obtain Lemma 7.3. $\square$

8 Concluding Remarks

The study on noncolliding BM and determinantal processes reported in this paper will be extended in several directions. We would like to give some of future problems below.

(1) In Sect. 4 we let the initial configuration ν_0 be the GUE-eigenvalue distribution, and shown that the system is determinantal. If we let ν_0 be the eigenvalue distribution in the Gaussian orthogonal ensemble (GOE),

$$
\nu_0^{\text{GOE}}(\boldsymbol{x}^{(0)}) = \frac{1}{C_N'} t_0^{-N(N+1)/4} e^{-|\boldsymbol{x}^{(0)}|^2/2t_0} h_N(\boldsymbol{x}^{(0)}), \quad \boldsymbol{x}^{(0)} \in \mathbf{W}_N, \quad (8.1)
$$

(3.11) becomes

$$\nu_t(\boldsymbol{x}) \propto h_N(\boldsymbol{x}) \int_{\mathbf{W}_N} f_N(t - t_0, \boldsymbol{x} | \boldsymbol{x}^{(0)}) d\boldsymbol{x}^{(0)}$$

$$= \frac{1}{N!} h_N(\boldsymbol{x}) \int_{\mathbf{R}^N} f_N(t - t_0, \boldsymbol{x} | \boldsymbol{x}^{(0)}) \, \mathrm{sgn}(h_N(\boldsymbol{x}^{(0)})) d\boldsymbol{x}^{(0)}. \qquad (8.2)$$

Instead of the Heine identity (4.11), we should use the de Bruin identity [25]

$$\int_{\mathbf{W}_N} d\boldsymbol{x} \det_{1 \le j, k \le N} [g_j(x_k)] = \mathrm{Pf}_{1 \le j, k \le N} \left[\int_{\mathbf{R}} dx \int_{\mathbf{R}} d\widetilde{x} \, \mathrm{sgn}(\widetilde{x} - x) g_j(x) g_k(\widetilde{x}) \right]$$

for integrable continuous functions g_j, $1 \le j \le N$. As shown in Appendix A of [57], for example, the generating function of multitime correlation functions is then expressed by the Fredholm Pfaffian [79] and the system becomes a Pfaffian process, in the sense that any multitime correlation function is given by a Pfaffian. Such Pfaffian processes have been studied by many authors [17, 19, 34, 44, 82, 86, 87]. The systems studied in [37, 51, 69, 72] are also Pfaffian processes, since the 'quaternion determinantal expressions' of correlation functions, introduced and developed by Dyson, Mehta, Forrester, and Nagao [31, 65, 66, 68, 71], are readily transformed to Pfaffian expressions. As implied by Proposition 3.2, the system exhibits a transition from GOE distribution to GUE distribution [52, 67, 77]. Continuity of sample paths and general characterization of infinite Pfaffian processes will be interesting problems. The case with other initial distribution (in particular, when it has continuous parameters) will be interesting [55, 57].

(2) As explained in Sects. 1 and 3, the present noncolliding BM is the h-transform of the absorbing BM in the Weyl chamber (1.10) of type A_{N-1}. We can find appropriate h-transforms of the absorbing BMs in the Weyl chambers of types C_N and D_N. The obtained noncolliding diffusion processes are stochastic versions of non-standard random matrix ensembles, which were called the class C and class D, respectively, by Altland and Zirnbauer [3, 4, 99]. The stochastic version of the chiral GUE, realized by the noncolliding squared Bessel process, was studied by König and O'Connell [59], which is also obtained as an h-transform of the absorbing BM in the Weyl chamber of type C_N. See [41, 55] for more details. Systematic classifications of determinantal and Pfaffian processes will be important.

(3) There are many other examples of finite and infinite determinantal processes, which are not considered in the present paper. Markov processes on partitions (Young diagrams) have been studied and determinantal processes associated with other types of projections than ours have been reported [14, 15, 73]. The determinantal processes, whose kernels are expressed using multiple orthogonal functions (e.g. the Pearcey kernel), are discussed in [5, 20–24, 44, 74, 97]. The consideration given in Sects. 6 and 7 in the present paper should be generalized.

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