Determinantal Processes and Entire Functions

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Katori, M., Tanemura, H. : Non-equilibrium dynamics of Dyson's model with an Infinite number of particles, Commun. Math. Phys. 293 (2010) 469-497

Katori, M., Tanemura, H. : Zeros of Airy function and relaxation process, J. Stat. Phys. 136 (2009) 1177-1204

Katori, M., Tanemura, H. : Noncolliding squared Bessel processes and Weierstrass canonical products for entire functions, arXiv:1008.0144

1. The Dyson model as a Determinantal Processes

- **Dyson's Brownian motion (BM) model** $\{X_i(t)\}_{i=1}^n$

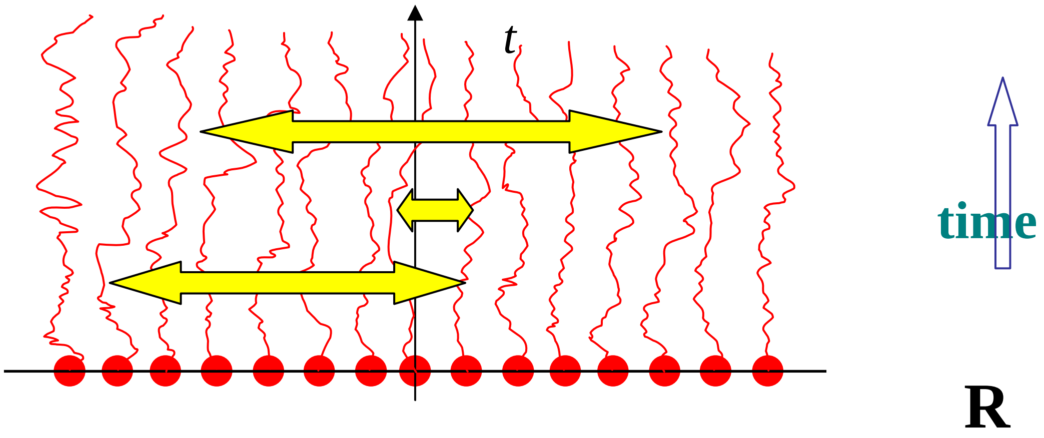
$$dX_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{1 \le j \le n, j \ne i} \frac{dt}{X_i(t) - X_j(t)}, \quad 1 \le i \le n, \quad t \in [0, \infty),$$

$\beta > 0$: a parameter indicating the strength of $1/x$ force,
$\{B_i(t)\}_{i=1}^n$: independent 1 dim. standard BMs, $B_i(0) = 0, 1 \le i \le n$.

- To understand the time-evolution of distributions of interacting particle systems on a large space-time scale (*thermodynamic and hydrodynamic limits*) is one of the main topics of statistical physics.

  — If the interactions among particles are **short ranged**, the standard theory is useful. *e.g.* Fritz (1987)

  — If they are **long ranged**, however, general theory has not yet been established and thus detailed study of model systems is required. *e.g.* Spohn (1987)



- In the present talk, we only consider the special case with $\beta = 2$.
- Let

$$\mathfrak{M} \equiv \text{the space of nonnegative integer-valued Radon measures on } \mathbb{R},$$
which is a Polish space with the vague topology.

$$\xi \in \mathfrak{M} \iff \xi(\cdot) = \sum_{i \in \mathbb{I}} \delta_{x_i}(\cdot) : \quad \text{unlabeled configuration}$$

with a finite or infinite index set $\mathbb{I}$,
a sequence of points in $\mathbb{R}$, $\boldsymbol{x} = (x_i)_{i \in \mathbb{I}}$,
satisfying $\xi(I) = \sharp\{x_i : x_i \in I\} < \infty$ for any compact subset $I \subset \mathbb{R}$.

We regard the Dyson model as **an $\mathfrak{M}$-valued diffusion process**

$$\Xi(t) = \sum_{i \in \mathbb{I}} \delta_{X_i(t)}, \quad t \in [0, \infty)$$

where $\{X_i(t)\}_{i \in \mathbb{I}}$ satisfy the SDEs

$$dX_i(t) = dB_i(t) + \sum_{1 \leq j \leq n, j \neq i} \frac{dt}{X_i(t) - X_j(t)}, \quad 1 \leq i \leq n, \quad t \in [0, \infty),$$

- The process under the initial configuration

$$\xi = \sum_{i \in \mathbb{I}} \delta_{x_i} \in \mathfrak{M}$$

is denoted by

$$(\Xi(t), \mathbb{P}_\xi).$$

We write the expectation with respect to $\mathbb{P}_\xi$ as $\mathbb{E}_\xi[\,\cdot\,]$.

- Note that

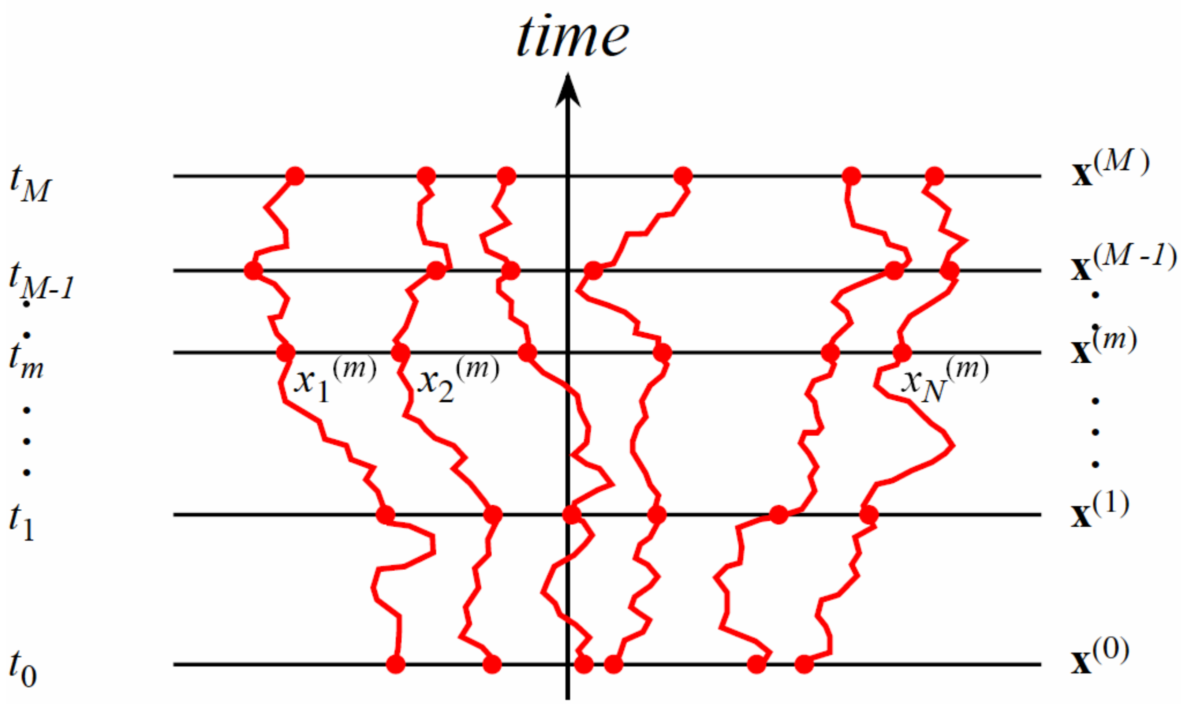
$$\xi(\mathbb{R}) = \Xi(t, \mathbb{R}) = \text{total number of particles}, \quad t \geq 0.$$

- Let

$$C_0(\mathbb{R}) = \text{the set of all continuous real-valued functions with compact supports},$$
$$M \in \mathbb{N} \equiv \{1, 2, \dots\},$$
$$\text{a sequence of times} \quad \boldsymbol{t} = (t_1, t_2, \dots, t_M) \quad \text{with} \quad 0 < t_1 < \cdots < t_M < \infty,$$
$$\text{a sequence of functions} \quad \boldsymbol{f} = (f_{t_1}, f_{t_2}, \dots, f_{t_M}) \in \mathrm{C}_0(\mathbb{R})^M.$$

- The **moment generating function** of multitime distribution of $(\Xi(t), \mathbb{P}_\xi)$

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] \equiv \mathbb{E}_\xi \left[\exp \left\{ \sum_{m=1}^{M} \int_{\mathbb{R}} f_{t_m}(x) \Xi(t_m, dx) \right\} \right].$$



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Remark 1.

Set $\chi_{t_m}(\cdot) = e^{f_{t_m}(\cdot)} - 1$, $1 \le m \le M$. Expand $\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}]$ w.r.t. χ_{t_m}'s. Then

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \sum_{\substack{N_m \ge 0, \\ 1 \le m \le M}} \int_{\prod_{m=1}^{M} \mathbb{W}_{N_m}^{A}} \prod_{m=1}^{M} \left\{ d\boldsymbol{x}_{N_m}^{(m)} \prod_{i=1}^{N_m} \chi_{t_m}\left(x_i^{(m)}\right) \right\} \rho_\xi\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\right)$$

with

multi-time correlation functions

$$\rho_\xi\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\right) = \int_{\prod_{m=1}^{M} \mathbb{R}^{N-N_m}} p_\xi\left(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)}\right) \prod_{m=1}^{M} \frac{1}{(N-N_m)!} \prod_{j=N_m+1}^{N} dx_j^{(m)}$$

where $\boldsymbol{x}_{N_m}^{(m)} = (x_1^{(m)}, \dots, x_{N_m}^{(m)})$, $N_m \le N$ and $d\boldsymbol{x}_{N_m}^{(m)} = \prod_{i=1}^{N_m} dx_i^{(m)}$, $1 \le m \le M$, and $p_\xi\left(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)}\right)$ denotes the multitime probability density.

Theorem 1

The Dyson model starting from **any fixed configuration** $\xi \in \mathfrak{M}$ with $\xi(\mathbb{R}) \in \mathbb{N}$ is **determinantal** in the sense that

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \mathop{\mathrm{Det}}_{\substack{(s,t)\in(t_1,t_2,\dots,t_M)^2, \\ (x,y)\in\mathbb{R}^2}} \Big[\delta_{st}\delta_x(y) + \mathbb{K}_\xi(s,x;t,y)\chi_t(y) \Big]$$

Fredholm determinant

with the continuous kernel (called **the correlation kernel**)

$$\mathbb{K}_\xi(s,x;t,y) = \mathcal{G}_{s,t}(x,y) - \mathbf{1}(s>t)p_{t,s}(y,x),$$

where

$$\mathcal{G}_{s,t}(x,y) = \frac{1}{2\pi\sqrt{-1}} \oint_{\Gamma(\xi)} dz\, p_{0,s}(z,x) \int_{\mathbb{R}} dw\, p_{0,t}(w,-\sqrt{-1}y) \frac{1}{\sqrt{-1}w - z} \Phi_\xi^z(\sqrt{-1}w),$$

$$p_{s,t}(x,y) = \frac{e^{-(y-x)^2/\{2(t-s)\}}}{\sqrt{2\pi(t-s)}} \quad \text{(heat kernel)}$$

Weierstrass canonical product rep.
with genus 0

and

Entire function

$$\Phi_\xi^u(z) = \prod_{x\in\mathrm{supp}\ \xi\cap\{u\}^c} \left(1 - \frac{z-u}{x-u}\right)^{\xi(\{x\})}$$

with supp $\xi = \{x \in \mathbb{R} : \xi(\{x\}) > 0\}$. Here $\Gamma(\xi)$ is a closed contour on the complex plane $\mathbb{C}$ encircling the points in supp ξ on the real line $\mathbb{R}$ once in the positive direction.

Remark 2A.

By definition of **Fredholm determinant**

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \operatorname*{Det}_{\substack{(s,t)\in(t_1,t_2,\dots,t_M)^2,\\ (x,y)\in\mathbb{R}^2}} \Big[\delta_{st}\delta_x(y) + \mathbb{K}_\xi(s,x;t,y)\chi_t(y)\Big]$$

$$= \sum_{\substack{N_m\geq 0,\\ 1\leq m\leq M}} \int_{\prod_{m=1}^M \mathbb{W}_{N_m}^{\mathrm{A}}} \prod_{m=1}^{M} \left\{ d\boldsymbol{x}_{N_m}^{(m)} \prod_{i=1}^{N_m} \chi_{t_m}\left(x_i^{(m)}\right)\right\} \det_{\substack{1\leq i\leq N_m, 1\leq j\leq N_n,\\ 1\leq m,n\leq M}} \left[\mathbb{K}_\xi(t_m, x_i^{(m)}; t_n, x_j^{(n)})\right].$$

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \sum_{\substack{N_m\geq 0,\\ 1\leq m\leq M}} \int_{\prod_{m=1}^M \mathbb{W}_{N_m}^{\mathrm{A}}} \prod_{m=1}^{M} \left\{ d\boldsymbol{x}_{N_m}^{(m)} \prod_{i=1}^{N_m} \chi_{t_m}\left(x_i^{(m)}\right)\right\} \rho_\xi\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\right)$$

Any **multitime correlation function** is given by a determinant

$$\rho_\xi\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\right) = \det_{\substack{1\leq i\leq N_m, 1\leq j\leq N_n,\\ 1\leq m,n\leq M}} \left[\mathbb{K}_\xi(t_m, x_i^{(m)}; t_n, x_j^{(n)})\right].$$

Remark 2B.

If we consider the particle distribution **at a single time** $t > 0$;

$$M = 1, \qquad \boldsymbol{t} = (t), \qquad \boldsymbol{f} = (f), \quad \chi_t(\,\cdot\,) = e^{f_t(\,\cdot\,)} - 1,$$

then

$$\Psi^t_\xi[f] = \operatorname*{Det}_{(x,y)\in\mathbb{R}^2} \Big[\delta_x(y) + \mathbb{K}^t_\xi(x,y)\chi_t(y)\Big]$$

with

$$\mathbb{K}^t_\xi(x,y) = \frac{1}{2\pi\sqrt{-1}} \oint_{\Gamma(\xi)} dz\, p_{0,t}(z,x) \int_{\mathbb{R}} dw\, p_{0,t}(w, -\sqrt{-1}y) \frac{1}{\sqrt{-1}w - z} \Phi^z_\xi(\sqrt{-1}w).$$

Any single-time distribution is a **determinantal (Fermion) point process** with the spatial correlations

$$\rho^t_\xi\Big(\boldsymbol{x}_{N_1}\Big) = \det_{1\le i,j\le N_1} \Big[\mathbb{K}^t_\xi(x_i,x_j)\Big], \quad t \in [0,\infty), \quad N_1 \le \xi(\mathbb{R}).$$

See Soshnikov (2000), Shirai-Takahashi (2003), Hough-Krishnapur-Peres-Virág (2009).

Remark 2C.

In particular, if the initial configuration is of N-multiple concentrated on the origin:

$$\xi = N\delta_0 = \xi(\mathbb{R})\delta_0 \quad (i.e.\ \text{all particles start from the origin}),$$

then

$$\mathbb{K}_{N\delta_0}(s,x;t,y) = \begin{cases} \dfrac{1}{\sqrt{2s}} \displaystyle\sum_{k=0}^{N-1} \left(\frac{t}{s}\right)^{k/2} \varphi_k\left(\frac{x}{\sqrt{2s}}\right) \varphi_k\left(\frac{y}{\sqrt{2t}}\right) & \text{if} \quad s \le t \\ -\dfrac{1}{\sqrt{2s}} \displaystyle\sum_{k=N}^{\infty} \left(\frac{t}{s}\right)^{k/2} \varphi_k\left(\frac{x}{\sqrt{2s}}\right) \varphi_k\left(\frac{y}{\sqrt{2t}}\right) & \text{if} \quad s > t, \end{cases}$$

where

$$\varphi_k(\zeta) = \frac{1}{\sqrt{\sqrt{\pi}2^k k!}} e^{-\zeta^2/2} H_k(\zeta), \quad k \in \mathbb{N}_0 = \{0,1,2,\dots\} \quad \text{(Hermite functions)}.$$

the extended Hermite kernel Eynard-Mehta (1998)

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the extended Hermite kernel **Eynard-Mehta (1998)**

$$\begin{aligned} &\mathbb{K}_\xi(s,x;t,y) + \mathbf{1}(s>t) p_{s,t}(y,x) \\ &= \frac{1}{2\pi\sqrt{-1}} \oint_{\Gamma(\xi)} dz\, p_{0,s}(z,x) \int_{\mathbb{R}} dw\, p_{0,t}(w, -\sqrt{-1}y) \frac{1}{\sqrt{-1}w - z} \Phi_\xi^z(\sqrt{-1}w) \\ &= \frac{1}{\sqrt{s}} \sum_{k=0}^{N-1} \left(\frac{t}{s}\right)^{k/2} H_k^{(+)}\left(\frac{x}{\sqrt{s}}; \frac{1}{\sqrt{s}} \circ \xi\right) H_k^{(-)}\left(\frac{y}{\sqrt{t}}; \frac{1}{\sqrt{t}} \circ \xi\right). \end{aligned}$$

multiple Hermite polynomials (see Ismail (2005), Bleher-Kuijlaars (2005)).

2. Construction of Infinite Particle Systems

Consider the special case treated in **Remark 2C**.

$$\xi = N\delta_0 = \xi(\mathbb{R})\delta_0 \quad (i.e. \text{ all particles start from the origin}),$$

then

$$\mathbb{K}_{N\delta_0}(s,x;t,y) = \begin{cases} \dfrac{1}{\sqrt{2s}} \displaystyle\sum_{k=0}^{N-1} \left(\frac{t}{s}\right)^{k/2} \varphi_k\left(\frac{x}{\sqrt{2s}}\right) \varphi_k\left(\frac{y}{\sqrt{2t}}\right) & \text{if} \quad s \le t \\ -\dfrac{1}{\sqrt{2s}} \displaystyle\sum_{k=N}^{\infty} \left(\frac{t}{s}\right)^{k/2} \varphi_k\left(\frac{x}{\sqrt{2s}}\right) \varphi_k\left(\frac{y}{\sqrt{2t}}\right) & \text{if} \quad s > t, \end{cases}$$

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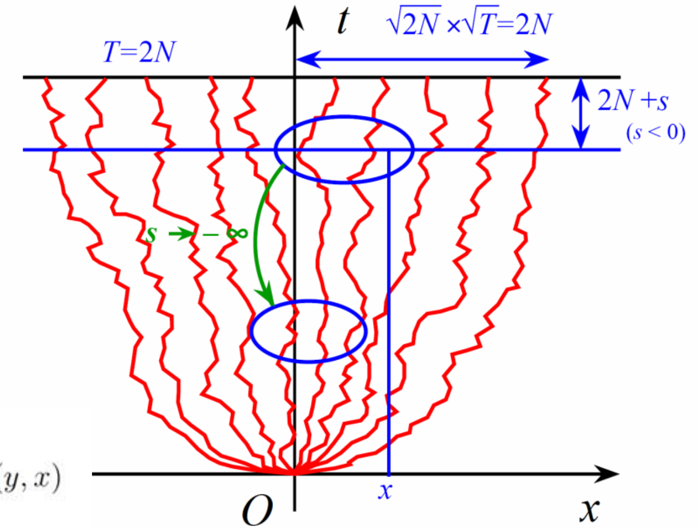


$$\xi(\mathbb{R}) = N \to \infty$$

$$\text{time} = \frac{2N}{\pi^2} + t$$

$$\begin{aligned} \mathbf{K}_{\sin}(s,x;t,y) &= \frac{1}{2\pi}\int_{|k|\le\pi} dk\, e^{k^2(t-s)/2 + ik(y-x)} - \mathbf{1}(s>t) p_{t,s}(y,x) \\ &= \begin{cases} \displaystyle\int_0^1 du\, e^{\pi^2 u^2 (t-s)/2} \cos\{\pi u (y-x)\} & \text{if } t > s \\ K_{\sin}(x,y) & \text{if } t = s \\ \displaystyle -\int_1^\infty du\, e^{\pi^2 u^2 (t-s)/2} \cos\{\pi u (y-x)\} & \text{if } t < s, \end{cases} \end{aligned}$$

$s, t \ge 0, x, y \in \mathbb{R}$ with

$$K_{\sin}(x,y) = \frac{1}{2\pi}\int_{|k|\le\pi} dk\, e^{ik(y-x)} = \frac{\sin\{\pi(y-x)\}}{\pi(y-x)}, \quad x, y \in \mathbb{R}.$$

The extended sine-kernel (Nagao-Forrester (1998)).

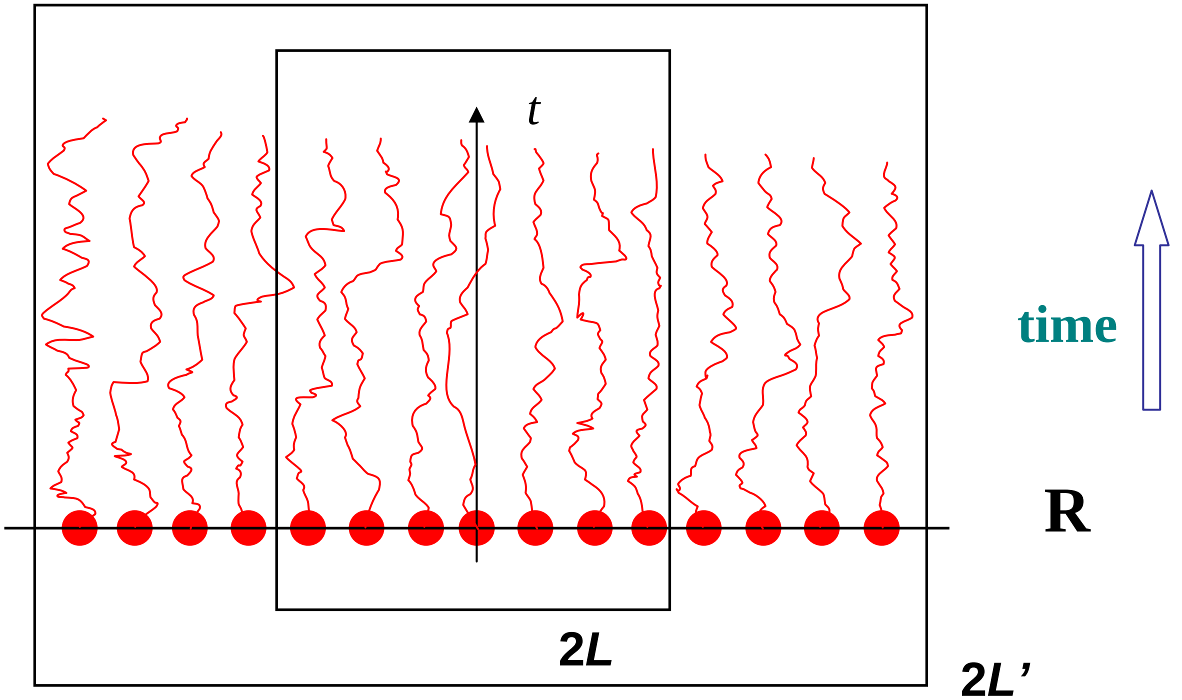
It is expected that the determinantal process having an infinite number of particels with the extended sine-kernel describes the equilibrium dynamics of the infinite particle system constructed by Osada (1996) by the Dirichlet form approach.

- Consider this infinite particle system from the view-point of SDEs:

$$dX_i(t) = dB_i(t) + \sum_{1 \le j \le N j \ne i} \frac{dt}{X_i(t) - X_j(t)}, \quad 1 \le i \le N\,, \;\; t \in [0, \infty).$$

- Since the $1/x$ force is not summable, in the infinite-particle limit $N \to \infty$ the sum in the above SDEs should be regarded as an **improper sum**, in the sense that for $X_i(t) \in [-L, L]$ the summation is restricted to j's such that $X_j(t) \in [-L, L]$ and then the limit $L \to \infty$ is taken.

- It is expected that the dynamics with infinite number of particles can exist only for initial configurations having the same asymptotic density to the right and left.



In our formula,

$$\lim_{N\to\infty} N\delta_0 \notin \mathfrak{M}.$$

Another route to infinite particle systems.

- For $L > 0, \alpha > 0$ and $\xi \in \mathfrak{M}$ we put

$$M(\xi, L) = \int_{[-L,L]\setminus\{0\}} \frac{\xi(dx)}{x}, \qquad M_\alpha(\xi, L) = \left(\int_{[-L,L]\setminus\{0\}} \frac{\xi(dx)}{|x|^\alpha}\right)^{1/\alpha},$$

  and

$$M(\xi) = \lim_{L\to\infty} M(\xi, L), \quad M_\alpha(\xi) = \lim_{L\to\infty} M_\alpha(\xi, L),$$

  if the limits finitely exist.

- We have introduced the following conditions for initial configurations $\xi \in \mathfrak{M}$:

  (**C.1**) there exists $C_0 > 0$ such that $|M(\xi, L)| < C_0$, $L > 0$,

  (**C.2**) (i) there exist $\alpha \in (1, 2)$ and $C_1 > 0$ such that $M_\alpha(\xi) \le C_1$,
  (ii) there exist $\beta > 0$ and $C_2 > 0$ such that

$$M_1(\tau_{-a^2}\xi^{\langle 2\rangle}) \le C_2(\max\{|a|, 1\})^{-\beta} \quad \forall a \in \text{supp } \xi.$$

- Set $\mathfrak{M}_0 = \Big\{\xi \in \mathfrak{M} : \xi(\{x\}) \le 1 \text{ for any } x \in \mathbb{R}\Big\}.$

- It was shown that, if $\xi \in \mathfrak{M}_0$ satisfies the conditions (**C.1**) and (**C.2**), then for $a \in \mathbb{R}$ and $z \in \mathbb{C}$,

$$\Phi_\xi^a(z) \equiv \lim_{L\to\infty} \Phi_{\xi\cap[a-L,a+L]}^a(z) \quad \text{finitely exists,}$$

and

$$|\Phi_\xi^a(z)| \le C \exp\left\{ c(|a|^\theta + |z|^\theta) \right\} \left| \frac{z}{a} \right|^{\xi(\{0\})} \left| \frac{a}{a-z} \right|, \quad a \in \text{supp}\ \xi, \ z \in \mathbb{C},$$

for some $c, C > 0$ and $\theta \in (\max\{\alpha, (2-\beta)\}, 2)$, which are determined by the constants C_0, C_1, C_2 and the indices α, β in the conditions.

- Then even if $\xi(\mathbb{R}) = \infty$, under the conditions (**C.1**) and (**C.2**), $\mathbb{K}_\xi$ is well-defined as a correlation kernel and dynamics of the Dyson model with an infinite number of particles $(\Xi(t), \mathbb{P}_\xi)$ exists as a determinantal process.

- We note that in the case that $\xi \in \mathfrak{M}_0$ satisfies the conditions (**C.1**) and (**C.2**) with constants C_0, C_1, C_2 and indices α and β, then $\xi \cap [-L, L], \forall L > 0$ does as well. Then we can obtain the convergence of moment generating functions

$$\Psi_{\xi\cap[-L,L]}^{\boldsymbol{t}}[\boldsymbol{f}] \to \Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] \quad \text{as} \quad L \to \infty,$$

which implies the convergence of the probability measures

$$\mathbb{P}_{\xi\cap[-L,L]} \to \mathbb{P}_\xi \quad \text{as} \quad L \to \infty$$

in the sense of finite dimensional distributions.

- Another way to the equilibrium dynamics of infinite particle system with the extended sine kernel.

- Set the initial configuration as

$$\xi_{\mathbb{Z}}(\cdot) \equiv \sum_{\ell \in \mathbb{Z}} \delta_\ell(\cdot),$$

  that is, the configuration in which every point of $\mathbb{Z}$ is occupied by one particle.

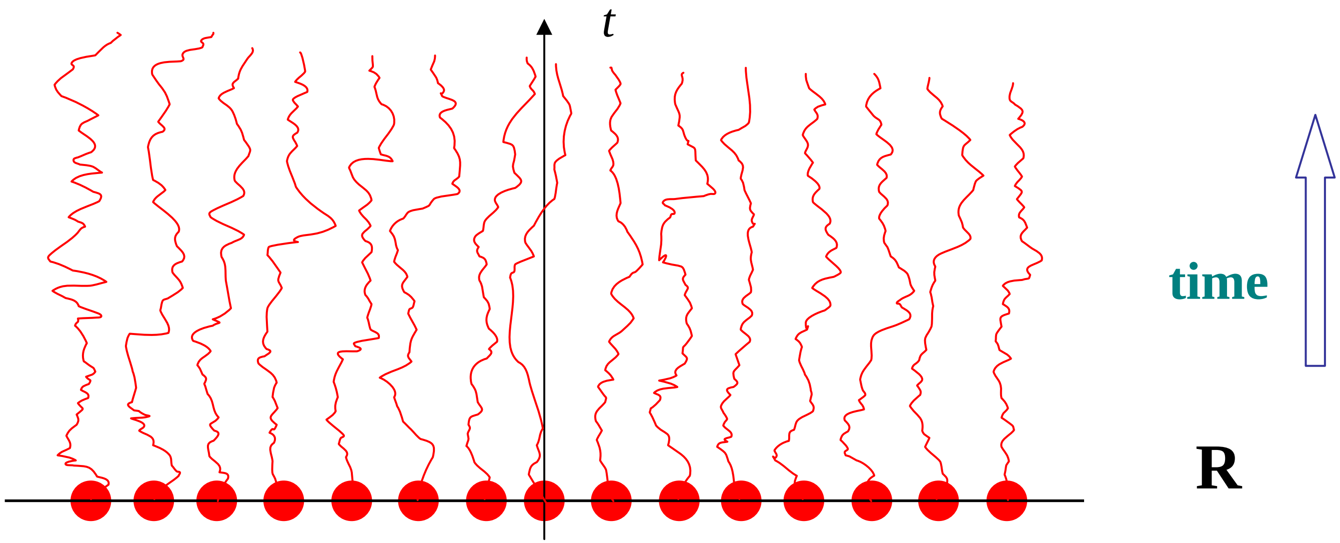
- This configuration $\xi_{\mathbb{Z}}$ satisfies our conditions and Dyson's model starting from $\xi_{\mathbb{Z}}$ is determinantal with the kernel

$$\begin{aligned}
\mathbb{K}_{\xi_{\mathbb{Z}}}(s,x;t,y) &= \mathbf{K}_{\sin}(s,x;t,y) \\
&\quad + \frac{1}{2\pi} \int_{|k|\le\pi} dk\, e^{k^2(t-s)/2+ik(y-x)} \Big\{ \vartheta_3(x-iks, 2\pi i s) - 1 \Big\} \\
&= \mathbf{K}_{\sin}(s,x;t,y) \\
&\quad + \sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{2\pi i x\ell - 2\pi^2 s\ell^2} \int_0^1 du\, e^{\pi^2 u^2(t-s)/2} \cos\Big[\pi u \{(y-x) - 2\pi i s\ell\}\Big],
\end{aligned}$$

  $s,t \ge 0, x,y \in \mathbb{R}$, where ϑ_3 is a version of the Jacobi theta function defined by

$$\vartheta_3(v,\tau) = \sum_{\ell\in\mathbb{Z}} e^{2\pi i v\ell + \pi i \tau \ell^2}, \quad \Im\tau > 0.$$

- The lattice structure $\mathbb{K}_{\xi_{\mathbb{Z}}}(s, x+n; t, y+n) = \mathbb{K}^{\xi^{\mathbb{Z}}}(s, x; t, y), \forall n \in \mathbb{Z}, s, t \geq 0$ is clear by the periodicity, $\vartheta_3(v+n, \tau) = \vartheta_3(v, \tau), \forall n \in \mathbb{Z}$.
- We can prove
$$\lim_{u \to \infty} \mathbb{K}_{\xi_{\mathbb{Z}}}(u+s, x; u+t, y) = \mathbf{K}_{\sin}(s, x; t, y).$$
- The **relaxation process** starting from $\xi_{\mathbb{Z}}$ to the stationary state, which is the determinantal point process with the sine kernel.



3. Inhomogeneous Infinite Particle Systems

- **Problem:** How we can control Dyson's model with infinite number of particles starting from *asymmetric* initial configurations.

- The motivation is again coming from the **random matrix theory** as follows.
  Consider the *Airy function*

$$\mathrm{Ai}(z) = \frac{1}{2\pi} \int_{\mathbb{R}} dk\, e^{\sqrt{-1}(zk+k^3/3)}.$$

It is a solution of Airy's equation $f''(z) - zf(z) = 0$ with the asymptotics on the real axis $\mathbb{R}$

$$\mathrm{Ai}(x) \simeq \frac{1}{2\sqrt{\pi}x^{1/4}} \exp\left(-\frac{2}{3}x^{3/2}\right), \quad \mathrm{Ai}(-x) \simeq \frac{1}{\sqrt{\pi}x^{1/4}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right) \quad \text{in} \quad x \to +\infty.$$

- In the GUE random matrix theory, the following scaling limit has been extensively studied:

$$\lim_{N\to\infty} \mu^{\mathrm{GUE}}_{N,N^{1/3}}(2N^{2/3} + \cdot) = \mu_{\mathrm{Ai}}(\cdot),$$

where μ_{Ai} is the determinantal point process such that the correlation kernel is given by (Tracy-Widom (1994)),

$$\begin{aligned} K_{\mathrm{Ai}}(y|x) &= \int_0^\infty du\, \mathrm{Ai}(u+x)\mathrm{Ai}(u+y) \\ &= \begin{cases} \dfrac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}, & x \neq y \in \mathbb{R} \\ (\mathrm{Ai}'(x))^2 - x(\mathrm{Ai}(x))^2, & x = y \in \mathbb{R}. \end{cases} \end{aligned}$$

It is called the *soft-edge scaling limit*, since $x^2/2t \simeq (2N^{2/3})^2/(2N^{1/3}) = 2N$ marks the right edge of semicircle-shaped profile of the GUE eigenvalue distribution.

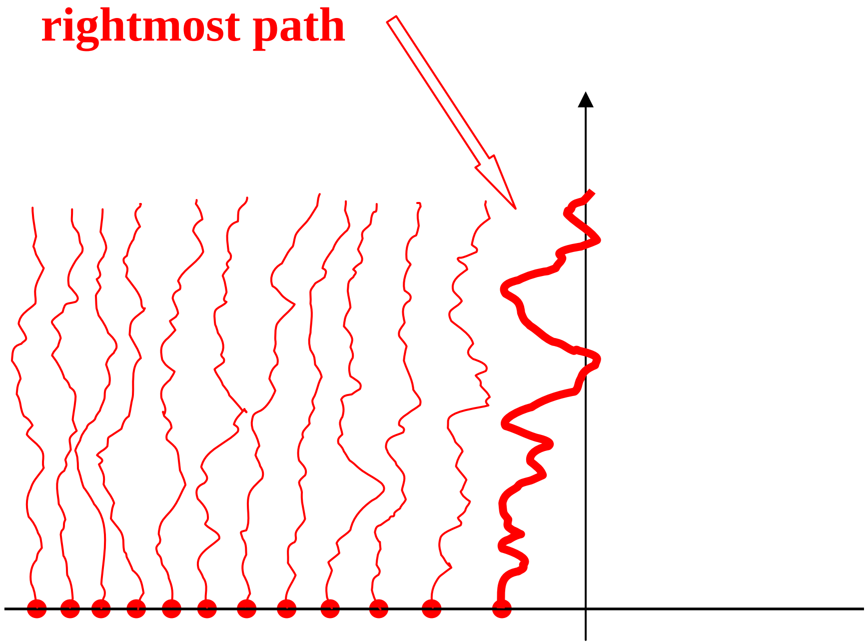
- The particle distribution μ_{Ai} with the *Airy kernel* is highly asymmetric: As a matter of fact, the particle density $\rho_{\mathrm{Ai}}(x) = K_{\mathrm{Ai}}(x|x)$ decays rapidly to zero as $x \to \infty$, but it diverges

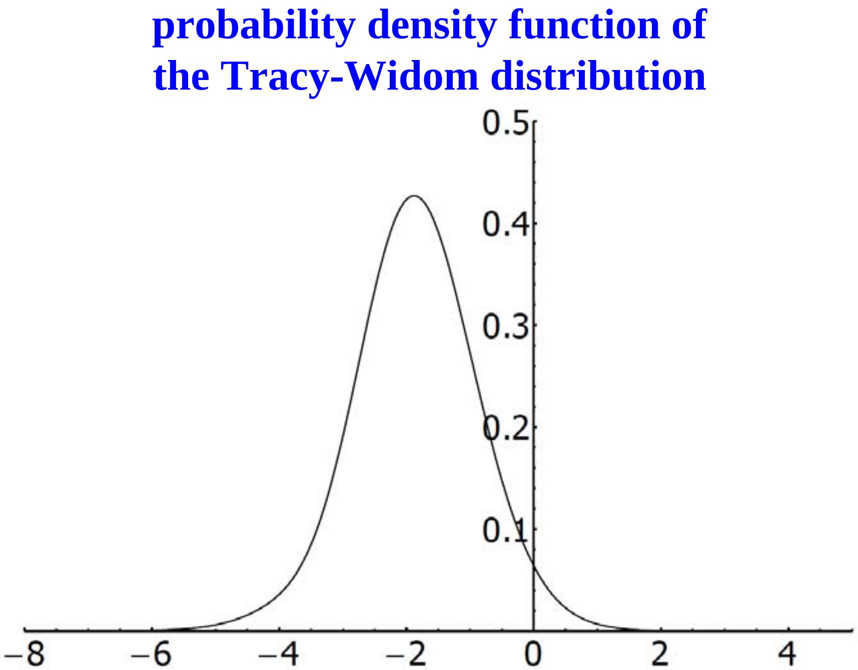
$$\rho_{\mathrm{Ai}}(x) \simeq \frac{1}{\pi}(-x)^{1/2} \to \infty \quad \text{as} \quad x \to -\infty.$$

- Let R be the position of the rightmost particle on $\mathbb{R}$ in μ_{Ai}. Then its distribution is given by the celebrated *Tracy-Widom distribution* (Tracy-Widom (1994))

$$\mu_{\mathrm{Ai}}(R < x) = \exp\left[-\int_x^\infty (y-x)(q(y))^2 dy\right],$$

where $q(x)$ is the unique solution of the Painlevé II equation $q'' = xq + 2q^3$ satisfying the boundary condition $q(x) \simeq \mathrm{Ai}(x)$ in $x \to \infty$.





- As an explicit answer to the above questions, we have presented a **relaxation process** with infinite number of particles converging to the stationary state μ_{Ai} in $t \to \infty$. Its initial configuration is given by

$$\xi_{\mathcal{A}}(\cdot) = \sum_{a \in \mathcal{A}} \delta_a(\cdot) = \sum_{i=1}^{\infty} \delta_{a_i}(\cdot),$$

  in which every **zero of the Airy function** is occupied by one particle.

- This special choice of the initial configuration is due to the fact that the zeros of the Airy function are located only on the negative part of the real axis $\mathbb{R}$,

$$\mathcal{A} \equiv \mathrm{Ai}^{-1}(0) = \Big\{ a_i, i \in \mathbb{N} : \mathrm{Ai}(a_i) = 0,\ 0 > a_1 > a_2 > \cdots \Big\},$$

  with the values $a_1 = -2.33\ldots, a_2 = -4.08\ldots, a_3 = -5.52\ldots, a_4 = -6.78\ldots,$ and that they admit the asymptotics $a_i \simeq -\left(\frac{3\pi}{2}\right)^{2/3} i^{2/3}$ in $i \to \infty$.

- Then the average density of zeros of the Airy function around x, denoted by $\rho_{\mathrm{Ai}^{-1}(0)}(x)$, behaves as

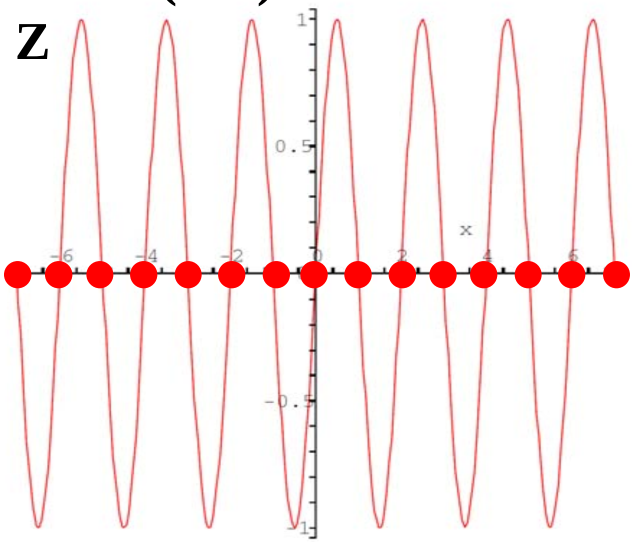
$$\rho_{\mathrm{Ai}^{-1}(0)}(x) \simeq \frac{1}{\pi}(-x)^{1/2} \to \infty \quad \text{as} \quad x \to -\infty,$$

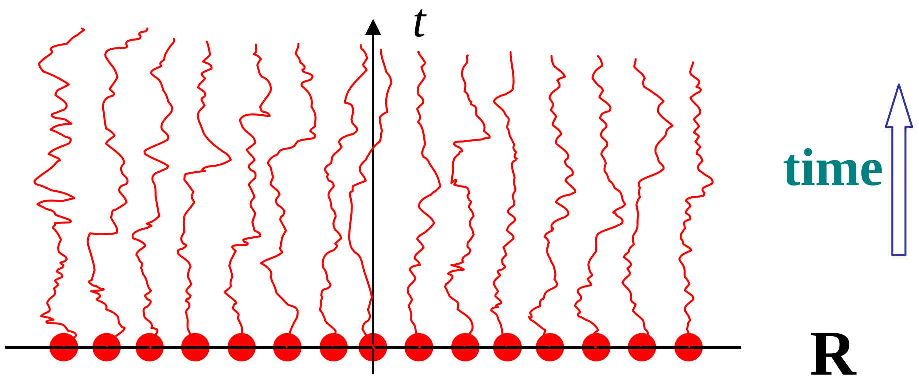
  which coincides with

$$\rho_{\mathrm{Ai}}(x) \simeq \frac{1}{\pi}(-x)^{1/2} \to \infty \quad \text{as} \quad x \to -\infty.$$

zeros of $\sin(\pi x)$

$= \mathbf{Z}$

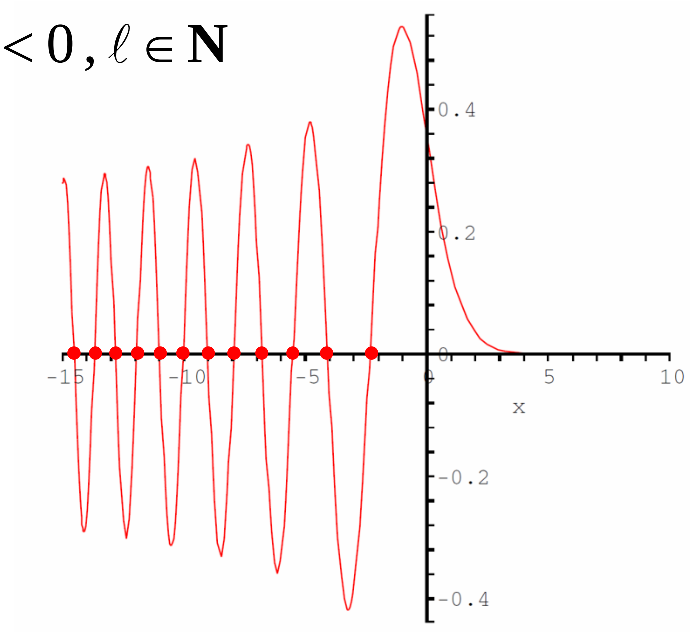


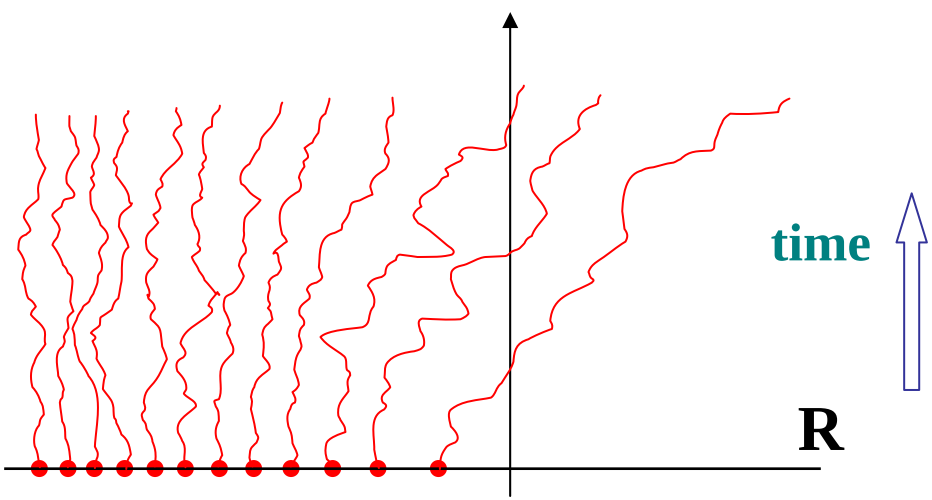


relaxation to 'sine process'

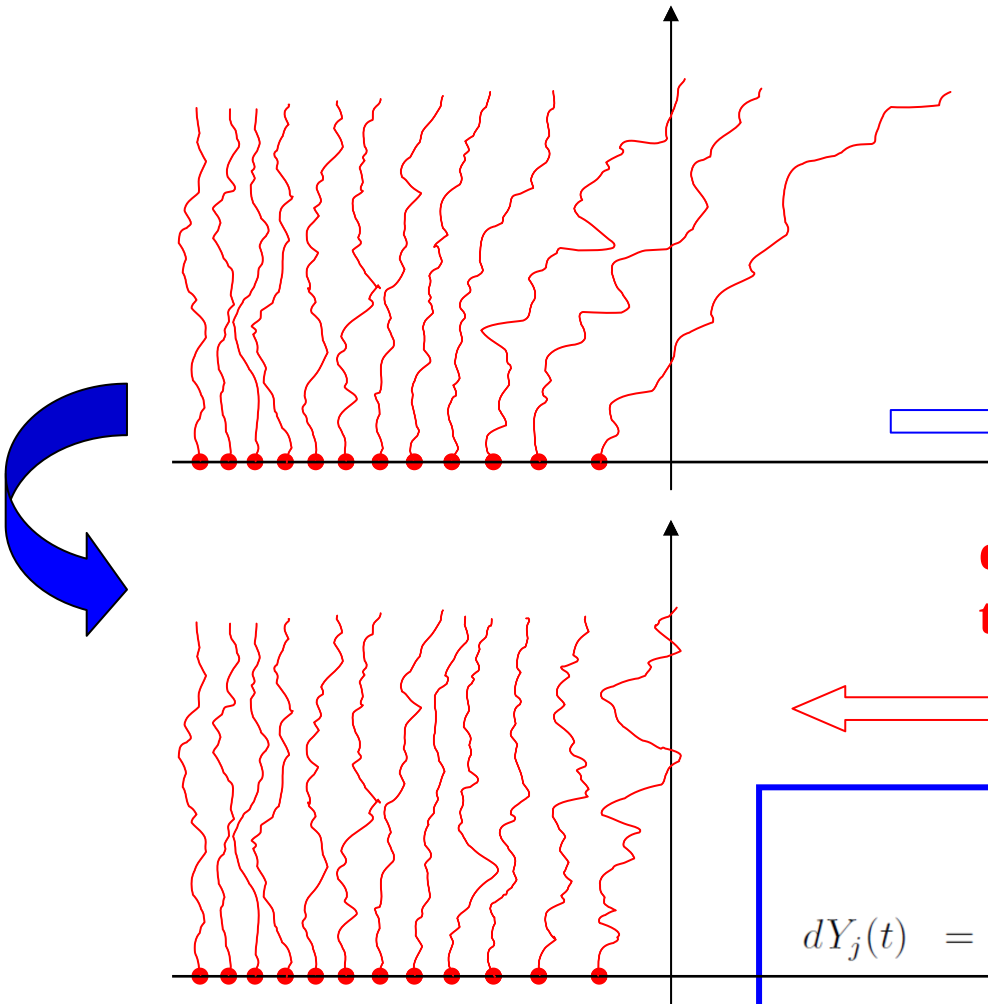
Airy zeros

$a_\ell < 0\,,\, \ell \in \mathbf{N}$





relaxation to 'Airy process' ???



confinement in the negative region R- at time *t* = 0

strong repulsive forces due to noncolliding condition

drift terms to make a stationary process

$$\boldsymbol{Y}(t) = \big(Y_1(t), Y_2(t), \dots, Y_N(t)\big)$$

$$dY_j(t) = dB_j(t) + \left(\frac{t}{2} + D_{\mathcal{A}_N}\right) dt + \sum_{\substack{1\le k\le N\\ k\ne j}} \frac{dt}{Y_j(t)-Y_k(t)}$$

$$D_{\mathcal{A}_N} = d_1 + \sum_{\ell=1}^{N} \frac{1}{a_\ell} \simeq -\left(\frac{12}{\pi^2}\right)^{1/3} \underline{N^{1/3} \to -\infty \quad \text{as} \quad N \to \infty}$$

- The approximation of our process with a finite number of particles $N < \infty$ is given by

$$\Xi_{\mathcal{A}}(t) = \sum_{i=1}^{N} \delta_{Y_i(t)}$$

with

$$Y_i(t) = X_i(t) + \frac{t^2}{4} + D_{\mathcal{A}_N} t, \quad 1 \le i \le N, \quad t \in [0, \infty),$$

associated with the solution $\boldsymbol{X}(t) = (X_1(t), \dots, X_N(t))$ of Dyson's model, where

$$D_{\mathcal{A}_N} = d_1 + \sum_{\ell=1}^{N} \frac{1}{a_\ell}.$$

Here $d_1 = \text{Ai}'(0)/\text{Ai}(0)$ and $\mathcal{A}_N \equiv \Big\{0 > a_1 > \cdots > a_N\Big\} \subset \mathcal{A}$ is the sequence of the first N zeros of the Airy function.

- In other words, $\boldsymbol{Y}(t) = (Y_1(t), Y_2(t), \dots, Y_N(t))$ satisfies the following SDEs ;

$$\begin{aligned} dY_i(t) &= dB_i(t) + \left(\frac{t}{2} + D_{\mathcal{A}_N}\right) dt + \sum_{\substack{1 \le j \le N \\ j \ne i}} \frac{dt}{Y_i(t) - Y_j(t)} \\ &= dB_i(t) + \sum_{\substack{1 \le j \le N \\ j \ne i}} \left(\frac{1}{Y_i(t) - Y_j(t)} + \frac{1}{a_j}\right) dt + \left(\frac{t}{2} + d_1 + \frac{1}{a_i}\right) dt, \\ &\qquad\qquad 1 \le i \le N, \quad t \in [0, \infty). \end{aligned}$$

- Note that

$$D_{\mathcal{A}_N} \simeq -\left(\frac{12}{\pi^2}\right)^{1/3} N^{1/3} \to -\infty \quad \text{as} \quad N \to \infty.$$

$$\mathbb{K}_{\mathrm{Ai}}(s,x;t,y) = \sum_{a\in \mathrm{Ai}^{-1}(0)} \int_{\sqrt{-1}\mathbb{R}} \frac{dz}{\sqrt{-1}} p_{\mathrm{Ai}}(s,x|a) \frac{1}{z-a} \frac{\mathrm{Ai}(z)}{\mathrm{Ai}'(a)} p_{\mathrm{Ai}}(-t,z|y) - \mathbf{1}(s>t) p_{\mathrm{Ai}}(s-t,x|y)$$

$$= \int_0^\infty du \int_{\mathbb{R}} dw\, e^{-ut/2+ws/2} \mathrm{Ai}(u+y)\mathrm{Ai}(w+x) \sum_{\ell=1}^{\infty} \frac{\mathrm{Ai}(u+a_\ell)\mathrm{Ai}(w+a_\ell)}{(\mathrm{Ai}'(a_\ell))^2} - \mathbf{1}(s>t) p_{\mathrm{Ai}}(s-t,x|y)$$

$t \to \infty$

$s \to \infty$

$|t-s| < \infty$

Relaxation Process

$$\mathbf{K}_{\mathrm{Ai}}(t,y|x) = \begin{cases} \displaystyle\int_0^\infty du\, e^{-ut/2} \mathrm{Ai}(u+x)\mathrm{Ai}(u+y) & \text{if } t \geq 0 \\ \displaystyle -\int_{-\infty}^0 du\, e^{-ut/2} \mathrm{Ai}(u+x)\mathrm{Ai}(u+y) & \text{if } t < 0, \end{cases}$$

extended Airy kernel

Let $q_{s,t}(x,y) =$ transition probability density of $B(t)+t^2/4$

$$= p_{s,t}\left(\left(x-\frac{s^2}{4}\right),\left(y-\frac{t^2}{4}\right)\right)$$

and set $\widehat{g}(s,x) \equiv \exp\left\{-D_{\mathcal{A}_N}\left(\frac{D_{\mathcal{A}_N}s}{2}+\frac{s^2}{4}-x\right)\right\}.$

Then $p_{0,s}\left(x',\left(x-D_{\mathcal{A}_N}s-\frac{s^2}{4}\right)\right) = q_{0,s}(x',x)\times\widehat{g}(s,x)e^{-D_{\mathcal{A}_N}x'}$

Then

$$\mathbb{K}_\xi\left(s,x-D_{\mathcal{A}_N}s-\frac{s^2}{4};t,y-D_{\mathcal{A}_N}t-\frac{t^2}{4}\right)$$

$$=\frac{\widehat{g}(s,x)}{\widehat{g}(t,y)}\left[\int_{\mathbb{R}}\xi^N(dx')\int_{\sqrt{-1}\mathbb{R}}\frac{dy'}{\sqrt{-1}}\, q_{0,s}(x',x)e^{-D_{\mathcal{A}_N}x'}\Phi_\xi^{x'}(y')e^{D_{\mathcal{A}_N}y'}q_{t,0}(y,y')-\mathbf{1}(s>t)q_{t,s}(y,x)\right]$$

$$\equiv\frac{\widehat{g}(s,x)}{\widehat{g}(t,y)}\mathbb{K}_\xi^{\mathcal{A}}(s,x;t,y).$$

Weierstrass canonical product with genus 1

$$e^{-D_{\mathcal{A}_N}x'}\Phi_\xi^{x'}(y')e^{D_{\mathcal{A}_N}y'} = e^{d_1(y'-x')}\prod_{x\in\xi^N\cap\{x'\}^c}\left[\left(1-\frac{y'-x'}{x-x'}\right)\exp\left(\frac{y'-x'}{x}\right)\right]$$

$$\equiv \widehat{\Phi}_\xi^{x'}(y').$$

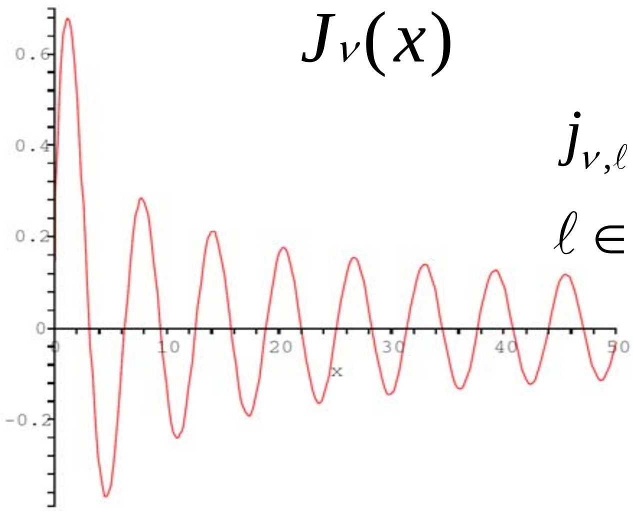
4. Noncolliding BESQ showing relaxation to determinantal process with extended Bessel kernel

SDEs for finite particle approximation

$$\boldsymbol{X}^{(\nu)}(t) = \left(X_1^{(\nu)}(t), X_2^{(\nu)}(t), \dots, X_N^{(\nu)}(t) \right) \in \mathbb{W}_N^{\mathrm{C}} \quad \text{Weyl chamber of type } C_N, \quad t \in [0, \infty)$$

$$dX_j^{(\nu)}(t) = 2\sqrt{X_j^{(\nu)}} dB_j(t) + 2 \left\{ N + \nu + \sum_{\substack{1 \le k \le N \\ k \ne j}} \frac{X_j^{(\nu)}(t) + X_k^{(\nu)}(t)}{X_j^{(\nu)}(t) - X_k^{(\nu)}(t)} \right\} dt, \quad 1 \le j \le N, \quad t \in [0, \infty),$$

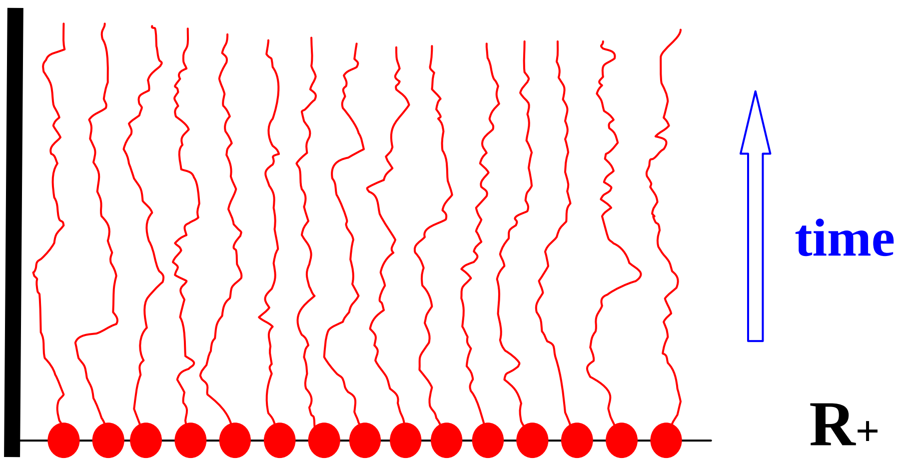
where $B_j(t)$ are independent one-dimensional standard BMs



$J_\nu(x)$

$j_{\nu,\ell} \Rightarrow j_{\nu,\ell}{}^2$

$\ell \in \mathbf{N}$



$$
\begin{aligned}
\mathbb{K}_{J_\nu}(s,x;t,y) &= \sum_{\ell=1}^{\infty}\int_{-\infty}^{0} dz\, p_{J_\nu}(s,x|j_{\nu,\ell}^2)\frac{2j_{\nu,\ell}}{z-j_{\nu,\ell}^2}\frac{J_\nu(\sqrt{z})}{J_{\nu+1}(j_{\nu,\ell})}p_{J_\nu}(-t,z|y) \\
&\quad -\mathbf{1}(s>t)p_{J_\nu}(s-t,x|y) \\
&= \int_0^1 du \int_0^{\infty} dw\, e^{ut/2-2ws} J_\nu(\sqrt{uy})J_\nu(2\sqrt{wx})\sum_{\ell=1}^{\infty}\frac{J_\nu(2\sqrt{w}j_{\nu,\ell})J_\nu(\sqrt{u}j_{\nu,\ell})}{(J_{\nu+1}(j_{\nu,\ell}))^2} \\
&\quad -\mathbf{1}(s>t)p_{J_\nu}(s-t,x|y)
\end{aligned}
$$

$t \to \infty$

$s \to \infty$

$|t-s| < \infty$

Relaxation Process

$$
\mathbf{K}_{J_\nu}(t-s,y|x) = \begin{cases}
\displaystyle\int_0^1 du\, e^{-2u(s-t)} J_\nu(2\sqrt{ux})J_\nu(2\sqrt{uy}) & \text{if} \quad s<t \\[2ex]
\displaystyle\frac{J_\nu(2\sqrt{x})\sqrt{y}J'_\nu(2\sqrt{y})-\sqrt{x}J'_\nu(2\sqrt{x})J_\nu(2\sqrt{y})}{x-y} & \text{if} \quad t=s \\[2ex]
\displaystyle -\int_1^{\infty} du\, e^{-2u(s-t)} J_\nu(2\sqrt{ux})J_\nu(2\sqrt{uy}) & \text{if} \quad s>t.
\end{cases}
$$

extended Bessel kernel

5. Concluding Remarks

Theory of Entire Functions

order of growth ρ_f

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r} \quad \text{for} \quad M_f(r) = \max_{|z|=r} |f(z)|$$

$$\Longrightarrow \quad \max_{|z|=r} |f(z)| \sim \exp(r^{\rho_f})$$

Weierstrass primary factors

$$G(u,p) = \begin{cases} 1-u & \text{if} \quad p=0 \\ (1-u)\exp\left[u + \dfrac{u^2}{2} + \cdots + \dfrac{u^p}{p}\right] & \text{if} \quad p \in \mathbb{N}. \end{cases}$$

Weierstrass canonical product of genus p

$$\Pi_p(\xi, z) = \prod_{x\in\xi\cap\{0\}^c} G\left(\frac{z}{x}, p\right), \quad z \in \mathbb{C}$$

Hadamard theorem

Any entire function f of finite order $\rho_f < \infty$ can be represented by

$$f(z) = z^m e^{P_q(z)} \Pi_p(\xi_f, z),$$

$p =$ a nonnegative integer less than or equal to ρ_f,

$P_q(z) =$ a polynomial in z of degree $q \le \rho_f$,

$m =$the multiplicity of the root at the origin,

and $\xi_f = \displaystyle\sum_{x \in f^{-1}(0) \cap \{0\}^c} \delta_x .$

$$\sin \pi z = \pi z \Pi_0(\xi_{\mathbb{Z}}, z) = \pi z \prod_{x \in \xi_{\mathbb{Z}} \cap \{0\}^c} \left(1 - \frac{z}{x}\right) = \pi z \prod_{\ell \in \mathbb{Z}, \ell \neq 0} \left(1 - \frac{z}{\ell}\right)$$

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \Pi_0(\xi_{J_\nu}^{\langle 2 \rangle}, z^2) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_{x \in \xi_{J_\nu} \cap \{0\}^c} \left(1 - \frac{z^2}{x^2}\right) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_{\ell=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,\ell}^2}\right)$$

$$\mathrm{Ai}(z) = e^{d_0 + d_1 z} \Pi_1(\xi_{\mathcal{A}}, z) = e^{d_0 + d_1 z} \prod_{x \in \mathcal{A}} \left[\left(1 - \frac{z}{x}\right) e^{z/x}\right] = e^{d_0 + d_1 z} \prod_{\ell=1}^{\infty} \left[\left(1 - \frac{z}{a_\ell}\right) e^{z/a_\ell}\right]$$

General Theory for Entire Functions and Infinite Particle Systems ?