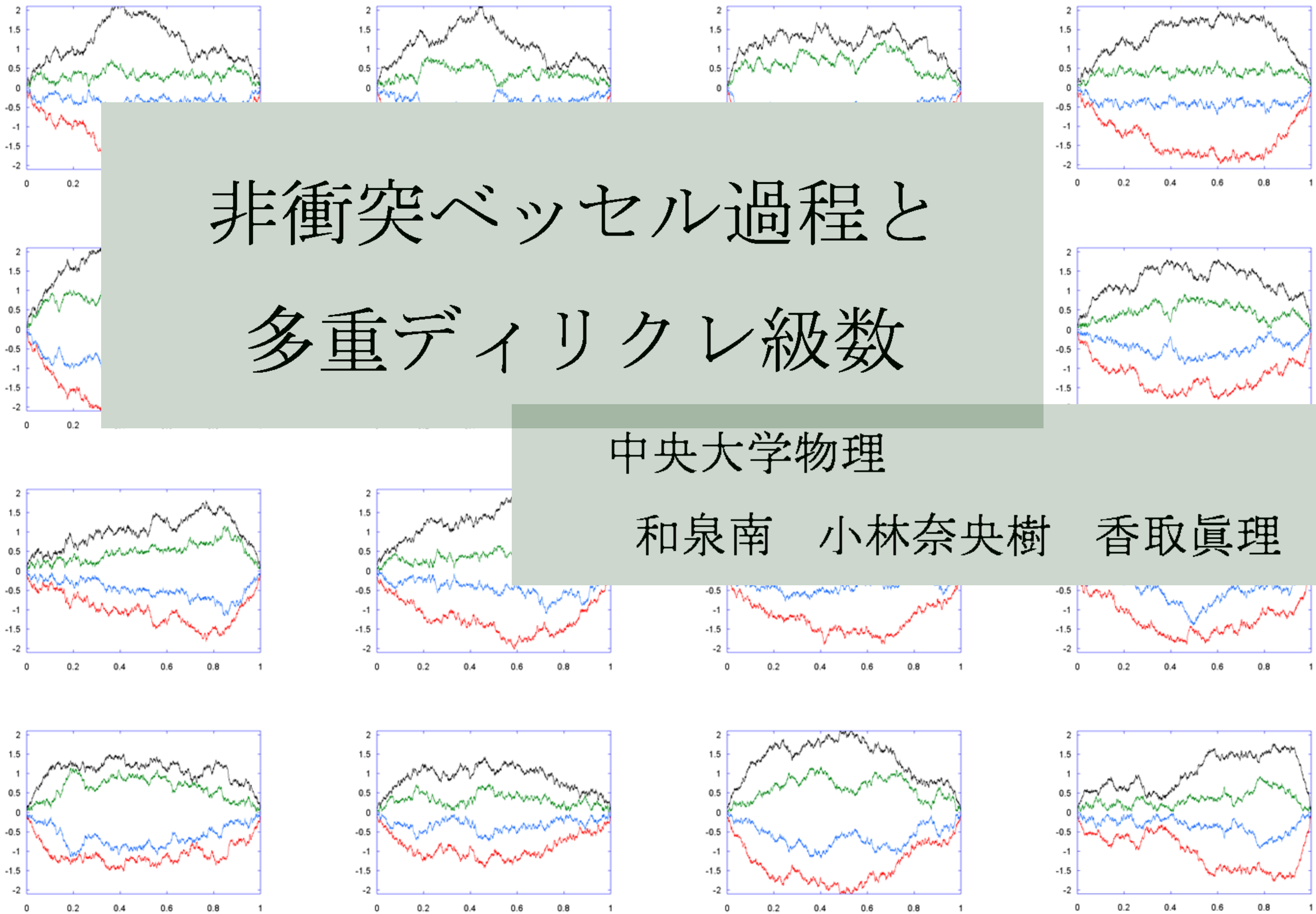

非衝突ベッセル過程と多重ディリクレ級数

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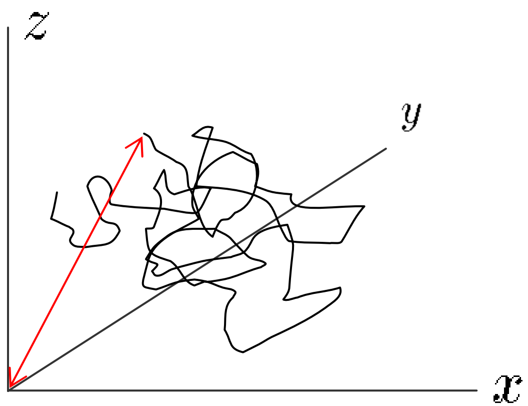
3 次元ベッセル過程　BES_3

$$X(t) \equiv |\mathbf{B}(t)| = \sqrt{B_1(t)^2 + B_2(t)^2 + B_3(t)^2}$$



伊藤の公式

$$dX(t) = dB(t) + \frac{1}{X(t)}dt, \quad t > 0, \quad X(0) = x$$

3 次元ベッセル橋 $\widetilde{X}(t), t \in [0,1]$

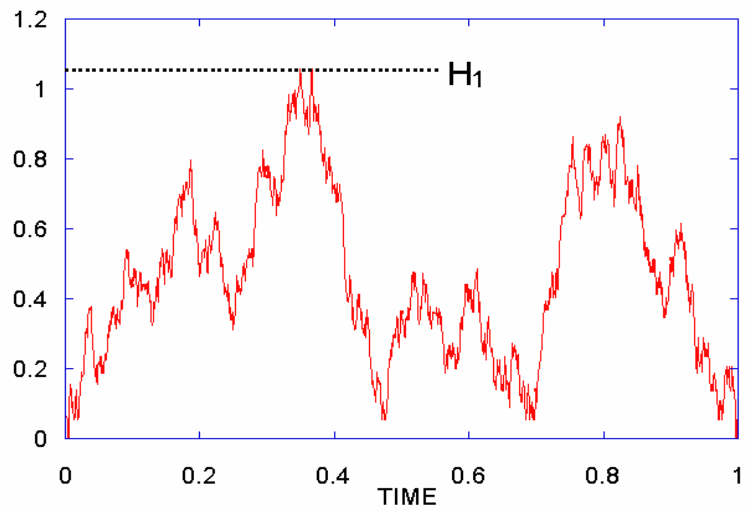
$$x = X(0) = 0, X(1) = 0$$

$$H_1 \equiv \max_{0<t<1} \widetilde{X}(t)$$

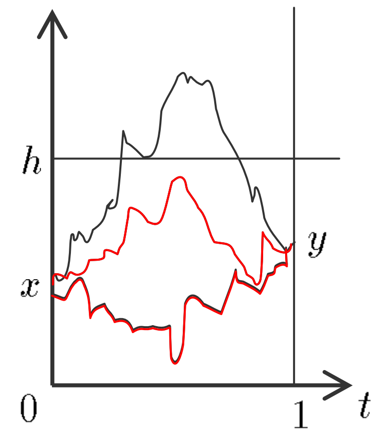
$$p(h) = \frac{d}{dh}\mathrm{Prob}(H_1 < h)$$

Moment

$$\mathbf{E}[x^s] = \int_0^\infty dx\, x^s\, p(x)$$



$$\mathrm{Prob}(H_1 < h) = \lim_{x\to 0, y\to 0} \frac{p_2^h(1,y|x)}{p_1(1,y|x)}$$



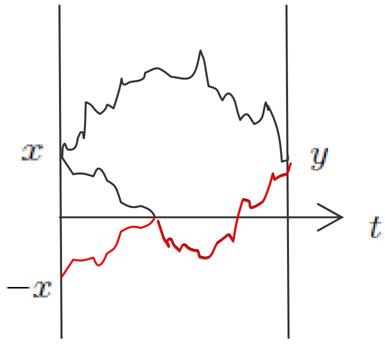
一次元ブラウン運動の推移確率密度

$$p(t,y|x) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(y-x)^2}{2t}\right\}, \quad x, y \in \mathbf{R},\ t \geq 0$$

原点に吸収壁がある場合（反射原理より）

$$\begin{aligned} p_1(t,y|x) &= p(t,y|x) - p(t,y|-x) \\ &= \frac{1}{\sqrt{2\pi t}}\left(e^{-(y-x)^2/2t} - e^{-(y+x)^2/2t}\right) \end{aligned}$$

$$x, y \in \mathbf{R}_+,\ t \geq 0$$



原点と $x = h > 0$ に吸収壁がある場合

$$\begin{aligned} p_2^h(t,y|x) &= \sum_{n=-\infty}^{\infty} \left\{p(t,y|x+2hn) - p(t,y|-x+2hn)\right\} \\ &= \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \left[\exp\left\{-\frac{1}{2t}(y-(x+2hn))^2\right\} - \exp\left\{-\frac{1}{2t}(y-(-x+2hn))^2\right\}\right] \end{aligned} \quad x, y \in (0,h), t \geq 0$$

$$\mathrm{Prob}(H_1 < h) = \sum_{n=-\infty}^{\infty} (1 - 4h^2n^2)e^{-2h^2n^2}$$

香取眞理, 種村秀紀著「ランダム行列と非衝突過程」
(小嶋泉編「物理数学への誘い 6」第 6 話, 遊星社,2006)

H_1 の s 次モーメント $\mathbf{E}[H_1^s]$

$$\mathbf{E}[H_1^s] = 2\left(\frac{\pi}{2}\right)^{s/2}\xi(s), \quad s \in \mathbf{C}$$

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

$$\Gamma(x) = \int_0^\infty du\, u^{x-1}e^{-u}, \quad \mathrm{Re}\, x > 0$$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$ Riemann Zeta Function

P. Biane, J. Pitman, and M. Yor, *Bull. Amer. Math. Soc.* **38** (2001) 435-465.

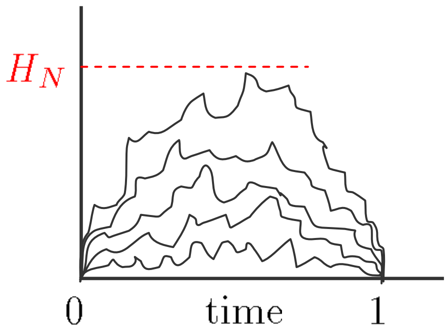
非衝突ベッセル橋（N粒子）

$$0 < \widetilde{X}_1(t) < \widetilde{X}_2(t) < \cdots < \widetilde{X}_N(t), \quad 0 < t < 1$$ Noncolliding

$$H_N = \max_{0 \le t \le 1} \widetilde{X}_N(t)$$

$$p_N(h) = \frac{d}{dh}\mathrm{Prob}(H_N < h)$$

$$E[H_N^s] = \int_0^\infty dh\, h^s p_N(h)$$



$$\mathrm{Prob}(H_N < h) = \lim_{x_j \to 0, y_j \to 0, 1 \le j \le N} \frac{\det_{1 \le j,k \le N} [p_2^h(1, y_j|x_k)]}{\det_{1 \le j,k \le N} [p_1(1, y_j|x_k)]}$$

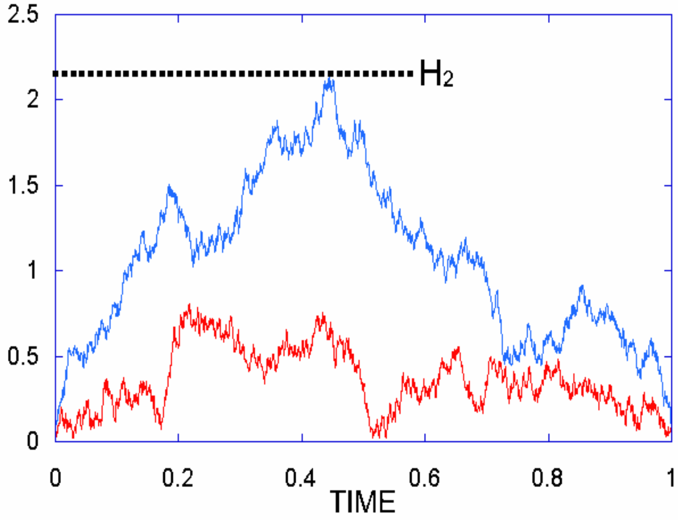
Karlin-McGregor の公式
推移確率を成分とした行列式をとると、
非衝突な推移確率になる

$$= \lim_{x_j \to 0, y_j \to 0, 1 \le j \le N} \frac{q_h^{(N)}(1, \boldsymbol{y}|\boldsymbol{x})}{q^{(N)}(1, \boldsymbol{y}|\boldsymbol{x})}$$

$N=2$ の場合

$$\mathrm{Prob}(H_2 < h) = \lim_{x_j \to 0, y_j \to 0, 1 \le j \le 2} \frac{\det_{1 \le j,k \le 2} [p_2^h(1, y_j | x_k)]}{\det_{1 \le j,k \le 2} [p_1(1, y_j | x_k)]}$$

$$= \lim_{x_j \to 0, 1 \le j \le 2} \frac{q_h^{(2)}(1, \boldsymbol{x}|\boldsymbol{x})}{q^{(2)}(1, \boldsymbol{x}|\boldsymbol{x})}$$



$$q^{(2)}(1, \boldsymbol{x}|\boldsymbol{x}) = \frac{1}{3\pi} x_1^2 x_2^2 (x_1 - x_2)^2 (x_1 + x_2)^2 + O(x_1^{10}, x_2^{10})$$

$$q_h^{(2)}(1, \boldsymbol{x}|\boldsymbol{x}) = \frac{1}{9\pi} x_1^2 x_2^2 (x_1 - x_2)^2 (x_1 + x_2)^2 \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} e^{-2h^2(n_1^2+n_2^2)} Q_h(n_1, n_2) + O(x_1^9, x_2^9)$$

$$\begin{aligned} Q_h(n_1, n_2) &= 3 - 48h^2 n_1^2 + 72h^4 n_1^4 + 72h^4 n_1^2 n_2^2 - 32h^6 n_1^6 - 96h^6 n_1^4 n_2^2 \\ &\quad + 128h^8 n_1^6 n_2^2 - 128h^8 n_1^4 n_2^4 \end{aligned}$$

$$\mathbf{E}[H_2^s] = \frac{2^{-s/2}}{24} s\Big[(s-1)(s^2 - 2s + 12)\widetilde{Z}_{s/2}(0) - 4(s+4)(s+6)\widetilde{Z}_{s/2}(1) + 64\widetilde{Z}_{s/2}(2)\Big]$$

$$\widetilde{Z}_a(b) = \Gamma(a+2b) Z(2b, 2b; a+2b)$$

$$Z(\alpha, \beta; \gamma) \equiv \sum_{(n_1, n_2) \in \mathbf{Z}^2 \setminus \{(0,0)\}} \frac{n_1^\alpha n_2^\beta}{(n_1^2 + n_2^2)^\gamma}$$

double Dirichlet series

関数方程式　functional equation

さらにヤコビのテータ関数を用いて $\vartheta(u)$ を次のように定義する

$$\vartheta(u) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u}, \quad u > 0$$

$$\mathbf{E}[H_1^s] = 2\left(\frac{\pi}{2}\right)^{s/2} \xi(s), \quad s \in \mathbf{C}$$

$$\xi(s) = \frac{1}{2} + \frac{1}{4}s(s-1)\int_1^\infty du\,(u^{s/2-1} + u^{(1-s)/2-1})(\vartheta(u) - 1)$$

$$\xi(1-s) = \xi(s), \quad s \in \mathbf{C}$$

$$\mathbf{E}[H_2^s] = \left(\frac{\pi}{2}\right)^{s/2}\left[\frac{1}{24}(1-s)(s^2-2s+12)(2-sK_0(s)) - 4s\left(\vartheta(1)\vartheta'(1) + 2s\vartheta'(1)^2\right) + s\xi_2(s)\right], \quad s \in \mathbf{C}$$

$$K_0(s) = \int_1^\infty du\, u^{s/2-1}\{\vartheta(u)^2 - 1\}$$

$$\xi_2(s) = -\frac{1}{6}\left\{(s+4)(s+6)\int_1^\infty du\, u^{s/2+1}\vartheta'(u)^2 + ((2-s)+4)((2-s)+6)\int_1^\infty du\, u^{(2-s)/2+1}\vartheta'(u)^2\right\} + \frac{8}{3}\int_1^\infty du\,(u^{s/2+3} + u^{(2-s)/2+3})\vartheta''(u)^2 + \frac{1}{12}s(s-2)\vartheta(1)^2$$

$$\xi_2(2-s) = \xi_2(s), \quad s \in \mathbf{C}$$

H_1^s と H_2^s の s 次モーメントの数値計算結果

表 1: Numerical values of moments

s	0	1	2	3	4	5
$\mathbf{E}[H_1^s]$	1.0	1.253314	1.644934	2.259832	3.246969	4.873485
$\mathbf{E}[H_2^s]$	1.0	1.822625	3.395156	6.463823	12.576665	25.005999

Katori-Izumi-Kobayashi, arXiv math.PR/0711.1710
to be published in J. Stat. Phys.

一般の N の場合

$$\mathrm{Prob}(Y_N < y) = \frac{(-1)^N}{2^{N^2}\prod_{j=1}^{N}(2j-1)!} \det_{1\le j,k\le N}\left[\sum_{n=-\infty}^{\infty} H_{2j+2k-2}(\sqrt{\pi}ny)e^{-\pi n^2 y^2}\right]$$

エルミート多項式を成分にもつ $N\times N$ の行列式 など

$Y_N < y \Leftrightarrow H_N < h$ with $y = h\sqrt{2/\pi}$

$H_n(x)$ エルミート多項式

$N=1$

$$\mathrm{Prob}(Y_1 < y) = -\frac{1}{2}\sum_{n=-\infty}^{\infty} H_2(\sqrt{\pi}ny)e^{-\pi n^2 y^2}$$

$$\mathrm{Prob}(H_1 < h) = \sum_{n=-\infty}^{\infty}(1-4h^2n^2)e^{-2h^2n^2}$$

⇒ 一致

$N=2$

$$\mathrm{Prob}(Y_2 < y) = \frac{1}{2^4\times 3!}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} e^{-\pi(n_1^2+n_2^2)y^2}\det\begin{bmatrix} H_2(\sqrt{\pi}n_1y) & H_4(\sqrt{\pi}n_1y) \\ H_4(\sqrt{\pi}n_2y) & H_6(\sqrt{\pi}n_2y)\end{bmatrix}$$

$$\mathrm{Prob}(H_2 < h) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} e^{-2h^2(n_1^2+n_2^2)}\Big\{1-16h^2n_1^2+24h^4n_1^4+24h^4n_1^2n_2^2-\frac{32}{3}h^6n_1^6$$
$$-32h^6n_1^4n_2^2+\frac{128}{3}h^8n_1^6n_2^2-\frac{128}{3}h^8n_1^4n_2^4\Big\}.$$

⇒ 一致

Katori, Izumi, Kobayashi, in preparation

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