Noncolliding Diffusion Processes and Determinantal Processes

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1. Noncolliding Diffusion Processe (finite systems)

N = # of particles < ∞, $Bj(t)$'s = independent one-dim. standard BMs

[A] Brownian motions (BMs) conditioned never to collide

= Dyson's BM model (with β=2)

$$\mathbf{X}(t) = (X_1(t), X_2(t), \cdots, X_N(t)) \in \mathbf{W}_N^{\mathrm{A}} = \{\mathbf{x} : x_1 < x_2 < \cdots < x_N\} \text{ (Weyl chamber of type A)}$$

$$dX_j(t) = dB_j(t) + \sum_{1 \le k \le N : k \neq j} \frac{1}{X_j(t) - X_k(t)} dt, \quad 1 \le j \le N, \quad t \in [0, \infty)$$

[B] Squared Bessel processes with parameter ν (BESQν) (dim. d=2(ν+1))

conditioned never to collide = Laguerre process (Koenig and O'Connell)

$$\mathbf{X}(t) = (X_1(t), X_2(t), \cdots, X_N(t)) \in \mathbf{W}_N^{\mathrm{C}} = \{\mathbf{x} : 0 < x_1 < x_2 < \cdots < x_N\} \text{(Weyl chamber of type C)}$$

$$dX_j(t) = 2\sqrt{X_j(t)} dB_j(t) + 2\left\{(N + \nu) + \sum_{1 \le k \le N, k \neq j} \frac{X_j(t) + X_k(t)}{X_j(t) - X_k(t)}\right\} dt$$

$$1 \le j \le N, \quad t \in [0, \infty)$$

Initial Distributions

initial times $t = t_0$

$$\nu_0(\mathbf{x}^{(0)})d\mathbf{x}^{(0)} \equiv \mathbf{P}\Big(X_1(t_0) \in [x_1^{(0)}, x_1^{(0)} + dx_1^{(0)}], \cdots, X_N(t_0) \in [x_N^{(0)}, x_N^{(0)} + dx_N^{(0)}]\Big)$$
$$= \mathbf{P}\Big(\mathbf{X}(t_0) \in d\mathbf{x}^{(0)}\Big)$$

$$\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \cdots, x_N^{(0)}), \qquad d\mathbf{x}^{(0)} = \prod_{j=1}^{N} dx_j^{(0)}, \qquad |\,\mathbf{x}^{(0)}\,|^2 = \sum_{j=1}^{N} (x_j^{(0)})^2$$

$$h(\mathbf{x}^{(0)}) = \prod_{1 \le j < k \le N} (x_k^{(0)} - x_j^{(0)}) \qquad \cdots \text{ product of differences } \quad \text{(Vandermonde determinant)}$$

[A] Noncolliding BM

GUE - eigenvalue distribution with variance t_0

$$\nu_0(\mathbf{x}^{(0)}) = C_N^{\ -1} t_0^{-N^2/2}\, \mathrm{e}^{-|\mathbf{x}^{(0)}|^2/2t_0}\, h(\mathbf{x}^{(0)})^2 \qquad \text{with} \quad C_N = (2\pi)^{N/2} \prod_{j=1}^{N} \Gamma(j)$$

[B] Noncolliding BESQ$_\nu$

$\nu \in \{0,1,2,\cdots\}$ 'squared of ' chiral GUE

$\nu = 1/2, -1/2$ 'squared of ' class C, class D (Altland - Zirnbauer)

$$\nu_0(\mathbf{x}^{(0)}) = \hat{C}_N^{\ -1} t_0^{-N(N+\nu)}\, \mathrm{e}^{-|\mathbf{x}^{(0)}|^2/2t_0} \prod_{j=1}^{N} (x_j^{(0)})^{\nu}\, h(\mathbf{x}^{(0)})^2 \quad \text{with} \quad \hat{C}_N = (2\pi)^{N(N+\nu)} \prod_{j=1}^{N} \{\Gamma(j)\Gamma(j+\nu)\}$$

multi-time probability density function

multi-time $t_0 < t_1 < \cdots < t_M$

$$p_N\left(t_0, \mathbf{x}^{(0)}; t_1, \mathbf{x}^{(1)}; \cdots; t_M, \mathbf{x}^{(M)}\right) \prod_{m=0}^{M} d\mathbf{x}^{(m)}$$

$$\equiv \mathbf{P}\left(\mathbf{X}(t_0) \in d\mathbf{x}^{(0)}, \mathbf{X}(t_1) \in d\mathbf{x}^{(1)}, \cdots, \mathbf{X}(t_M) \in d\mathbf{x}^{(M)}\right)$$

$$= \frac{h_N(\mathbf{x}^{(M)})}{h_N(\mathbf{x}^{(0)})} \prod_{m=0}^{M-1} \det_{1 \le j,k \le N}\left[p(t_{m+1} - t_m; x_j^{(m+1)} \mid x_k^{(m)})\right] \nu_0(\mathbf{x}^{(0)}) \prod_{m=0}^{M} d\mathbf{x}^{(m)}$$

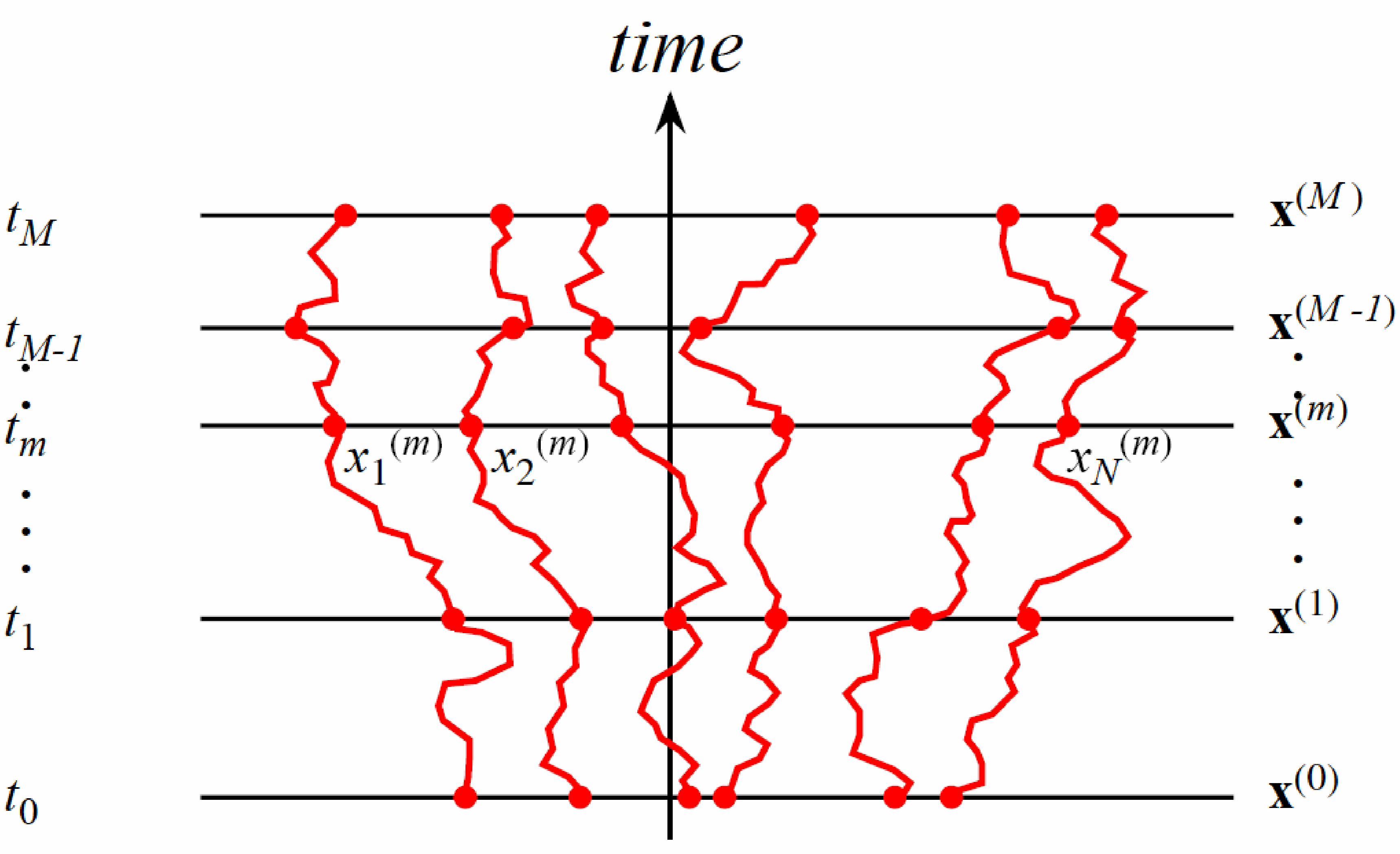
where

$$p(t, y \mid x) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(x-y)^2}{2t}\right\} \quad \text{for [A] noncolliding BM}$$

or

$$p(t, y \mid x) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\nu/2} \exp\left\{-\frac{x+y}{2t}\right\} I_\nu\left(\frac{\sqrt{xy}}{t}\right) \quad \text{for [B] noncolliding } \mathrm{BESQ}_\nu$$

Karlin-McGregor formula + *h*-transform of Doob



time
t_M
t_{M-1}
t_m
t_1
t_0
$x_1^{(m)}$
$x_2^{(m)}$
$x_N^{(m)}$
$\mathbf{x}^{(M)}$
$\mathbf{x}^{(M-1)}$
$\mathbf{x}^{(m)}$
$\mathbf{x}^{(1)}$
$\mathbf{x}^{(0)}$

$$p_N\left(t_0, \mathbf{x}^{(0)}; t_1, \mathbf{x}^{(1)}; \cdots; t_M, \mathbf{x}^{(M)}\right)$$

$$= \frac{h_N(\mathbf{x}^{(M)})}{h_N(\mathbf{x}^{(0)})} \prod_{m=0}^{M-1} \det_{1 \le j,k \le N}\left[p(t_{m+1} - t_m; x_j^{(m+1)} \mid x_k^{(m)})\right] \nu_0(\mathbf{x}^{(0)})$$

The probability law is invariant under any permutation of particle indices.

We can regard that particles are **identical and indistinguishable**.

X = the space of countable subset ξ of $\mathbf{R}$ satisfying $\#(\xi \cap K) < \infty$

for any compact subset K

the state space : $\mathbf{W}_N \;\Rightarrow\; \mathsf{X}$

$\mathbf{x} = (x_1, x_2, \cdots, x_N) \;\Rightarrow\; \{\mathbf{x}\} = \{x_j\}_{j=1}^{N}$

the process $\mathbf{X}(t) \;\Rightarrow\; \Xi(t) = \{\mathbf{X}(t)\}$

$$p_N\left(t_0, \mathbf{x}^{(0)}; t_1, \mathbf{x}^{(1)}; \cdots; t_M, \mathbf{x}^{(M)}\right) \Rightarrow \mathrm{p}_N\left(t_0, \{\mathbf{x}^{(0)}\}; t_1, \{\mathbf{x}^{(1)}\}; \cdots; t_M, \{\mathbf{x}^{(M)}\}\right)$$

multi-time correlation function

For $\mathbf{x}^{(m)} \in \mathbf{R}^N, 0 \le m \le M, N' = 1, 2, \cdots, N$

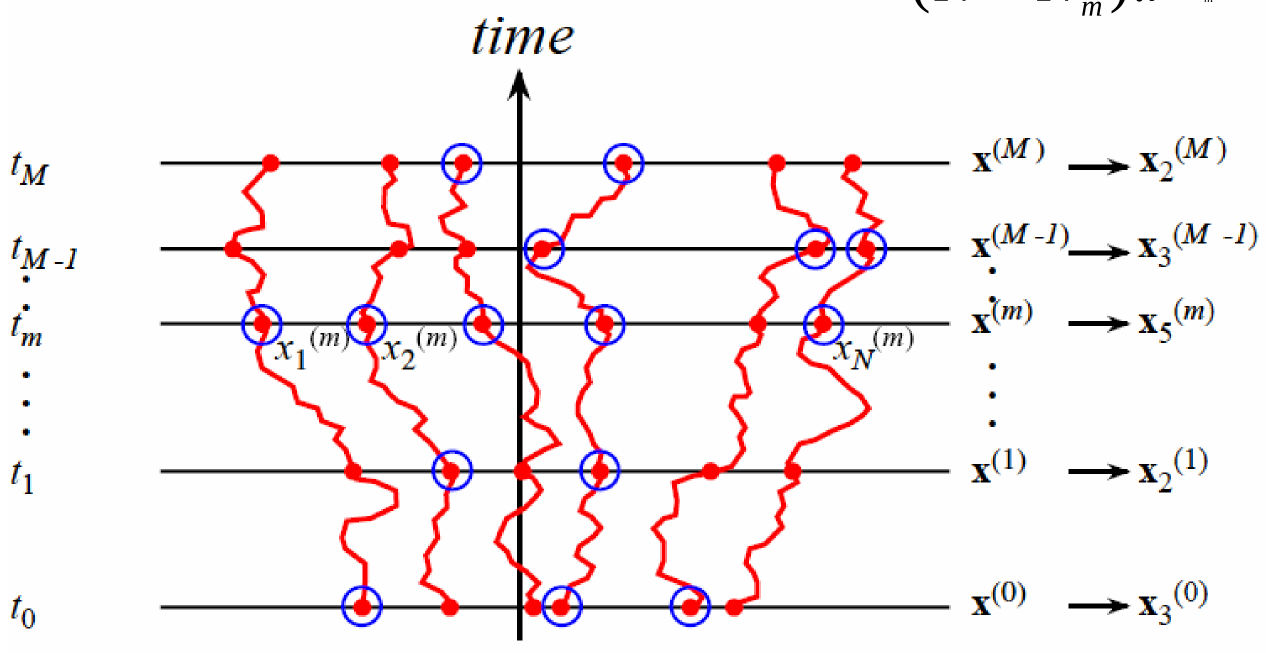
put $\mathbf{x}^{(m)}_{N'} = (x^{(m)}_1, x^{(m)}_2, \cdots, x^{(m)}_{N'})$.

For $(N_0, N_1, \cdots, N_M)$, $0 \le N_m \le N$ $(0 \le m \le M)$,

the $(N_0, N_1, \cdots, N_M)$ – multitime correlation function is defined as

$$\rho_N\left(t_0, \{\mathbf{x}^{(0)}_{N_0}\}; t_1, \{\mathbf{x}^{(1)}_{N_1}\}; \cdots, t_M, \{\mathbf{x}^{(M)}_{N_M}\}\right)$$

$$\equiv \int_{\prod_{m=0}^{M} \mathbf{R}^{N-N_m}} \mathrm{p}_N\left(t_0, \{\mathbf{x}^{(0)}\}; t_1, \{\mathbf{x}^{(1)}\}; \cdots, t_M, \{\mathbf{x}^{(M)}\}\right) \prod_{m=0}^{M} \frac{1}{(N - N_m)!} \prod_{j=N_m+1}^{N} dx^{(m)}_j$$



Theorem 1

The noncolliding diffusion processes are (finite) **determinantal processes** in the sense that any multitime correlation function is given by a determinant specified by matrix - kernel K_N;

$$\rho_N\left(t_0,\{\mathbf{x}^{(0)}_{N_0}\};t_1,\{\mathbf{x}^{(1)}_{N_1}\};\cdots,t_M,\{\mathbf{x}^{(M)}_{N_M}\}\right)$$
$$=\det_{1\le j\le N_m,1\le k\le N_n,0\le m,n\le M}\left[K_N\left(t_m,x_j^{(m)};t_n,x_k^{(n)}\right)\right]$$

For **[A] noncolliding BM**,

$$K_N(s,x;t,y)=\begin{cases}\dfrac{1}{\sqrt{2s}}\displaystyle\sum_{k=0}^{N-1}\left(\frac{t}{s}\right)^{k/2}\varphi_k\left(\frac{x}{\sqrt{2s}}\right)\varphi_k\left(\frac{y}{\sqrt{2t}}\right) & \text{if } s\le t\\ -\dfrac{1}{\sqrt{2s}}\displaystyle\sum_{k=N}^{\infty}\left(\frac{t}{s}\right)^{k/2}\varphi_k\left(\frac{x}{\sqrt{2s}}\right)\varphi_k\left(\frac{y}{\sqrt{2t}}\right) & \text{if } s>t\end{cases}$$

where $\{\varphi_k\}_{k=0}^{\infty}$ are the orthonormal **Hermite** functions;

$$\varphi_k(\zeta)=\frac{1}{\sqrt{\sqrt{\pi}\,2^k k!}}\mathrm{e}^{-\zeta^2/2}H_k(\zeta)$$

For **[B] noncolliding BESQ$_\nu$**,

$$K_N(s,x;t,y)=\begin{cases}\dfrac{1}{2s}\displaystyle\sum_{k=0}^{N-1}\left(\frac{t}{s}\right)^{k}\varphi_k^{\nu}\left(\frac{x}{2s}\right)\varphi_k^{\nu}\left(\frac{y}{2t}\right) & \text{if } s\le t\\ -\dfrac{1}{2s}\displaystyle\sum_{k=N}^{\infty}\left(\frac{t}{s}\right)^{k}\varphi_k^{\nu}\left(\frac{x}{2s}\right)\varphi_k^{\nu}\left(\frac{y}{2t}\right) & \text{if } s>t\end{cases}$$

where $\{\varphi_k^{\nu}\}_{k=0}^{\infty}$ are the orthonormal **Laguerre** functions;

$$\varphi_k^{\nu}(\zeta)=\sqrt{\frac{\Gamma(k+1)}{\Gamma(k+1+\nu)}}\zeta^{\nu/2}\mathrm{e}^{-\zeta/2}L_k^{\nu}(\zeta)$$

Proof.

$C_0(\mathbf{R}) =$ set of all continuous real functions with compact supports.

$\mathbf{f} = (f_1, f_2, \cdots, f_M) \in C_0(\mathbf{R})$

$\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_M) \in \mathbf{R}^M$

generating function for multitime correlation function for $\Xi(t) = \{X(t)\}$.

$$\psi_N[\chi] = \mathbf{E}\left[\exp\left\{\sum_{m=0}^{M} \theta_m \sum_{j_m=1}^{N} f_m(X_{j_m}(t_m))\right\}\right]$$

with $\chi_m(x) = \exp(\theta_m f_m(x)) - 1.$

We can show that

$\psi_N[\chi] = \mathrm{Det}\left[\delta_{mn}\delta(x-y) + K_N(t_m, x; t_n, y)\chi_n(y)\right]$ (Fredholm determinant)

diffusion equations and (imaginary time) Schrodinger equations

diffusione equation $\dfrac{\partial}{\partial t}u(t,x)=\dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}u(t,x)$

Set $\tau=\log t,\ \zeta=\dfrac{x}{\sqrt{2t}}$, and $u(t,x)=\mathrm{e}^{-(\tau+\zeta^2)/2}U(\tau,\zeta)$.

$$\Rightarrow\ \frac{\partial}{\partial\tau}U(\tau,\zeta)=-\frac{1}{2}\left(\mathbf{H}_H-\frac{1}{2}\right)U(\tau,\zeta)$$

$$\mathbf{H}_H=-\frac{1}{2}\frac{\partial^2}{\partial\zeta^2}+\frac{1}{2}\zeta^2\quad\text{(harmonic oscillator)},\quad\left(\mathbf{H}_H-\frac{1}{2}\right)H_k(\zeta)=k\,H_k(\zeta),\qquad k=0,1,2,\cdots$$

forward Kolmogorov equation for BESQ$_\nu$

$$\frac{\partial}{\partial t}u(t,x)=2x\frac{\partial^2}{\partial x^2}u(t,x)-2(\nu-1)\frac{\partial}{\partial x}u(t,x)$$

Set $\tau=\log t,\ \zeta=\dfrac{x}{2t}$, and $u(t,x)=\mathrm{e}^{-(\tau+\zeta)/2}\,\zeta^{\nu/2}U(\tau,\zeta)$.

$$\Rightarrow\ \frac{\partial}{\partial\tau}U(\tau,\zeta)=-\mathbf{H}_L U(\tau,\zeta)$$

$$\mathbf{H}_L=-\frac{\partial}{\partial\zeta}\zeta\frac{\partial}{\partial\zeta}+\frac{1}{4}\left(\sqrt{\zeta}-\frac{\nu}{\sqrt{\zeta}}\right)^2-\frac{1}{2},\quad \mathbf{H}_L\varphi_k^\nu(\zeta)=k\,\varphi_k^\nu(\zeta),\qquad k=0,1,2,\cdots$$

$$\varphi_k^\nu(\zeta)=\sqrt{\frac{\Gamma(k+1)}{\Gamma(k+1+\nu)}}\zeta^{\nu/2}\,\mathrm{e}^{-\zeta/2}\,L_k^\nu(\zeta)$$

2. Infinite Particle Limits

Theorem 2 If we take the appropriate **scaling limits ($N \to \infty$, $t \to \infty$)**, We will have the **infinite determinantal processes** governed by the following matrix-kernels.

$$\rho\left(t_0, \{\mathbf{x}_{N_0}^{(0)}\}; t_1, \{\mathbf{x}_{N_1}^{(1)}\}; \cdots, t_M, \{\mathbf{x}_{N_M}^{(M)}\}\right) = \det_{1 \le j \le N_m, 1 \le k \le N_n, 0 \le m, n \le M} \left[K\left(t_m, x_j^{(m)}; t_n, x_k^{(n)}\right) \right]$$

[A1] **bulk scaling limit** of noncolliding BM

$$x : \text{finite}, \quad t \approx N \to \infty$$

$$K(s, x; t, y) = \begin{cases} \dfrac{1}{\pi} \displaystyle\int_0^1 du \, e^{(t-s)u^2} \cos(u(x-y)) & \text{if } s \le t \\ -\dfrac{1}{\pi} \displaystyle\int_1^\infty du \, e^{-(s-t)u^2} \cos(u(x-y)) & \text{if } s > t \end{cases}$$

extended sine kernel

[A2] **soft-edge scaling limit** of noncolliding BM

$$x \approx 2N^{2/3}, \quad t \approx N^{1/3}, \quad N \to \infty$$

$$K(s, x; t, y) = \begin{cases} \displaystyle\int_{-\infty}^0 d\lambda \, e^{(t-s)\lambda} \mathrm{Ai}(x-\lambda) \, \mathrm{Ai}(y-\lambda) & \text{if } s \le t \\ -\displaystyle\int_0^\infty d\lambda \, e^{-(s-t)\lambda} \mathrm{Ai}(x-\lambda) \, \mathrm{Ai}(y-\lambda) & \text{if } s > t \end{cases}$$

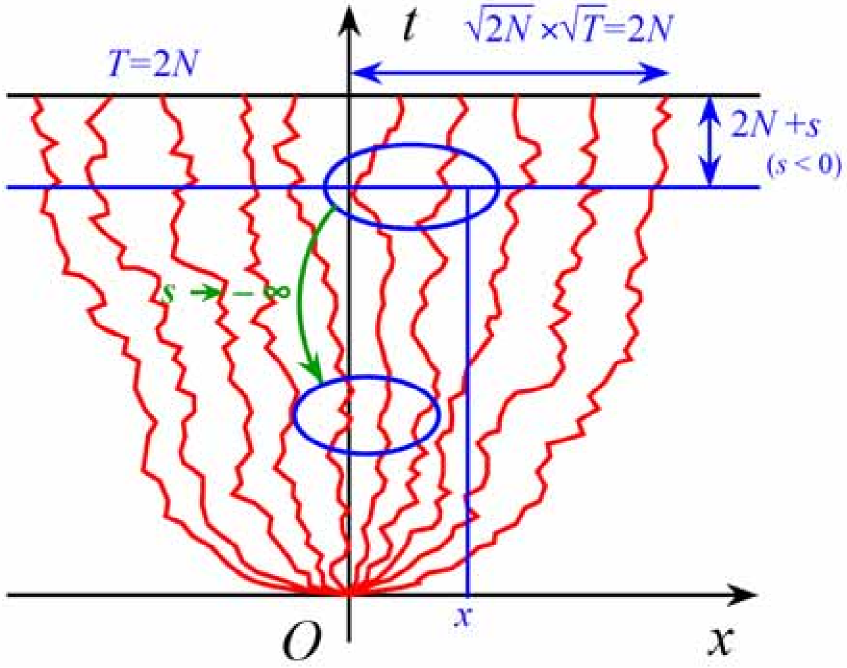
extended Airy kernel

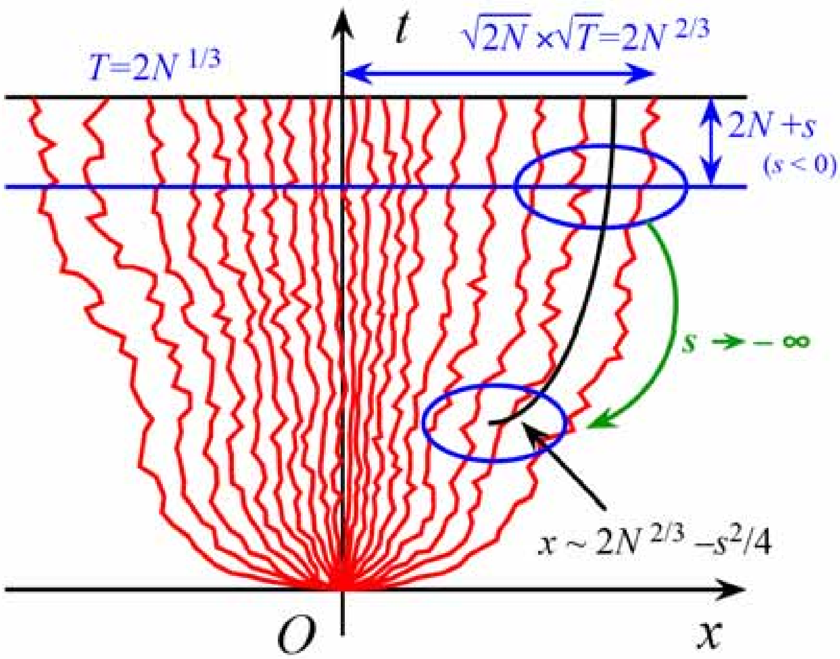
[B] **hard - edge scaling limit** of noncolliding BESQ_ν

x : finite, $t \approx N/2 \to \infty$

$$K(s,x;t,y) = \begin{cases} \displaystyle\int_0^1 d\lambda\, \mathrm{e}^{(t-s)\lambda} J_\nu(2\sqrt{\lambda x}\,) J_\nu(2\sqrt{\lambda y}\,) & \text{if} \quad s \le t \\ \displaystyle -\int_1^\infty d\lambda\, \mathrm{e}^{-(s-t)\lambda} J_\nu(2\sqrt{\lambda x}\,) J_\nu(2\sqrt{\lambda y}\,) & \text{if} \quad s > t \end{cases}$$

extended Bessel kernel





bulk scaling limit⇒sine kernel

soft-edge scaling limit⇒Airy kernel

Proof.

$$\varphi_k(\zeta) = \frac{1}{\sqrt{\sqrt{\pi}\,2^k\,k!}}\,\mathrm{e}^{-\zeta^2/2}\,H_k(\zeta)$$

$$\varphi_k^{\nu}(\zeta) = \sqrt{\frac{\Gamma(k+1)}{\Gamma(k+1+\nu)}}\,\zeta^{\nu/2}\,\mathrm{e}^{-\zeta/2}\,L_k^{\nu}(\zeta)$$

The following asymptotics are used.

$$\lim_{\ell\to\infty}(-1)^{\ell}\,\ell^{1/4}\varphi_{2\ell}\left(\frac{u}{2\sqrt{\ell}}\right) = \frac{1}{\sqrt{\pi}}\cos u$$

$$\lim_{\ell\to\infty}(-1)^{\ell}\,\ell^{1/4}\varphi_{2\ell+1}\left(\frac{u}{2\sqrt{\ell}}\right) = \frac{1}{\sqrt{\pi}}\sin u$$

$$\lim_{\ell\to\infty} 2^{-1/4}\,\ell^{1/12}\varphi_{\ell}\left(\sqrt{2\ell} + \frac{u}{\sqrt{2}}\,\ell^{-1/6}\right) = \mathrm{Ai}(u)\ ,\ u \in \mathbf{R}$$

$$\lim_{\ell\to\infty} \varphi_k^{\nu}\left(\frac{u}{\ell}\right) = J_{\nu}(2\sqrt{u})\,,\ u \in \mathbf{R}_+$$

generalized eigenfunctions

trigonomet ric functions

$$S_+(\sqrt{\lambda}x) = \frac{1}{\sqrt{2\pi}\lambda^{1/4}} \sin\left(\sqrt{\lambda}x\right),\ S_-(\sqrt{\lambda}x) = \frac{1}{\sqrt{2\pi}\lambda^{1/4}} \cos\left(\sqrt{\lambda}x\right)$$

$$\mathbf{H}_{\sin} S_\pm(\sqrt{\lambda}x) = \lambda\, S_\pm(\sqrt{\lambda}x), \quad \lambda \in \sigma(\mathbf{H}_{\sin}) = \mathbf{R}_+ \ \text{ with } \ \mathbf{H}_{\sin} = -\frac{\partial^2}{\partial x^2}$$

Airy functions

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\ \mathrm{e}^{\sqrt{-1}(xk + k^3/3)}$$

$$\mathbf{H}_{\mathrm{Ai}} \mathrm{Ai}(x-\lambda) = \lambda\ \mathrm{Ai}(x-\lambda), \quad \lambda \in \sigma(\mathbf{H}_{\mathrm{Ai}}) = \mathbf{R} \ \text{ with } \ \mathbf{H}_{\mathrm{Ai}} = -\frac{\partial^2}{\partial x^2} + x$$

Bessel functions

$$J_\nu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+\nu}}{\Gamma(n+1)\Gamma(n+1+\nu)}$$

$$\mathbf{H}_{\mathrm{J}} J_\nu(2\sqrt{\lambda x}) = \lambda\ J_\nu(2\sqrt{\lambda x}), \quad \lambda \in \sigma(\mathbf{H}_{\mathrm{J}}) = \mathbf{R}_+ \ \text{ with } \mathbf{H}_{\mathrm{J}} = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x} + \frac{\nu^2}{4x}$$

3. Possible General Form of Determinantal Processes

(1) (Effective) Hamiltonian

$$\mathbf{H} = -a(x)\frac{\partial^2}{\partial x^2} - b(x)\frac{\partial}{\partial x} - c(x), \qquad a(x) \neq 0 \text{ for } x \in \Lambda$$

$\downarrow$ Liouville transformation

① $x \to z = \int_0^x a(y)^{-1/2} dy$

② $\mathbf{H} \to r^{-1}\mathbf{H}\, r, \qquad r = \exp\left\{-\frac{1}{2}\int_0^z \tilde{b}(u)du\right\}$

Strum - Liouville operator

$$\mathbf{H} = -\frac{\partial^2}{\partial x^2} + q(x)$$

Eamples

$$q(x) = \begin{cases} \frac{1}{16}(x^2 - 4) & \text{for [A] noncolliding BM } (\Lambda = \mathbf{R}) \\ \frac{1}{16}\left\{x^2 + 16\left(\nu^2 - \frac{1}{4}\right)\frac{1}{x^2} - 8(\nu+1)\right\} & \text{for [B] non colliding BESQ}_\nu \ (\Lambda = \mathbf{R}_+) \\ 0 & \text{for [A1] sine-kernel } (\Lambda = \mathbf{R}) \\ x & \text{for [A2] Airy-kernel } (\Lambda = \mathbf{R}) \\ \left(\nu^2 - \frac{1}{4}\right)\frac{1}{x^2} & \text{for [B1] Bessel-kernel } (\Lambda = \mathbf{R}_+) \end{cases}$$

(2) Eigenvalue (spectrum) and Eigenfunctions

$$\mathbf{H}\varphi_\ell(x) = \lambda_\ell \varphi_\ell(x), \quad \ell = 0,1,2,\cdots$$

(Schrodinger) equation

$$\frac{\partial}{\partial t}\varphi_\ell(t,x) = -\mathbf{H}\varphi_\ell(t,x) \;\Rightarrow\; \varphi_\ell(t,x) = \varphi_\ell(x)\mathrm{e}^{-\lambda_\ell t}, \quad \ell = 0,1,2,\cdots$$

Let $\overline{\varphi}_\ell(t,x) = \varphi_\ell(x)\mathrm{e}^{\lambda_\ell t}$.

(3) A Specified Level $\ell^* \Leftrightarrow \lambda_{\ell^*}$

(4) Matrix - Kernel

$$K(s,x;t,y) = \begin{cases} \sum_{\ell=0}^{\ell^*-1} \varphi_\ell(s,x)\overline{\varphi}_\ell(t,y) & \text{if } s \le t \\ -\sum_{\ell=\ell^*}^{\infty} \varphi_\ell(s,x)\overline{\varphi}_\ell(t,y) & \text{if } s > t \end{cases}$$

(5) For any $M \ge 0$, any $N_0 > 0, N_1 > 0, \cdots, N_M > 0$,

the $(N_0, N_1, \cdots, N_M)$ - multitime correlation function is given by the determinant

$$\rho\left(t_0, \{\mathbf{x}^{(0)}_{N_0}\}; t_1, \{\mathbf{x}^{(1)}_{N_1}\}; \cdots, t_M, \{\mathbf{x}^{(M)}_{N_M}\}\right) = \det_{1\le j\le N_m, 1\le k\le N_n, 0\le m,n\le M}\left[K\left(t_m, x^{(m)}_j; t_n, x^{(n)}_k\right)\right].$$

Determinantal Processes

Problem

Well defined ?

(Q1) total positivity
(Q2) continuity
(Q3) Markov property

[A] noncolliding BM
[B] noncolliding BESQ ν
[A1] bulk limit of [A] (ext. sine-kernel)
[A2] soft-edge limit of [A] (ext. Airy-kernel)
[B1] hard-edge limit of [B] (ext. Bessel-kernel)

Remarks

(a) For the five examples [A],[B],[A1],[A2],[B1],
(Q1) is OK, by the Karlin-McGregor formula.

(b) For the finite systems [A], [B], (Q2) and (Q3) are OK,
since they are conditional diffusion processes.

(c) For the examples of infinite systems [A1],[A2], [B1],
(Q2) was proved. (Katori and Tanemura (2007) J. Stat. Phys.)

(d) But even for these three examples of infinite systems,
(Q3) is not proved.

(cf : Dirichlet form was determined in [KT07], but …)

4. Equal-Time Case

Let $s=t$.

$$K(t,x;t,y)=\sum_{\ell=0}^{\ell^*-1}\varphi_\ell(t,x)\overline{\varphi_\ell}(t,y)=\sum_{\ell=0}^{\ell^*-1}\varphi_\ell(x)\varphi_\ell(y)$$

For simplicity, set $N=\ell^*-1$.

$$\rho\left(\{\mathbf{x}_{\ell^*-1}\}\right)=\det_{1\le j,k\le \ell^*-1}\left[\sum_{\ell=0}^{\ell^*-1}\varphi_\ell(x_j)\varphi_\ell(x_k)\right]=\left(\det_{1\le \ell+1,\, j\le N}\left[\varphi_\ell(x_j)\right]\right)^2\ge 0$$

That is,

$$\rho\left(\{\mathbf{x}_{\ell^*-1}\}\right)=\left(\Psi\left(\{\mathbf{x}_{\ell^*-1}\}\right)\right)^2 \quad \text{with}$$

$$\Psi_{\{0,1,\cdots,\ell^*-1\}}\left(\mathbf{x}_{\ell^*-1}\right)=\det_{1\le \ell+1,\, j\le \ell^*-1}\left[\varphi_\ell(x_j)\right]=\det\begin{bmatrix}\varphi_0(x_1) & \varphi_1(x_1) & \cdots & \varphi_{\ell^*-1}(x_1)\\ \varphi_0(x_2) & \varphi_1(x_2) & \cdots & \varphi_{\ell^*-1}(x_2)\\ \vdots & & & \vdots\\ \varphi_0(x_{\ell^*-1}) & \varphi_{\ell^*-1}(x_{\ell^*-1}) & \cdots & \varphi_{\ell^*-1}(x_{\ell^*-1})\end{bmatrix}$$

Slater determinant for many-body wave function of **free fermions**

References: Determinantal point processes on the space.

Soshinikov, Russ. Math. Surv. (2000).

Shirai-Takahashi, J. Func. Anal. (2003).

5. Multi-Time Cases

$$K(s,x;t,y)=\begin{cases}\displaystyle\sum_{\ell=0}^{\ell^*-1}\varphi_\ell(s,x)\overline{\varphi_\ell}(t,y) & \text{if} \quad s\le t\\ \displaystyle-\sum_{\ell=\ell^*}^{\infty}\varphi_\ell(s,x)\overline{\varphi_\ell}(t,y) & \text{if} \quad s>t\end{cases}$$

$$=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\varphi_p(s,x)g(s,p;t,q)\overline{\varphi_q}(t,y)$$

with

$$g(s,p;t,q)=\delta_{pq}\left[\mathbf{1}_{\{p\le\ell^*-1,\, s\le t\}}-\mathbf{1}_{\{p\ge\ell^*,\, s>t\}}\right]$$

Then

$$\rho\left(t_0,\{\mathbf{x}_{N_0}^{(0)}\};t_1,\{\mathbf{x}_{N_1}^{(1)}\};\cdots,t_M,\{\mathbf{x}_{N_M}^{(M)}\}\right)=\det\left[\Phi\, g\, \Phi^*\right]$$

Example (Two times ($s<t$), $N1=N2=2$)

$$\rho(s,\{x_1,x_2\};t,\{y_1,y_2\})$$

$$= \det\begin{bmatrix} \sum_{\ell=0}^{\ell^*-1}(\varphi_\ell(x_1))^2 & \sum_{\ell=0}^{\ell^*-1}\varphi_\ell(x_1)\varphi_\ell(x_2) & \sum_{\ell=0}^{\ell^*-1}e^{(t-s)\lambda_\ell}\,\varphi_\ell(x_1)\varphi_\ell(y_1) & \sum_{\ell=0}^{\ell^*-1}e^{(t-s)\lambda_\ell}\,\varphi_\ell(x_1)\varphi_\ell(y_2) \\ \sum_{\ell=0}^{\ell^*-1}\varphi_\ell(x_2)\varphi_\ell(x_1) & \sum_{\ell=0}^{\ell^*-1}(\varphi_\ell(x_2))^2 & \sum_{\ell=0}^{\ell^*-1}e^{(t-s)\lambda_\ell}\,\varphi_\ell(x_2)\varphi_\ell(y_1) & \sum_{\ell=0}^{\ell^*-1}e^{(t-s)\lambda_\ell}\,\varphi_\ell(x_2)\varphi_\ell(y_2) \\ -\sum_{\ell=\ell^*}^{\infty}e^{-(t-s)\lambda_\ell}\,\varphi_\ell(y_1)\varphi_\ell(x_1) & -\sum_{\ell=\ell^*}^{\infty}e^{-(t-s)\lambda_\ell}\,\varphi_\ell(y_1)\varphi_\ell(x_2) & \sum_{\ell=0}^{\ell^*-1}(\varphi_\ell(y_1))^2 & \sum_{\ell=0}^{\ell^*-1}\varphi_\ell(y_1)\varphi_\ell(y_2) \\ -\sum_{\ell=\ell^*}^{\infty}e^{-(t-s)\lambda_\ell}\,\varphi_\ell(y_2)\varphi_\ell(x_1) & -\sum_{\ell=\ell^*}^{\infty}e^{-(t-s)\lambda_\ell}\,\varphi_\ell(y_2)\varphi_\ell(x_2) & \sum_{\ell=0}^{\ell^*-1}\varphi_\ell(y_2)\varphi_\ell(y_1) & \sum_{\ell=0}^{\ell^*-1}(\varphi_\ell(y_2))^2 \end{bmatrix}$$

$$= \det\Bigg[\begin{pmatrix} \varphi_0(s,x_1) & \varphi_1(s,x_1) & \cdots & \varphi_{\ell^*}(s,x_1) & \varphi_{\ell^*+1}(s,x_1) & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \varphi_0(s,x_2) & \varphi_0(s,x_2) & \cdots & \varphi_{\ell^*}(s,x_2) & \varphi_{\ell^*+1}(s,x_2) & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \varphi_0(t,y_1) & \varphi_1(t,y_1) & \cdots & \varphi_{\ell^*}(t,y_1) & \varphi_{\ell^*+1}(t,y_1) & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \varphi_0(t,y_2) & \varphi_1(t,y_2) & \cdots & \varphi_{\ell^*}(t,y_2) & \varphi_{\ell^*+1}(t,y_2) & \cdots \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & & & & & & 1 & & & & & \\ & 1 & & & & & & 1 & & & & \\ & & \ddots & & & & & & \ddots & & & \\ & & & 0 & & & & & & 0 & & \\ & & & & 0 & & & & & & 0 & \\ & & & & & \ddots & & & & & & \ddots \\ 0 & & & & & & 1 & & & & & \\ & 0 & & & & & & 1 & & & & \\ & & \ddots & & & & & & \ddots & & & \\ & & & -1 & & & & & & 0 & & \\ & & & & -1 & & & & & & 0 & \\ & & & & & \ddots & & & & & & \ddots \end{pmatrix} \times \begin{pmatrix} \overline{\varphi}_0(s,x_1) & \overline{\varphi}_0(s,x_2) & 0 & 0 \\ \overline{\varphi}_1(s,x_1) & \overline{\varphi}_0(s,x_2) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \overline{\varphi}_{\ell^*}(s,x_1) & \overline{\varphi}_{\ell^*}(s,x_2) & 0 & 0 \\ \overline{\varphi}_{\ell^*+1}(s,x_1) & \overline{\varphi}_{\ell^*+1}(s,x_2) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \overline{\varphi}_0(t,y_1) & \overline{\varphi}_0(t,y_2) \\ 0 & 0 & \overline{\varphi}_1(t,y_1) & \overline{\varphi}_1(t,y_2) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \overline{\varphi}_{\ell^*}(t,y_1) & \overline{\varphi}_{\ell^*}(t,y_2) \\ 0 & 0 & \overline{\varphi}_{\ell^*+1}(t,y_1) & \overline{\varphi}_{\ell^*+1}(t,y_2) \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \Bigg]$$

$$\rho\left(t_0,\{\mathbf{x}_{N_0}^{(0)}\};t_1,\{\mathbf{x}_{N_1}^{(1)}\};\cdots,t_M,\{\mathbf{x}_{N_M}^{(M)}\}\right)=\det_{1\le j\le N_m,1\le k\le N_n,0\le m,n\le M}\left[K\left(t_m,x_j^{(m)};t_n,x_k^{(n)}\right)\right]$$

$$=\det\left[\Phi\ \mathrm{g}\ \Phi^*\right]$$

where $\Phi:\ \sum_{m=0}^{M} N_m \times \infty$ matrix

$\mathrm{g}:\ \infty\times\infty$ matrix

$\Phi^* = \overline{\Phi}^T:\ \infty\times\sum_{m=0}^{M} N_m$ matrix

$$\det\left[\Phi\ \mathrm{g}\ \Phi^*\right]=\det[\Phi]\times\det[\mathrm{g}]\times\det\left[\Phi^*\right] \quad ???$$

Remark

If we approximate g by a finite matrix, say $L\times L$ matrix, by truncation

$$\mathrm{g}\approx \mathrm{g}_L$$

we will have $\det[\mathrm{g}_L]=0$ many times,

since the matrix g is so sparse.

Expansion by minors:

In general, for $A: s\times r$ matrix, $B: r\times s$ matrix, with $s<r$

$$\det[AB]=\sum_J \det\left[A_{\{J\}}\right]\times\det\left[B_{\{J\}}\right]$$

where summation is over all subsets $J=\{j_1, j_2,\cdots, j_s\}$ of $\{1,2,\cdots,r\}$

s.t. $j_1<j_2<\cdots<j_s$

$A_{\{J\}}$: $s\times s$ submatrix of A, where colums have indices in J

$B_{\{J\}}$: $s\times s$ submatrix of B, where rows have indices in J

$\det\left[A_{\{J\}}\right]$: minor of det [A]

$\det\left[B_{\{J\}}\right]$: minor of det [B]

This expansion formula by minors may be used for the case $r=\infty$.

We will use this formula twice.

Again Example (Two times ($s<t$), $N_1 = N_2 = 2$)

$$
\begin{aligned}
&\rho\left(s,\{x_1,x_2\};t,\{y_1,y_2\}\right) \\
&= \sum_{0\le\ell_1<\ell_2\le\ell^*-1}\ \sum_{0\le\ell_3<\ell_4\le\ell^*-1} \det\begin{bmatrix} \varphi_{\ell_1}(s,x_1) & \varphi_{\ell_2}(s,x_1) & & \\ \varphi_{\ell_1}(s,x_2) & \varphi_{\ell_2}(s,x_2) & & \\ & & \varphi_{\ell_3}(t,y_1) & \varphi_{\ell_4}(t,y_1) \\ & & \varphi_{\ell_3}(t,y_2) & \varphi_{\ell_4}(t,y_2) \end{bmatrix} \times \det\begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix} \times \det\begin{bmatrix} \overline{\varphi}_{\ell_1}(s,x_1) & \overline{\varphi}_{\ell_1}(s,x_2) & & \\ \overline{\varphi}_{\ell_2}(s,x_1) & \overline{\varphi}_{\ell_2}(s,x_2) & & \\ & & \overline{\varphi}_{\ell_3}(t,y_1) & \overline{\varphi}_{\ell_3}(t,y_2) \\ & & \overline{\varphi}_{\ell_4}(t,y_1) & \overline{\varphi}_{\ell_4}(t,y_2) \end{bmatrix} \\
&+ \sum_{\ell_1=0}^{\ell^*-1}\sum_{\ell_2=0}^{\ell^*-1}\sum_{\ell_3=0}^{\ell^*-1}\sum_{\ell_4=\ell^*}^{\infty} \det\begin{bmatrix} \varphi_{\ell_1}(s,x_1) & \varphi_{\ell_2}(s,x_1) & & \\ \varphi_{\ell_1}(s,x_2) & \varphi_{\ell_2}(s,x_2) & & \\ & & \varphi_{\ell_3}(t,y_1) & \varphi_{\ell_4}(t,y_1) \\ & & \varphi_{\ell_3}(t,y_2) & \varphi_{\ell_4}(t,y_2) \end{bmatrix} \times \det\begin{bmatrix} 1 & & 1 & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{bmatrix} \times \det\begin{bmatrix} \overline{\varphi}_{\ell_1}(s,x_1) & \overline{\varphi}_{\ell_1}(s,x_2) & & \\ \overline{\varphi}_{\ell_4}(s,x_1) & \overline{\varphi}_{\ell_4}(s,x_2) & & \\ & & \overline{\varphi}_{\ell_3}(t,y_1) & \overline{\varphi}_{\ell_3}(t,y_2) \\ & & \overline{\varphi}_{\ell_2}(t,y_1) & \overline{\varphi}_{\ell_2}(t,y_2) \end{bmatrix} \\
&+ \sum_{0\le\ell_1<\ell_2\le\ell^*-1}\ \sum_{\ell^*\le\ell_3<\ell_4} \det\begin{bmatrix} \varphi_{\ell_1}(s,x_1) & \varphi_{\ell_2}(s,x_1) & & \\ \varphi_{\ell_1}(s,x_2) & \varphi_{\ell_2}(s,x_2) & & \\ & & \varphi_{\ell_3}(t,y_1) & \varphi_{\ell_4}(t,y_1) \\ & & \varphi_{\ell_3}(t,y_2) & \varphi_{\ell_4}(t,y_2) \end{bmatrix} \times \det\begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix} \times \det\begin{bmatrix} \overline{\varphi}_{\ell_3}(s,x_1) & \overline{\varphi}_{\ell_3}(s,x_2) & & \\ \overline{\varphi}_{\ell_4}(s,x_1) & \overline{\varphi}_{\ell_4}(s,x_2) & & \\ & & \overline{\varphi}_{\ell_1}(t,y_1) & \overline{\varphi}_{\ell_1}(t,y_2) \\ & & \overline{\varphi}_{\ell_2}(t,y_1) & \overline{\varphi}_{\ell_2}(t,y_2) \end{bmatrix}
\end{aligned}
$$

$$\rho\left(s,\{x_1,x_2\};t,\{y_1,y_2\}\right)$$

$$= \sum_{0\le\ell_1<\ell_2\le\ell^*-1}\left(\det\begin{bmatrix}\varphi_{\ell_1}(x_1) & \varphi_{\ell_2}(x_1)\\ \varphi_{\ell_1}(x_2) & \varphi_{\ell_2}(x_2)\end{bmatrix}\right)^2 \sum_{0\le\ell_3<\ell_4\le\ell^*-1}\left(\det\begin{bmatrix}\varphi_{\ell_3}(y_1) & \varphi_{\ell_4}(y_1)\\ \varphi_{\ell_3}(y_2) & \varphi_{\ell_4}(y_2)\end{bmatrix}\right)^2$$

$$+ \sum_{\ell_1=0}^{\ell^*-1}\sum_{\ell_2=0}^{\ell^*-1}\sum_{\ell_3=0}^{\ell^*-1}\sum_{\ell_4=\ell^*}^{\infty} \exp\left[-(\lambda_{\ell_4}-\lambda_{\ell_2})(t-s)\right]\det\begin{bmatrix}\varphi_{\ell_1}(x_1) & \varphi_{\ell_2}(x_1)\\ \varphi_{\ell_1}(x_2) & \varphi_{\ell_2}(x_2)\end{bmatrix}\det\begin{bmatrix}\varphi_{\ell_1}(x_1) & \varphi_{\ell_4}(x_1)\\ \varphi_{\ell_1}(x_2) & \varphi_{\ell_4}(x_2)\end{bmatrix}$$

$$\times\det\begin{bmatrix}\varphi_{\ell_3}(y_1) & \varphi_{\ell_4}(y_1)\\ \varphi_{\ell_3}(y_2) & \varphi_{\ell_4}(y_2)\end{bmatrix}\det\begin{bmatrix}\varphi_{\ell_3}(y_1) & \varphi_{\ell_2}(y_1)\\ \varphi_{\ell_3}(y_2) & \varphi_{\ell_2}(y_2)\end{bmatrix}$$

$$+ \sum_{0\le\ell_1<\ell_2\le\ell^*-1}\ \sum_{\ell^*\le\ell_3<\ell_4}\exp\left[-\left\{(\lambda_{\ell_3}-\lambda_{\ell_1})+(\lambda_{\ell_4}-\lambda_{\ell_2})\right\}(t-s)\right]\det\begin{bmatrix}\varphi_{\ell_1}(x_1) & \varphi_{\ell_2}(x_1)\\ \varphi_{\ell_1}(x_2) & \varphi_{\ell_2}(x_2)\end{bmatrix}\det\begin{bmatrix}\varphi_{\ell_3}(x_1) & \varphi_{\ell_4}(x_1)\\ \varphi_{\ell_3}(x_2) & \varphi_{\ell_4}(x_2)\end{bmatrix}$$

$$\times\det\begin{bmatrix}\varphi_{\ell_3}(y_1) & \varphi_{\ell_4}(y_1)\\ \varphi_{\ell_3}(y_2) & \varphi_{\ell_4}(y_2)\end{bmatrix}\det\begin{bmatrix}\varphi_{\ell_1}(y_1) & \varphi_{\ell_2}(y_1)\\ \varphi_{\ell_1}(y_2) & \varphi_{\ell_2}(y_2)\end{bmatrix}$$

$$
\begin{aligned}
&\rho\left(s,\{x_1,x_2\};t,\{y_1,y_2\}\right) \\
&\quad = \sum_{0\le \ell_1<\ell_2\le \ell^*-1}\left(\Psi_{\{\ell_1,\ell_2\}}(\mathbf{x}_{N_1})\right)^2 \sum_{0\le \ell_3<\ell_4\le \ell^*-1}\left(\Psi_{\{\ell_3,\ell_4\}}(\mathbf{y}_{N_2})\right)^2 \\
&\quad + \sum_{\ell_1=0}^{\ell^*-1}\sum_{\ell_2=0}^{\ell^*-1}\sum_{\ell_3=0}^{\ell^*-1}\sum_{\ell_4=\ell^*}^{\infty} \mathrm{e}^{-(\lambda_{\ell_4}-\lambda_{\ell_2})(t-s)}\, \Psi_{\{\ell_1,\ell_2\}}(\mathbf{x}_{N_1})\Psi_{\{\ell_1,\ell_4\}}(\mathbf{x}_{N_1})\Psi_{\{\ell_3,\ell_4\}}(\mathbf{y}_{N_2})\Psi_{\{\ell_3,\ell_2\}}(\mathbf{y}_{N_2}) \\
&\quad + \sum_{0\le \ell_1<\ell_2\le \ell^*-1}\ \sum_{\ell^*\le \ell_3<\ell_4} \mathrm{e}^{-\left\{(\lambda_{\ell_3}-\lambda_{\ell_1})+(\lambda_{\ell_4}-\lambda_{\ell_2})\right\}(t-s)}\, \Psi_{\{\ell_1,\ell_2\}}(\mathbf{x}_{N_1})\Psi_{\{\ell_3,\ell_4\}}(\mathbf{x}_{N_1})\Psi_{\{\ell_3,\ell_4\}}(\mathbf{y}_{N_2})\Psi_{\{\ell_1,\ell_2\}}(\mathbf{y}_{N_2})
\end{aligned}
$$

The time dependence is associated with level - changes between Slater - determinantal wave functions in different times.

6. Open Problems

1. Characterization of deterninantal processes:
conditions for total positivity ?
Markov property ?
In particular for infinite determinantal processes.

2. Understanding of (1+1)-dimensional free fermion systems.

cf. : Praehofer-Spohn, J.Stat.Phys.(2002)
Ferrari-Spohn, J. Stat. Phys. (2003)
quantum field theoretical description
Harnad-Orlov, Physica D (2007)
fermionic construction of tau function